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NILPOTENT QUOTIENTS OF FUNDAMENTAL GROUPS OF SPECIAL 3-MANIFOLDS WITH BOUNDARY

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We use a Betti number estimate of Freedman-Hain-Teichner to show that the maximal torsion-free nilpotent quotient of the fundamental group of a 3-manifold with boundary is either $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$. In particular we reobtain the Evans-Moser classification of 3-manifolds with boundary which have nilpotent fundamental groups.

I. Nilpotent quotients of groups are closely related to the rational homotopy type of $K(\pi, 1)$ spaces. The object which defines this relation is called a *Malcev completion* and is dual to the one-minimal model [**5**]. Because of its naturality and computability, the Malcev completion can be used to describe nilpotent quotients of $\pi_1(X)$ from homotopic information about X. This approach is adopted in [**2**], [**6**], to study nilpotent quotient groups of orientable closed 3-manifolds.

Two types of (closed, orientable) 3-manifolds are distinguished in [**2**]: the *special type* 3-manifolds have finite number of stages in their one-minimal models, or equivalently their rational lower central series stabilises, that is, they admit a maximal torsion-free nilpotent quotient. The other 3-manifolds are of *general type.* It is shown in [**2**] that for 3-manifolds of special type the only possible maximal torsion-free nilpotent quotient groups of the fundamental group are 1, $\mathbf{Z}$ or the Heisenberg groups $H_n, n \in \mathbf{Z}$. By contrast [**4**] shows that in the general type case, the one-minimal models grow exponentially.

In this note we extend the special case to 3-manifolds with boundary. We find that the maximal torsion-free nilpotent quotient of the fundamental group of such a manifold must be $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$. If moreover the fundamental group itself is nilpotent, it follows that the 3-manifold is (essentially) either the solid torus or a thickened 2-torus.

II. Let us collect first a few definitions and facts about nilpotent groups and Malcev completions.

Let G be a group. Its *lower central series* is a decreasing series defined inductively by $\Gamma_2 = [G, G]$, $\Gamma_{r+1} = [G, \Gamma_r]$, with $[\,,]$ denoting commutators. We say G is *nilpotent (respectively, prenilpotent)* if $\Gamma_r = 1$ for some r (respectively, if $\Gamma_r = \Gamma_{r+1}$ for some r). It follows that G is prenilpotent if and only if it has a maximal nilpotent quotient.

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There are rational versions of the above definitions. The r-th term of the *rational lower central series* is defined by:

$$\Gamma_r{}^{\mathbf{Q}} = \mathrm{Rad}\,(\Gamma_r) = \{g \in G\,; \exists n \in \mathbf{Z}, n \neq 0,\ g^n \in \Gamma_r\}.$$

It has the defining property that $G/\Gamma_r{}^{\mathbf{Q}} = G/\Gamma_r/\mathrm{torsion}$.

We obtain the definition of *rationally (pre)nilpotent* groups by replacing Γ_r with $\Gamma_r{}^{\mathbf{Q}}$ in the above definitions. Note that G is rationally prenilpotent if and only if G has a maximal torsion-free nilpotent quotient.

We describe next the construction of the Malcev completion of a group G using nilpotent quotients. Let $N_r = G/\Gamma_r$. We have a tower of nilpotent quotients of G:

$$\ldots \to N_r \to N_{r-1} \to \ldots \to N_2 \to 0,$$

and at each stage, a central extension:

$$0 \to \Gamma_{r-1}/\Gamma_r \to N_r \to N_{r-1} \to 0.$$

Central extensions are classified by elements k_r in $H^2(\text{base; fibre})$. The tower of extensions above can be tensored with $\mathbf{Q}$ inductively (starting with $N_2 \otimes \mathbf{Q}$ which since N_2 is Abelian can be defined as the usual $\otimes\mathbf{Q}$) and such that the new invariants are precisely $k_r \otimes \mathbf{Q}$ in the corresponding $H^2 \otimes \mathbf{Q}$.

This new, rationalised tower is called the *Malcev completion* of G, denoted $G \otimes \mathbf{Q}$. Viewed as a Postnikov tower, it leads to a one-minimal model as in [**5**].

If G is nilpotent we have a natural embedding of groups $G/\mathit{torsion} \longrightarrow G \otimes \mathbf{Q}$. Further $G \otimes \mathbf{Q}$ is uniquely divisible (that is, for every $x \in G \otimes \mathbf{Q}$ and every $n \in \mathbf{Z}, n \neq 0$, there exists a unique $y \in G \otimes \mathbf{Q}$ such that $x = y^n$), and is the minimal uniquely divisible group containing G/tors.

In conclusion let us note that the above are related by : G is rationally prenilpotent if and only if $G \otimes \mathbf{Q}$ is a finite stage tower if and only if G has a finite-stage one-minimal model.

III. With the above definitions, a (connected) 3-manifold is called of *special type* if its fundamental group is rationally prenilpotent, that is, if it admits a maximal torsion-free nilpotent quotient.

Let M be a compact, oriented 3-manifold with boundary. In order to have a genuinely non-empty boundary we shall assume $\partial\big(\widehat{M}\big) \neq \emptyset$, where $\widehat{M}$ is obtained from M by capping-off all the S^2-boundary components with 3-cells. This is equivalent to assuming that M itself has at least one boundary component which is not a sphere. Also, replacing M with $\widehat{M}$ does not change the fundamental group.

The classification in the boundary case is given by:

THEOREM 1. *Let M be a compact orientable 3-manifold such that $\widehat{M}$ has non-empty boundary. If M is of special type then the maximal torsion-free nilpotent quotient of $\pi_1(M)$ is $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$.*

For the proof we use the following Betti numbers estimate, which, together with a stronger version, is the central result of [**2**]:

THEOREM 2. [**2**]. *Let G be a nilpotent group with $b_1(G) < \infty$. Then either $G \otimes \mathbf{Q} = 1, \mathbf{Q}, \mathbf{Q} \oplus \mathbf{Q}$, or $b_2(G) > b_1(G)^{2/4}$.*

Here $G \otimes \mathbf{Q}$ is the Malcev completion of G, while $b_i(G)$ denote the rational Betti numbers of G.

We shall also use *Stallings' exact sequence* from homological algebra, which says:

If N is a normal subgroup of a group G, and if $Q = G/N$ is the quotient group, then there is a natural exact sequence:

$$H_2(G) \to H_2(Q) \to N/[G,N] \to H_1(G) \to H_1(Q) \to 0.$$

PROOF OF THEOREM 1: Let M be special, let $G = \pi_1(M)/\Gamma_r{}^{\mathbf{Q}}$ be the maximal torsion-free nilpotent quotient, with notation as in II. By maximality, $\Gamma_r{}^{\mathbf{Q}} = \Gamma_{r+1}{}^{\mathbf{Q}}$.

Since we may assume $M = \widehat{M}$, we have $g_i \geqslant 1$ for $1 \leqslant i \leqslant n$, where g_i denotes the genus of the i-th boundary component and n is the number of components of M. By Poincaré duality we have:

$$\chi(M) = \frac{1}{2}\chi(\partial M) = \sum_{i=1}^{n}(1 - g_i) \leqslant 0,$$

which implies $b_1(M) \geqslant b_2(M) + 1$.

We claim that also $b_1(G) \geqslant b_2(G) + 1$. (Note that this will imply $G \neq 0$). Since $b_1(G) = b_1\big(\pi_1(M)\big) = b_1(M)$, it suffices to show $b_2(M) \geqslant b_2(G)$. Indeed, $H_2(M) \to H_2\big(\pi_1(M)\big)$ is always onto, while

$$H_2\big(\pi_1(M)\big) \longrightarrow H_2(G) \longrightarrow \Gamma_r^{\mathbf{Q}}/\Gamma_{r+1}^{\mathbf{Q}} = 0$$

is exact by Stallings' exact sequence, tensored with $\mathbf{Q}$.

Next we apply Theorem 2 to the group G. Assuming $G \otimes \mathbf{Q} \neq \mathbf{Q}$ and $G \otimes \mathbf{Q} \neq \mathbf{Q} \oplus \mathbf{Q}$, from the above we obtain:

$$b_2(G) > \frac{1}{4}b_1(G)^2 \geqslant \frac{1}{4}\big(b_2(G) + 1\big)^2,$$

a contradiction. So $G \otimes \mathbf{Q} = \mathbf{Q}$ or $\mathbf{Q} \oplus \mathbf{Q}$. Finally this implies $G = \mathbf{Z}$ or $G = \mathbf{Z} \oplus \mathbf{Z}$: Indeed, since G is nilpotent, torsion free, the natural map $G \to G \otimes \mathbf{Q}$ is an embedding.

Because $G \otimes \mathbf{Q} = \mathbf{Q}^k$ (k=1,2) is Abelian and G is finitely-generated, it follows that $G = \mathbf{Z}^l$ for some ℓ. But $l = k$ by the minimality of the Malcev completion. $\square$

IV. As a particular case we reobtain the classification [**1**] of *nilpotent* manifold *groups*, for compact orientable 3-manifolds with non-empty boundary. But we note that M special does not imply $\pi_1(M)$ nilpotent, for example, if $\pi_1(M)$ is a perfect non-zero group.

In [**1**] the geometric approach is based on Waldhausen's hierarchies and leads to a complete classification of the *3-manifolds* themselves in the more general case when $\pi_1(M)$ is solvable. Using the above group-theoretic approach we can simplify the geometric argument and obtain an almost homotopy-theoretic proof in the nilpotent case.

Instead of M and $\widehat{M}$ we shall consider as usual the 3-manifold $\mathcal{P}(M)$ = the Poincare associate of M, which is uniquely determined by $\widehat{M} = \mathcal{P}(M)\sharp\Sigma$, where Σ is a maximal homotopy 3-sphere and $\sharp$ denotes connected sum. Again, the fundamental groups and the boundaries of $\widehat{M}$ and $\mathcal{P}(M)$ are the same.

LEMMA 1. *Let M be compact, orientable such that $\mathcal{P}(M)$ has non-empty boundary. If M is special then $\pi_1(M)$ is torsion-free and $\mathcal{P}(M)$ is a $K(\pi,1)$-space.*

PROOF: Since M is special, $\pi_1(M)$ is not a non-trivial free product. Then $\mathcal{P}(M)$ is prime [**3**, p.28]. Since $\partial\mathcal{P}(M) \neq \emptyset$, it follows that $\mathcal{P}(M)$ is irreducible and is in fact a $K(\pi,1)$. This and the fact that $\mathcal{P}(M)$ is a finite-dimensional CW-complex implies $\pi_1(M)$ is torsion-free [**3**, p.76]. $\square$

The following lemma is proved by a standard Van Kampen argument [**1**]:

LEMMA 2. *If $\partial\widehat{M}$ is compressible and $\pi_1(M)$ is nilpotent then $\mathcal{P}(\mathcal{M})$ is homeomorphic with $S^1 \times D$, the full torus.*

COROLLARY. *If M is a compact, oriented, 3-manifold such that $\partial\mathcal{P}(M) \neq \emptyset$ and if $\pi_1(M)$ is nilpotent then $\mathcal{P}(M)$ is homeomorphic to $D \times S^1$ or $S^1 \times S^1 \times I$.*

PROOF: By Theorem 1 and Lemma 1, $\pi_1(M) = \mathbf{Z}$ or $\pi_1(M) = \mathbf{Z} \oplus \mathbf{Z}$. Again let $M = \mathcal{P}(M)$.

If $\pi_1(M) = \mathbf{Z}$ then ∂M is compressible since $\mathrm{Ker}\big(\pi_1(\partial M) \longrightarrow \pi_1(M)\big) \neq 0$ and by the loop theorem there exists a compressing disk. By Lemma 2, $M = S^1 \times D$.

If $\pi_1(M) = \mathbf{Z} \oplus \mathbf{Z}$, by Stallings' theorem $\mathcal{P}(M)$ fibers over S^1 with fiber F, a closed connected 2-sided surface with boundary such that $\pi_1(F) = \mathbf{Z}$. Therefore F is an annulus $S^1 \times I$ and by orientability $\mathcal{P}(M) = S^1 \times S^1 \times I$. $\square$

On the other hand, Theorem 1 shows that special 3-manifolds with non-empty boundary have *formal* one-minimal models, that is, that they are determined from the cohomology only. By [**2**], *closed* 3-manifolds are not formal, because the Heisenberg groups H_n with $n \neq 0$ are not formal.

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