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Integral Formula for Spectral Flow for p-Summable Operators

Magdalena Cecilia Georgescu

Abstract. Fix a von Neumann algebra $\mathcal{N}$ equipped with a suitable trace τ. For a path of self-adjoint Breuer–Fredholm operators, the spectral flow measures the net amount of spectrum that moves from negative to non-negative. We consider specifically the case of paths of bounded perturbations of a fixed unbounded self-adjoint Breuer–Fredholm operator affiliated with $\mathcal{N}$. If the unbounded operator is p-summable (that is, its resolvents are contained in the ideal L^p), then it is possible to obtain an integral formula that calculates spectral flow. This integral formula was first proved by Carey and Phillips, building on earlier approaches of Phillips. Their proof was based on first obtaining a formula for the larger class of θ-summable operators, and then using Laplace transforms to obtain a p-summable formula. In this paper, we present a direct proof of the p-summable formula that is both shorter and simpler than theirs.

1 Introduction

In this paper, we present a different proof of the p-summable integral formula for spectral flow. Previous proofs relied on advanced machinery, which we avoid in our presentation. Following the quote of the result below, we discuss its history.

Theorem 1.1 ([CP04, Corollary 9.4]) *Let $(\mathcal{N}, D_0)$ be an odd p-summable Breuer–Fredholm module for the unital Banach $*$-algebra $\mathfrak{A}$, and let $P = \chi_{[0,\infty)}(D_0)$. Then for each $u \in \mathcal{U}(\mathfrak{A})$ for which the domain of D_0 is invariant and $[D_0, u]$ is bounded, PuP is a Breuer–Fredholm operator in $P\mathcal{N}P$.*

Define the constant $\widetilde{C}_{\frac{p}{2}} = \int_{-\infty}^{\infty}(1+x^2)^{-\frac{p}{2}}\,dx = \frac{\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})}{\Gamma(p/2)}$ (where Γ denotes the gamma function). Then

$$\alpha_D(X) = \frac{1}{\widetilde{C}_{p/2}}\tau(X(1+D^2)^{-\frac{p}{2}})$$

is an exact one-form on the manifold $D_0 + \mathcal{N}_{\mathrm{sa}}$.

Moreover, if $\{D_t^u\}$ is any piecewise C^1 path in $D_0 + \mathcal{N}_{sa}$ from D_0 to uD_0u^* (for example, the linear path connecting the two operators), then

$$\operatorname{ind}(PuP) = \operatorname{sf}(\{D_t^u\}) = \frac{1}{\widetilde{C}_{p/2}}\int_0^1 \tau\Big(\frac{d}{dt}(D_t^u)(1+(D_t^u)^2)^{-p/2}\Big)\,dt,$$

the integral of the above-mentioned exact one-form along the path $\{D_t^u\}$.

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Carey and Phillips [CP98] gave a proof of a similar result except with $p/2$ replaced by $m > p/2 + 1$, an integer, and only in the case when $\mathcal{N}$ is a semifinite factor, and [CP04] showed the result as stated above. We exhibit a recipe that allows us to produce spectral flow formulas for operators whose resolvents are contained in certain types of ideals, which we then follow to obtain a proof of Theorem 1.1; for details, see Section 2. The results presented here are part of the author's Ph.D. thesis [Geo13, Chapter 3] completed at the University of Victoria under the supervision of John Phillips.

First, a short summary of how Theorem 1.1 was proved in [CP04]. Connes introduced a generalization of p-summable modules called θ-summable modules, whereby the module $(\mathcal{H}, D)$ is θ-summable if e^{-tD^2} is trace class for any $t > 0$ [Con89, §7]. A different and useful description of θ-summable is that $(1 + D^2)^{-1}$ belongs to a certain ideal of operators, a description of which can be given via generalized s-numbers [CP04, §2.2, Definition 2.4, Appendix B]. Carey and Phillips proved integral formulas for spectral flow in the context of θ-summable modules and used these formulas to derive the p-summable version stated in Theorem 1.1. Their approach relies on the fact that, if $(\mathcal{H}, D_0)$ is p-summable, then it is θ-summable, and one can use a series of steps inspired by Laplace transforms and the knowledge from the θ-summable case to obtain the desired result.

A more general result about integrating one-forms to calculate spectral flow can be found in [CPS09]. While the thrust of the proof is the same (show that a desired one-form is exact in order to be able to relate an integral formula to the integral along a simpler path over which the value of the integral can be explicitly calculated), the proofs are quite different. The results set out a list of conditions that can be placed on a function in order to ensure that it calculates spectral flow for a given path; double operator integral techniques are used to show that the one-forms under consideration are exact. The p-summable formula can then be obtained by approximation of the integral.

The p-summable integral formula for spectral flow is a key step of the proof of the Local Index Theorem [CPRS06a, Lemmas 5.6 5.7]. In order to streamline the proof of the Local Index Theorem, it was hoped that the p-summable case could be proved directly, without having to rely on the θ-summable results. As already mentioned, the proof of Theorem 1.1 required that the power of $(1 + D_t^2)^{-1}$ appearing in the integrand be an integer $m > p/2+1$ [CP98, Theorem 2.17]. As a consequence of this assumption, the proof has features that do not generalize to the case when the power $p/2$ is a real number; nonetheless, the goal was to find a proof similar to the one in [CP98], which we feel is achieved by this approach.

Here is a quick overview of the layout of the paper. For readability and in order to establish notation, we start with some background definitions, self-explained by the subsection titles. Section 2 describes the approach to the problem, as well as the kinds of situations to which this approach might generalize. Sections 3–5 prove the various steps in a general setting, and Section 6 applies them to the specific case of p-summable operators. Let us proceed to the mise-en-scène.

1.1 Breuer–Fredholm Operators

The theory of Breuer–Fredholm extends the idea of Fredholm operators to semifinite von Neumann algebras equipped with a trace τ; the particular approach we use is the one outlined in [PR94, Appendix B], whereby the index associated with a projection, used by Breuer in his introduction of the topic [Bre68], is replaced by the trace of the projection. The resulting theory is dependent on the choice of trace τ. The Breuer–Fredholm index will take values in the abelian group generated by the possible values the trace takes on finite trace projections.

If $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and τ is the usual trace, the Breuer–Fredholm operators correspond to the Fredholm operators. The theory describing the properties of Breuer–Fredholm operators is very similar to that for Fredholm operators in $\mathcal{B}(\mathcal{H})$. The role of the compact operators is played in this context by the ideal $\mathcal{K}_{\mathcal{N}}$.

Definition 1.2 Denote by $\mathcal{K}_{\mathcal{N}}$ the two-sided norm-closed ideal generated by projections of finite trace. The operators in $\mathcal{K}_{\mathcal{N}}$ are called *τ-compact*.

The usual results regarding the properties of the index carry over, as well as Atkinson's theorem, which states that the Breuer–Fredholm operators are invertible mod the τ-compacts [PR94, Theorem B1]. We will use π to denote the canonical projection from $\mathcal{N}$ onto the generalized Calkin algebra $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$.

A generalization of Breuer–Fredholm operators is obtained by considering operators in a skew corner of $\mathcal{N}$. We will encounter such operators in the definition of spectral flow. The idea of Breuer–Fredholm operators from $P\mathcal{H}$ to $Q\mathcal{H}$ (for P, Q projections in a von Neumann algebra $\mathcal{N}$) was introduced in [Phi97] for P and Q infinite and co-infinite projections in a factor, and developed in full generality in [CPRS06b]. In particular, if P and Q are projections such that $\|\pi(P) - \pi(Q)\| < 1$, then PQ as an operator from $Q\mathcal{H}$ to $P\mathcal{H}$ is Breuer–Fredholm, and one can calculate $\operatorname{ind}(PQ)$ using the definition given below.

Definition 1.3 Suppose $P, Q \in \mathcal{N}$ are projections. Say that $T \in P\mathcal{N}Q$ is $(P\text{-}Q)$-*Fredholm* if $\tau([\ker T \cap Q\mathcal{H}]) < \infty$, $\tau([\ker T^* \cap P\mathcal{H}]) < \infty$, and there exists a projection $P_1 < P$ such that $P_1\mathcal{H} \subset \operatorname{ran} T$ and $\tau(P - P_1) < \infty$. If $T \in P\mathcal{N}Q$ is $(P\text{-}Q)$-Fredholm, define $\operatorname{ind}_{(P\text{-}Q)}(T) = \tau([\ker T \cap Q\mathcal{H}]) - \tau([\ker T^* \cap P\mathcal{H}])$.

A summary of the various results for Breuer–Fredholm operators in a skew corner, with appropriate references, can be found in [BCP$^+$06]. As already mentioned, this kind of index will be needed when defining the spectral flow. Instead of using the $(P\text{-}Q)$-prefix, we will simply talk about an operator T being Breuer–Fredholm from $Q\mathcal{H}$ to $P\mathcal{H}$, or in $P\mathcal{N}Q$.

If D is unbounded, we say that D is Breuer–Fredholm if the bounded operator $D(1+|D|^2)^{-\frac{1}{2}}$ is Breuer–Fredholm [CPRS06b, Proposition 3.10]. In fact, we will often appeal to the Riesz transform, $T \mapsto T(1+|T|^2)^{-\frac{1}{2}}$, in order to reduce our unbounded operator problems to the bounded operator setting.

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1.2 Spectral Flow Definitions

In order to talk about spectral flow we will usually require a continuous path of self-adjoint Breuer–Fredholm operators. We then use the trace τ on $\mathcal{N}$ to get a measure of the net amount of spectrum that changes from negative to positive as we move along the path.

1.2.1 Spectral Flow for Bounded Operators

There are multiple interpretations of spectral flow; the one presented below is due to Phillips [Phi97]. The presentation there assumes that $\mathcal{N}$ is a factor; however, the approach generalizes to semifinite von Neumann algebras, especially once the appropriate Breuer–Fredholm theory is in place [CPRS06b, §3, Corollary 3.8].

Recall that π is used for the canonical map from $\mathcal{N}$ onto the generalized Calkin algebra, $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$. Denote by χ the characteristic function of the interval $[0,\infty)$. Given a path of self-adjoint Breuer–Fredholm operators $\{F_t\}$, $\chi(F_t)$ is not continuous, but $\pi(\chi(F_t))$ is.

Definition 1.4 Suppose $\{F_t\}$ is a continuous path of self-adjoint Breuer–Fredholm operators. Let $P_t = \chi(F_t)$. Choose finitely many points $0 = r_0 < r_1 < \cdots < r_n = 1$ such that for each $0 \le i \le n$ and each $u, v \in [r_i, r_{i+1}]$ we have $\|\pi(P_u) - \pi(P_v)\| < 1$. Define the *spectral flow* of the path to be $\operatorname{sf}(\{F_t\}) := \sum \operatorname{ind}(P_{r_i}P_{r_{i+1}})$, where $\operatorname{ind}(P_{r_i}P_{r_{i+1}})$ is the index of $P_{r_i}P_{r_{i+1}}$ as a Breuer–Fredholm operator in $P_{r_i}\mathcal{N}P_{r_{i+1}}$.

This definition is independent of the choice of partition $\{P_{r_i}\}$ [Phi97, Lemma 1.3, Definition 2.2]. If the von Neumann algebra $\mathcal{N}$ is finite, all projections have finite trace, so the spectral flow depends only on the endpoints of the path, *i.e.*, $\operatorname{sf}(\{F_t\}) = \operatorname{ind}(\chi(F_0)\chi(F_1))$. However, if $\mathcal{N}$ is not finite, then the spectral flow depends on the path, as can be seen from the construction of a loop whose spectral flow is non-zero [Phi96]. Note however that, even if $\mathcal{N}$ is not finite, if our path $\{F_t\}$ satisfies $F_t - F_0 \in \mathcal{K}_{\mathcal{N}}$ for all t, then once again the spectral flow only depends on the endpoints, as $\pi(F_t)$ is constant, and hence so is $\pi(P_t)$.

Remark 1.5 Let us consider what happens if we join a self-adjoint Breuer–Fredholm operator F to $2\chi_{[0,\infty)}(F) - 1$ via a straight line path. Intuitively, the positive part of the operator is joined to 1, and the negative part is joined to −1; we would expect no spectral flow along this path. Indeed, using Definition 1.4, one can check that if $\{F_t\}$ is a path for which $\chi_{[0,\infty)}(F_t)$ is constant, then the spectral flow of $\{F_t\}$ is 0. Let $P_t = \chi_{[0,\infty)}(F_t) =: P$. Then $\pi(P_t) = \pi(P)$ is constant for all t, so the spectral flow is equal to $\operatorname{ind}(P_0P_1) = \operatorname{ind}(P)$ as an operator from $P\mathcal{H}$ to $P\mathcal{H}$. As P is the identity on $P\mathcal{H}$, $\operatorname{ind}(P) = 0$, *i.e.*, the spectral flow is 0. Since the positive operators form a convex set, it should be clear that, by going from F to $2\chi_{[0,\infty)}(F) - 1$ via a straight line, the projection onto the positive part of the spectrum remains unchanged. So the spectral flow along this path is 0, as expected.

1.2.2 Spectral Flow for Unbounded Operators

In discussing paths of unbounded operators, we will restrict ourselves to bounded perturbations of a fixed self-adjoint operator affiliated with the von Neumann algebra $\mathcal{N}$. If $\{D_t\}$ is a path of unbounded self-adjoint operators, where $D_t = D_0 + A_t$ and $\{A_t\}$ is a norm-continuous path of bounded operators, we can apply the Riesz transform, $D \mapsto D(1+D^2)^{-\frac{1}{2}}$, to obtain a path of bounded (self-adjoint) operators. The resulting collection $\{F_t\}$ of operators obtained by applying the Riesz transform is continuous in t [CP98, Theorem A.8]. Moreover, if the D_t are Breuer–Fredholm, then by definition so are the operators $\{F_t\}$. Under these conditions, we can in this manner reduce the problem of calculating spectral flow for unbounded operators to the bounded case.

Definition 1.6 If $\{D_t\}$ is a path of (unbounded) self-adjoint Breuer–Fredholm operators such that $D_t = D_0 + A_t$ with $\{A_t\}$ a (norm-continuous) path of bounded operators, then $\mathrm{sf}(\{D_t\}) = \mathrm{sf}(\{D_t(1+D_t^2)^{-\frac{1}{2}}\})$.

Connection to Non-Commutative Geometry

Definition 1.7 ([CPRS08, Definition 3.1]) A *pre-Breuer–Fredholm module* for a unital Banach $*$-algebra $\mathfrak{A}$ is a pair $(\mathcal{N}, F_0)$, where $\mathfrak{A}$ is represented in the semifinite von Neumann algebra $\mathcal{N}$ via a $*$-homomorphism π (which can be assumed without loss of generality to be faithful) such that, if τ is a faithful, normal, semifinite trace on $\mathcal{N}$, then

- $F_0 \in \mathcal{N}$ is self-adjoint and $1 - F_0^2 \in \mathcal{K}_{\mathcal{N}}$, and
- $\{a \in \mathfrak{A} \mid [F_0, a] \in \mathcal{K}_{\mathcal{N}}\}$ is a dense $*$-subalgebra of $\mathfrak{A}$.

If, in addition to the above, $F_0^2 = 1$, we refer to $(\mathcal{N}, F_0)$ as a *Breuer–Fredholm module*, *i.e.*, we drop the "pre-".

Definition 1.8 ([CP04, Definition 2.4]) An *unbounded Breuer–Fredholm module* for $\mathfrak{A}$ is a pair $(\mathcal{N}, D_0)$ where, with $\mathfrak{A}$ and $\mathcal{N}$ as in the previous definition, D_0 is an unbounded self-adjoint operator affiliated with $\mathcal{N}$ such that

(i) $(1+D_0^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$, and
(ii) $\{a \in \mathfrak{A} \mid a(\mathrm{Dom}\, D_0) \subset \mathrm{Dom}\, D_0 \text{ and } [D_0, a] \in \mathcal{N}\}$ is a dense $*$-subalgebra of $\mathfrak{A}$.

If we can replace condition (i) by $(1+D_0^2)^{-\frac{1}{2}} \in \mathcal{L}^p$, we say that $(\mathcal{N}, D_0)$ is a *p-summable unbounded Breuer–Fredholm module*.

Fredholm modules are the building blocks of K-homology, which in turn is dual to K-theory. In the semifinite setting, the theory is not as fully developed; nonetheless, one can pair an odd K-homology class represented by an unbounded module $(\mathcal{N}, D_0)$ with an odd K-theory class represented by a unitary operator u. Denoting by P the spectral projection of D_0 corresponding to the interval $[0, \infty)$, the pairing is given by $\mathrm{ind}(PuP)$ (as an operator in $P\mathcal{N}P$). This index is equal to the spectral flow of any path $\{D_t\}$ from D_0 to uD_0u^* with $D_t \in D_0 + \mathcal{N}_{\mathrm{sa}}$, and is hence calculated by the integral formula in Theorem 1.1 in the case when $(\mathcal{N}, D_0)$ is p-summable and the path is piecewise $\mathcal{C}^1$. For more details on the connection between bounded and unbounded

Breuer–Fredholm modules, see [CP04, §2.4], and for the proof that $\mathrm{ind}(PuP)$ is the aforementioned spectral flow, see [Phi97, §3].

1.3 Ideals of Operators

We will use the generalized singular s-numbers of operators to discuss the ideals of $\mathcal{N}$ in which our work will take place. Recall that the singular numbers of a compact operator K are the eigenvalues of $|K|$, with multiplicity. The notion of singular numbers is generalized to (τ-measurable) operators affiliated with a semifinite von Neumann algebra by Fack and Kosaki [FK86]; in particular, bounded operators in $\mathcal{N}$ are τ-measurable. We highlight below the definitions and results from their paper that are relevant to our presentation.

Definition 1.9 ([FK86, Definition 2.1]) For $A \in \mathcal{N}$ and $t > 0$, the t-th *singular s-number* (or, briefly, s-number) is

$$\mu_t(A) = \inf\{\|AP\| : P \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-P) \le t\}.$$

It should be clear that $t \mapsto \mu_t(A)$ is a non-negative, decreasing function. The generalized s-numbers can be used to get a handle on certain classes of operators; for example, T is τ-measurable if and only if $\mu_t(T) < \infty$ for all $t > 0$. Also, it is well known that $S \in \mathcal{K}_{\mathcal{N}}$ if and only if S is bounded and $\mu_t(S) \to 0$ as $t \to \infty$. A consequence of this property of τ-compact operators is that, if P is a τ-compact projection, then $\tau(P) < \infty$. Note that in some definitions of the τ-compact operators the condition that S is bounded is dropped, but we wish to have the τ-compacts be a subset of $\mathcal{N}$.

We also make the slightly unpopular choice of restricting ourselves to bounded operators when it comes to discussing the $\mathcal{L}^p$ spaces; *i.e.*, by $\mathcal{L}^p$ we mean the ideal $\{T \in \mathcal{N} : \tau(|T|^p)^{1/p} < \infty\}$. The expression $\tau(|T|^p)^{1/p}$ is used to define a norm on $\mathcal{L}^p$, and the following result connects this expression to the generalized s-numbers of T; this will allow us to tie $\mathcal{L}^p$ spaces to the concept of a small power invariant ideal, which we will introduce later (Definition 1.15).

Theorem 1.10 ([FK86, Corollary 2.8]) *If f is a continuous, increasing function on $[0,\infty)$ with $f(0) = 0$ and T is a τ-measurable operator, then*

$$\tau(f(|T|)) = \int_0^\infty f(\mu_t(T))\,dt.$$

In particular, $\tau(|T|^p)^{1/p} = (\int_0^\infty \mu_t(T)^p\,dt)^{1/p}$ for $0 < p < \infty$.

Remark 1.11 In [FK86], L^p is defined as the set of all closed, densely-defined operators T affiliated with $\mathcal{N}$ for which $\tau(|T|^p)^{1/p} < \infty$, with $|||T||| = \tau(|T|^p)^{1/p}$. Using this definition, it is well known that L^p is a Banach space. If we wish our space $\mathcal{L}^p$ to consist of only the bounded operators, then we need to put a different norm on it if we want it to be complete: for example, $\|T\|_p = \max\{\|T\|, |||T|||\}$ [CP04].

The most commonly used ideal in the following will be $\mathcal{I} = \mathcal{L}^p$; however, we do state some theorems for more general ideals, so we wish to establish a few properties. In a general von Neumann algebra, an ideal $\mathcal{I}$ need not be contained in the τ-compact

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operators of $\mathcal{N}$, even if $\mathcal{I}$ is essential. For example, if $\mathcal{N} = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$, then $\mathcal{K}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$ is an essential ideal that is not contained in the compact operators, $\mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})$. As we will need our ideals to consist of compact operators, we will eventually have to explicitly assume it for the ideals under consideration (see Definition 1.15 for the properties required of our ideals).

Another issue that needs to be addressed is the norm we place on our ideals. In all cases under consideration, the norm will satisfy the properties of the following definition.

Definition 1.12 ([CP04, Definition A.2]) If $\mathcal{I}$ is a (two-sided) $*$-ideal in $\mathcal{N}$ that is complete in a norm $\|\cdot\|_{\mathcal{I}}$, then we call $\mathcal{I}$ an *invariant operator ideal* if

(i) $\|S\|_{\mathcal{I}} \geq \|S\|$ for all $S \in \mathcal{I}$,
(ii) $\|S^*\|_{\mathcal{I}} = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{I}$,
(iii) $\|ASB\|_{\mathcal{I}} \leq \|A\|\|S\|_{\mathcal{I}}\|B\|$ for all $S \in \mathcal{I}$, $A, B \in \mathcal{N}$.

Note that, using polar decomposition, property (ii) follows from (iii).

One of the properties that follows from the definition is a comparison of norms for positive operators. Dixmier proved the first part of the following result; as he does not concern himself with invariant operator ideals, we must refer to the proof in [Dix81] to obtain the norm inequality with which we conclude the result.

Theorem 1.13 ([Dix81, §1.1.6, Proposition 10]) *If $\mathcal{I}$ is an ideal in a von Neumann algebra $\mathcal{N}$, $0 \leq S \leq T$, and $T \in \mathcal{I}$, then $S \in \mathcal{I}$. Moreover, if $\mathcal{I}$ is an invariant operator ideal, then $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.*

To start with, we note that $\mathcal{L}^p$ for $p \geq 1$ is an invariant operator ideal, and this will be our main concern. There is, however, a wealth of examples of invariant operator ideals; we relegate the discussion thereof to Remark 1.16. We move on instead to consider powers of ideals. As seen above for $\mathcal{I} = \mathcal{L}^p$, we will often have a norm defined on our ideals; so, in particular, we need to consider if a suitable norm can be defined on the power of an ideal, and what the relationship is between an ideal and its powers.

Definition 1.14 Suppose that $\mathcal{I}$ is an ideal of operators. Define for $q \in (0, \infty)$,

$$\mathcal{I}^q = \left\{ T \in \mathcal{N} \;\middle|\; |T|^{1/q} \in \mathcal{I} \right\}.$$

Dixmier discussed powers of an ideal in detail [Dix52b, Dix52a]. For example, he showed that $\mathcal{I}^q$ is in turn an ideal [Dix52b, Propoition 1]. The notation is justified by the fact that $(\mathcal{I}^a)^b = \mathcal{I}^{ab}$ and $\mathcal{I}^a\mathcal{I}^b = \mathcal{I}^{a+b}$ [Dix52b, Proposition 2]. It can be shown that $\mathcal{I}^\alpha \subset \mathcal{I}^\beta$ for $0 < \beta \leq \alpha < \infty$ (see [Dix52a] for a proof); in particular we will need this for $\alpha = 1$ and $\beta < 1$.

We can define a norm on $\mathcal{I}^q$ (for $q < 1$) via $\|T\|_{\mathcal{I}^q} = (\||T|^{1/q}\|_{\mathcal{I}})^q$. Using terminology from the theory of non-commutative symmetric spaces, $\mathcal{I}^q$ is the $(1/q)$-convexification of $\mathcal{I}$; our setup is such that $\mathcal{I}^q$ will be invariant operator ideals in their own right. This is certainly true for $\mathcal{I} = \mathcal{L}^p$; note, moreover, that the norm $\|\cdot\|_{\mathcal{I}^q}$ defined above corresponds to the appropriate $\mathcal{L}^p$ norm when $\mathcal{I}$ is an $\mathcal{L}^p$ type ideal.

We observe that if $q < 1$, the inclusion $\mathcal{J} \hookrightarrow \mathcal{J}^q$ is continuous

$$\begin{aligned}\|T\|_{\mathcal{J}^q} &= \big(\| \,|T||T|^{\frac{1}{q}-1}\|_{\mathcal{J}}\big)^q \le \|T\|_{\mathcal{J}}^q \|\,|T|^{\frac{1}{q}-1}\|^q \le \|T\|_{\mathcal{J}}^q \|T\|^{1-q}\\ &\le \|T\|_{\mathcal{J}}^q \|T\|_{\mathcal{J}}^{1-q} = \|T\|_{\mathcal{J}}.\end{aligned}$$

Note that the above proof relies on the fact that $\mathcal{J}$ is an invariant operator ideal: for the first inequality, we use property (iii) of the definition, and in the second to last step, property (i).

Finally, we discuss Hölder's inequality as it applies to powers of ideals. For $\mathcal{J} = \mathcal{L}^1$, whose powers are $\mathcal{L}^p$ for various values of p, Hölder's inequality is a result of Dixmier [Dix53, Corollaries 2, 3, Theorem 6], [Dix52b]; we will use it to determine whether an operator is in $\mathcal{L}^p$ for some p, and to find upper bounds for $\mathcal{L}^p$ norms. For other ideals this theorem might need to be proved separately; again, Remark 1.16 discusses some general conditions under which Hölder's inequality is already known.

In the following definition we collect all the properties that we need to impose on the operator ideals under consideration in order for our proofs to work. We then follow it up with a discussion of some ideals that satisfy these properties.

Definition 1.15 Say that $\mathcal{J}$ is a *small power invariant operator ideal* if $\mathcal{J} \subset \mathcal{K}_{\mathcal{N}}$ is an invariant operator ideal (as defined in Definition 1.12) for which the following hold:

- The powers $\mathcal{J}^q$ for $0 < q < 1$ are also invariant operator ideals for the usual norm $\|A\|_{\mathcal{J}^q} = (\||A|^{\frac{1}{q}}\|_{\mathcal{J}})^q$.
- $\mathcal{J}$ and its powers satisfy Hölder's inequality: if $s_1, s_2, q \in (0,1]$ are such that $q = s_1 + s_2$, and $A \in \mathcal{J}^{s_1}$, $B \in \mathcal{J}^{s_2}$, then $\|AB\|_{\mathcal{J}^q} \le \|A\|_{\mathcal{J}^{s_1}} \|B\|_{\mathcal{J}^{s_2}}$.

Note that, given this setup, $\mathcal{J}^q \subset \mathcal{K}_{\mathcal{N}}$ must be true for all $q \in (0,1]$.

Remark 1.16 The example in which we are particularly interested is $\mathcal{L}^p$, but the question might well arise whether there are any other small power invariant ideals. Another example can be found in [CP04], where ideals Li^q make an appearance while proving θ-summable versions of the spectral flow formula (see [CP04, Appendix A, Lemma A.3] for the proof of Hölder's theorem for these ideals).

Both of these examples are in fact symmetric operator spaces as well. By definition, a *symmetric operator space* $\mathcal{E}$ on $\mathcal{N}$ is a linear subspace of the $*$-algebra of measurable operators affiliated with $\mathcal{N}$ such that, if $T \in \mathcal{E}$ and S is any measurable operator with $\mu_t(S) \le \mu_t(T)$ for all $t > 0$, then $S \in \mathcal{E}$ and $\|S\|_{\mathcal{E}} \le \|T\|_{\mathcal{E}}$.

Symmetric operator spaces are connected to invariant operator ideals, as we now describe. Suppose $\mathcal{E}$ is a symmetric operator space, $S \in \mathcal{E}$, and A, B are bounded. From the properties of s-numbers [FK86, Lemma 2.5], we have $\mu_t(ASB) \le \|A\|\mu_t(S)\|B\| = \mu_t(\|A\|S\|B\|)$. Since $\|A\|S\|B\| \in \mathcal{E}$, it follows from the definition of symmetric operator spaces that $ASB \in \mathcal{E}$ and $\|ASB\|_{\mathcal{E}} \le \|A\|\|S\|_{\mathcal{E}}\|B\|$. Finally, if we consider $\mathcal{E} \cap \mathcal{N}$ with norm $\max\{\|\cdot\|, \|\cdot\|_{\mathcal{E}}\}$, then it is easy to see that the resulting space is a two-sided ideal that satisfies properties (i) and (iii) in Definition 1.12, *i.e.*, $\mathcal{E} \cap \mathcal{N}$ is an invariant operator ideal. On the other hand, there are invariant operator ideals that cannot be constructed from a symmetric operator space in the manner just described. For example, $\mathcal{J} = \mathcal{K}(\mathcal{H}) \oplus 0$ is an invariant operator ideal of $\mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$ (and in fact is contained in the ideal of compact operators). It is easy to see that for $A \in \mathcal{K}(\mathcal{H})$, we

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have $\mu_t(0 \oplus A) = \mu_t(A \oplus 0)$, but although $A \oplus 0 \in \mathcal{J}$, we certainly do not have that $0 \oplus A \in \mathcal{J}$.

Now consider an invariant operator ideal $\mathcal{J}$ that does arise from a symmetric operator space, as indeed all practical examples seem to. Then, using existing results about symmetric operator spaces, one can conclude easily that $\mathcal{J}$ is in fact a small power invariant ideal. To start with, we must have $\mathcal{J} \subset \mathcal{K}_{\mathcal{N}}$. Otherwise, if there exists an operator A in $\mathcal{J}$ such that $A \notin \mathcal{K}_{\mathcal{N}}$, then $\mu_t(A) \not\to 0$, so in particular $\mu_t(A) > r$ for some $r > 0$. Since $\mu_t(r1) = r < \mu_t(A)$ for all t, this would imply that $r1 \in \mathcal{J}$ (by the definition of symmetric operator space), and so $\mathcal{J} = \mathcal{N}$. Therefore, as long as $\mathcal{J}$ is a proper ideal of $\mathcal{N}$, we must have $\mathcal{J} \subset \mathcal{K}_{\mathcal{N}}$.

Now for $0 < q \leq 1$ a real number, recall the definition $\mathcal{J}^q = \{T \in \mathcal{N} : |T|^{1/q} \in \mathcal{J}\}$ and that $\mathcal{J}^q$ is also an ideal [Dix52b, Proposition 1]. Define a norm on $\mathcal{J}^q$ via $\|T\|_{\mathcal{J}^q} = (\||T|^{1/q}\|_{\mathcal{J}})^q$; results from [KS08] allow us to conclude that $\|\cdot\|_{\mathcal{J}^q}$ is in fact a norm. Moreover, it is easy to check that $\mathcal{J}^q$ equipped with this norm also arises from a symmetric operator space; for a discussion of norms on symmetric operator spaces and p-convexification, see [DDS14]. By the discussion at the beginning of the remark, we can conclude that $\mathcal{J}^q$ equipped with the norm $\|\cdot\|_{\mathcal{J}^q}$ is an invariant operator ideal.

Finally, [Suk16] proved that the Hölder inequality holds for symmetric operator spaces, and so $\mathcal{J}$ satisfies all the conditions we require of our ideals. Many thanks to the referee for pointing out the references regarding symmetric operator spaces, which helped clarify this remark and give a much better idea of the context.

The rest of the article is concerned with our new proof of the p-summable integral formula, the steps of which are summarized in Section 2.

2 Outline of the Analytic Continuation Proof of p-summable Formulas

The idea is to show that the integral formula for spectral flow works with $\frac{p}{2}$ replaced by any large enough m; the larger power gives us extra maneouvering room, as we can split up the $(1+D^2)^{-m}$ factor and still have part of it be trace class (we do this, for example, during the proof that the one-form is closed). We then use analytic continuation of complex functions to show that the formula works for all $m \geq \frac{p}{2}$.

To show that the integral formula calculates the spectral flow, the proof follows the same main steps as for the finitely-summable case when the power is an integer [CP98], but changes the proof so it works for all (sufficiently large) real powers. Note that the proof is split into a bounded case and an unbounded case. One way of showing that a formula calculates spectral flow is to start with an integral formula which works for special linear paths of bounded operators and build up the general formula from there. The bounded case and the unbounded case are connected by the Riesz transform, $D \mapsto D(1+D^2)^{-\frac{1}{2}}$, which maps unbounded operators to bounded operators. The following two points explain how we decide what the corresponding bounded case should be.

- Given a summability condition on D_0, say $(1+D_0^2)^{-1} \in \mathcal{J}$ for some operator ideal $\mathcal{J}$, the Riesz transform maps paths in $D_0 + \mathcal{N}_{sa}$ to paths in $F_0 + \mathcal{S}$, where

$F_0 = D_0(1+D_0^2)^{-\frac{1}{2}}$ and $\mathcal{S}$ is a Banach space whose exact nature will be revealed in Section 4.

- For appropriate choices of function k, if $\{F_t\} \subset F_0 + \mathcal{S}$ is the image of the path $\{D_t\}$ under the Riesz transform, and $\{F_t\}$ is differentiable, then

$$\tau\Big(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\Big) = \tau\Big(\frac{d}{dt}(F_t)(1-F_t^2)^{-\frac{3}{2}}k(1-F_t^2)\Big).$$

Suppose $k(x)$ is a suitable function and we want to show that the integral

$$\int_0^1 \tau\Big(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\Big)\,dt$$

can be used to calculate spectral flow (in particular, we want to do this for $k(x) = x^m$, where m can be any large enough real number). The course of action indicated by the above observations is clear:

(1) Modify the desired formula, replacing $(1+D_t^2)^{-1}$ by $1-F_t^2$, and add a factor of $(1-F_t^2)^{-\frac{3}{2}}$ to obtain the corresponding integral formula in the bounded case, *i.e.*, consider the integral $\int_0^1 \tau(\frac{d}{dt}(F_t)h(1-F_t^2))\,dt$, where $h(x) = x^{-\frac{3}{2}}k(x)$. Some care might be required in doing this:
   (a) The operator $(1+D_t^2)^{-1}$ is always positive, but there will be bounded operators $F \in F_0 + \mathcal{S}$ for which $1-F^2$ is not positive. Thus, in order for $h(1-F^2)$ to make sense for all F in our manifold, we might need to replace $1-F_t^2$ by $|1-F_t^2|$ (so consider $h(|x|)$ instead of $h(x)$). This is indeed the case for $h(x) = x^{m-3/2}$ (the function in which we are ultimately interested), as the proof needs to work for any large enough real m. Note that this is one of the reasons the proof for general real powers p differs from the one for integer powers presented in [CP98].
   (b) The extra factor of $|1-F_t^2|^{-\frac{3}{2}}$ might very well cause problems, as there is no reason to suppose that $1-F_t^2$ is invertible. However, for suitable functions k, say if $\lim_{x\to 0} x^{-\frac{3}{2}}k(x) = 0$, we can still make sense of the expression

$$(1-F_t^2)^{-\frac{3}{2}}k(1-F_t^2).$$

   Namely, define

$$l(x) = \begin{cases} x^{-\frac{3}{2}}k(x) & \text{for } x \neq 0,\\ 0 & \text{for } x = 0.\end{cases}$$

   Then l is a continuous function on $\mathbb{R}$, and whenever we write $h(F)$ we really mean $l(F)$. This is, for example, the approach taken in [CP04], for the function $k(x) = e^{-\frac{1}{x}}$.

(2) Show that the one-form $\alpha_F\colon X \mapsto \tau(Xh(1-F_t^2))$ is exact, so its integral is independent of the path over which it is calculated (one way to do this is to show that the one-form is closed, which is the approach we take).

(3) Show that the integral formula holds in the bounded case. It is during this step that we find the suitable normalizing factor for the one-form, and also where we must introduce correction terms for a path whose endpoints are not unitarily equivalent.

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(4) Reduce the unbounded case to the bounded case. The vital point here is that, if $D_t = D_0 + A_t$ describes a C^1 path in $D_0 + \mathcal{N}_{sa}$, then its Riesz image $\{F_t\}$ should be C^1 in $F_0 + \mathcal{S}$ for an appropriately chosen Banach space $\mathcal{S}$. Then the equality of traces mentioned earlier allows us to rewrite the bounded formula into a formula depending on $\{D_t\}$.

The above program should explain how we arrive at the formulae we consider, and summarize a possible approach for proving similar results. Along the way, we try to present results in some generality. We start by presenting, in Section 3, a set of conditions that are sufficient to prove that $\alpha_T\colon X \mapsto \tau(Xg(T)^k)$ is a closed one-form on an affine operator space $T_0 + \mathcal{S}$. The choice of function g will be different for the bounded case and the unbounded case. We mention conditions under which the integral formula

$$\int \tau\Big(\frac{d}{dt}(F_t)h(1-F_t^2)\Big)\,dt$$

can be suitably modified to calculate spectral flow for paths of bounded operators $\{F_t\} \in F_0 + \mathcal{S}$. As part of this process, both a constant normalization factor and correction terms emerge. We then use this to obtain, for unbounded operators, an integral formula of the type

$$\text{const} \int \tau\Big(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\Big)\,dt + \text{ correction terms.}$$

This is the content of Section 4. We show in Section 5 that, with some conditions on the function g, the complex function $z \mapsto \int_0^1 \tau(Ag(D_t)^z)\,dt$ is analytic, and we explain the idea behind the analytic continuation step.

Finally, in Section 6, we put these various facts together to prove the integral formula stated for p-summable operators in Theorem 1.1. The first step (getting a one-form that we will use in the spectral flow formula) is accomplished by Corollary 6.1. The second and third step (getting an initial spectral flow formula, and using analytic continuation to improve the power used and obtain the desired final formula) are both implemented in the proof of Theorem 6.4.

3 Closed Forms of the Type $\tau(Xg(T)^q)$

In the p-summable formula proof, we will have reason to consider two one-forms of the same type (one for the bounded case and one for the unbounded case), so the purpose of this section is to prove that these one-forms are closed and, hence, exact in a more general context. As always, we have a von Neumann algebra $\mathcal{N}$ equipped with a faithful normal semifinite trace τ. The one-forms are defined on a manifold of the type $T_0 + \mathcal{S}$ for a fixed self-adjoint operator T_0 and a real Banach space $\mathcal{S} \subset \mathcal{N}_{sa}$. We will have either $T_0 \in \mathcal{N}_{sa}$ or, if T_0 is unbounded, then T_0 is affiliated with $\mathcal{N}$; in practice, T_0 will have additional properties dictated by the context of the spectral flow problem. The one-forms are of the type $\alpha_T(X) = C\tau(Xg(T)^q)$, where C is a constant, $q > 3$ is a fixed real number, $T \in T_0 + \mathcal{S}$, X is in the tangent space at T of the manifold (namely $\mathcal{S}$), and $g\colon T_0 + \mathcal{S} \to \mathcal{N}$ is a suitable function.

In order to define the one-form and show that the one-form is closed using the technique below, we will need to place some restrictions on the function g. In all the

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cases in which we are interested, we have that $\|X\|_{\mathcal{S}} \geq \|X\|$ for $X \in \mathcal{S}$, where $\|\cdot\|$ denotes the operator norm; hence, the identity function from $(\mathcal{S}, \|\cdot\|_{\mathcal{S}})$ to $(\mathcal{S}, \|\cdot\|)$ is continuous. In order to simplify the presentation, we state conditions in terms of the operator norm instead of the norm on $\mathcal{S}$, and for all practical purposes ignore the actual norm on $\mathcal{S}$.

We explain the reasoning behind the restrictions on g in Remark 3.2. The main result of this section, which we will prove in the sequel, is the following proposition.

Proposition 3.1 *Suppose T_0 is a fixed self-adjoint operator affiliated with $\mathcal{N}$, $\mathcal{S}$ is a real Banach space such that $\mathcal{S} \subset \mathcal{N}_{\mathrm{sa}}$ and $\|X\|_{\mathcal{S}} \geq \|X\|$ for $X \in \mathcal{S}$, and that for some $q > 3$, g is a function satisfying the following conditions:*

(i) *$g(T)$ is a positive operator in $\mathcal{N}$ for all $T \in T_0 + \mathcal{S}$ (so $g(T)^t$ is defined and in $\mathcal{N}$ for all real $t \geq \frac{q-3}{2}$).*

(ii) *$\tau(g(T)^t) < \infty$, for $t \geq \frac{q-3}{2}$ and for all $T \in T_0 + \mathcal{S}$.*

(iii) *For each fixed $t \geq \frac{q-3}{2}$ and $T \in T_0 + \mathcal{S}$ and $X \in \mathcal{S}$, the function from $\mathbb{R}$ to $\mathcal{L}^1$ defined by $s \mapsto g(T + sX)^t$ is continuous. Note that it is sufficient to prove continuity at $s = 0$ for each T and X, as continuity at some other r then follows by replacing T with $T + rX$ (which is also an element of $T_0 + \mathcal{S}$).*

(iv) *For each $T \in T_0 + \mathcal{S}$ and $X \in \mathcal{S}$, the function $s \mapsto g(T + sX)$ is differentiable in the operator norm. As above, differentiability at $s = 0$ is sufficient to ensure differentiability, and hence continuity, at any $r \in \mathbb{R}$.*

The derivative of $s \mapsto g(T + sX)$ at $s = 0$ plays an important role, and it is useful to introduce a notation for the difference quotient. Namely, define the function

$$d_{g,s}(T, X) = \begin{cases} \dfrac{g(T + sX) - g(T)}{s} & \text{for } s \neq 0, \\ \lim\limits_{s \to 0} \dfrac{g(T + sX) - g(T)}{s} & \text{for } s = 0. \end{cases}$$

Note that $s \mapsto d_{g,s}(T, X)$ is a continuous function (since $s \mapsto g(T + sX)$ is itself continuous, and differentiable at $s = 0$). Further restrictions on g are in fact placed on this function.

(v) *$d_{g,0}(T, X)$ (the derivative of $s \mapsto g(T + sX)$ at $s = 0$) can be written as a sum, where each term is of the following type:*
- *$g_1(T)Xg_1(T)$ with $T \mapsto g_1(T)$ continuous from $T_0 + \mathcal{S}$ to $\mathcal{N}$, or*
- *$g_1(T)Xg_2(T)$ with $T \mapsto g_1(T)$ and $T \mapsto g_2(T)$ both continuous from $T_0 + \mathcal{S}$ to $\mathcal{N}$; each such term (when $g_1 \neq g_2$) can be paired up with a corresponding term of the form $g_2(T)Xg_1(T)$*

Then, for any $C > 0$, $\alpha_T(X) = C\tau(Xg(T)^q)$ is a closed one-form on the manifold $T_0 + \mathcal{S}$, i.e., *$d\alpha = 0$. It follows that α is exact; that is, the integral of α is independent of the path of integration.*

Remark 3.2 Here, we explain whence the conditions on g arise. Note the repeated appearance of $\frac{q-3}{2}$ instead of the (perhaps) expected q. This is a consequence of the method of proof; we will want to split $g(T)^q$ into a product $g(T)^a g(T)^b$, and we need to ensure that at least one of these two factors is trace class (see the proof of

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Lemma 3.4). Property (v) might also look a bit strange; it is needed so that when we use the trace property on some of the resulting expressions, we get a permutation of the same terms (see the calculations following Lemma 3.5). Examples of spaces $T_0 + \mathcal{S}$ and functions g that satisfy these conditions can be found in Sections 6.1 and 6.2.

From the definition of α and conditions on g it is easy to check that for each fixed T, α_T is a bounded linear functional on the tangent space at T. These conditions might be insufficient to show that α is C^1 when considered as a map from the manifold $T_0 + \mathcal{S}$ to the cotangent space; however, this is in fact irrelevant to our purposes. Our goal is to calculate the exterior derivative and hence conclude that the integral of α is independent of path, for which we really only need directional derivatives.

The rest of this section is dedicated to proving Proposition 3.1. Before we proceed, note that if $\alpha_T(X)$ is closed, then so is $C\alpha_T(X)$ for any constant C, so in the following we assume without loss of generality that $C = 1$.

The definition of exterior derivative applied to the one-form α gives us that

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]),$$

where $[X,Y]$ is the bracket product of X and Y viewed as constant vector fields. One can easily check that the flows of X and Y commute, so it follows that $[X,Y] = 0$. Hence, showing that the one-form is closed, reduces to showing that the derivatives in the direction X of $\alpha_T(Y)$ and in the direction Y of $\alpha_T(X)$ are equal,

$$\frac{d}{ds}\Big|_{s=0}\tau(Yg(T+sX)^q) = \frac{d}{ds}\Big|_{s=0}\tau(Xg(T+sY)^q).$$

We start with the left-hand side and manipulate it until we can conclude that it is the same as the right-hand side. By definition,

$$\frac{d}{ds}\Big|_{s=0}\tau(Yg(T+sX)^q) = \lim_{s\to 0}\tau\Big(Y\frac{1}{s}[g(T+sX)^q - g(T)^q]\Big).$$

Let $n = \lfloor q \rfloor$ and $r = \frac{q}{n+1}$ (note that $0 < r < 1$). It is easy to check (by expanding) that

$$g(T+sX)^q - g(T)^q = \sum_{i=0}^{n} g(T+sX)^{r(n-i)}[g(T+sX)^r - g(T)^r]g(T)^{ri}. \tag{3.1}$$

Since $0 < r < 1$ we can use an integral formula of Pedersen's [Ped79, p. 8] for small powers of a positive operator to write

$$g(T+sX)^r = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}[(1+\lambda g(T+sX))^{-1}g(T+sX)]\,d\lambda,$$

and similarly for $g(T)^r$. Apply the resolvent formula to get the following expression for our derivative calculation

$$\frac{d}{ds}\Big|_{s=0}\tau(Yg(T+sX)^q) = \lim_{s\to 0}\tau\Big(Y\frac{1}{s}\sum_{i=0}^{n}\Big(g(T+sX)^{r(n-i)}\frac{\sin(r\pi)}{\pi}\int_0^\infty I_\lambda\,d\lambda\, g(T)^{ri}\Big)\Big),$$

where the integrand $I_\lambda = \lambda^{-r}(1+\lambda g(T+sX))^{-1}(g(T+sX)-g(T))(1+\lambda g(T))^{-1}$.

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We combine the difference $g(T+sX)-g(T)$ from the integrand with the factor of $\frac{1}{s}$ appearing in the limit, thereby obtaining the difference quotient at 0 of the function $s \mapsto g(T+sX)$. Recall that this difference quotient was denoted by $d_{g,s}(T,X)$, and the derivative at 0 by $d_{g,0}(T,X)$. In order to make the formulas easier to read, we will also use $R_\lambda(A)$ for $(1+\lambda A)^{-1}$; by design, $g(T)$ is a positive operator for $T \in T_0 + \mathcal{S}$, by condition (i) of Proposition 3.1, and it is with these operators that the notation R_λ will mainly be used. In addition, we remark here that if A is a positive operator, then $\|R_\lambda(A)\| \le 1$, an inequality that will be needed often. We introduce these notations into our last derivative calculation to get

$$\frac{d}{ds}\Big|_{s=0} \tau(Yg(T+sX)^q) = \lim_{s\to 0} \tau\Big(Y \sum_{i=0}^{n} g(T+sX)^{r(n-i)} \times \frac{\sin(r\pi)}{\pi}\Big(\int_0^\infty \lambda^{-r} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T)\, d\lambda \Big) g(T)^{ri} \Big).$$

Note that, in each term, either $g(T+sX)^{r(n-i)}$ or $g(T)^{ri}$ must be trace class (since $r(n-i)+ri = rn = q-r > q-1$, so at least one of $r(n-i)$ and ri must be greater than or equal to $\frac{q-3}{2}$). This will be important in the proofs below. In order to avoid having to distinguish which of $g(T+sX)^{r(n-i)}$ and $g(T)^{ri}$ is trace class (which will not be relevant beyond the fact that one of them is), we introduce a special notation to deal with both cases at once. Namely, for an operator A denote by $\|A^t\|_\star$ the $\mathcal{L}^1$ norm if t is large enough that $A^t \in \mathcal{L}^1$, and the operator norm otherwise.

To start with, since one of $g(T+sX)^{r(n-i)}$ and $g(T)^{ri}$ is trace class, we can pull the trace into the sum (each term is also trace class). Moreover, $g(T+sX)^{r(n-i)}$ and $g(T)^{ri}$ are bounded operators, so we can pull them into the integral without changing the value of the expression. We get

$$\frac{d}{ds}\Big|_{s=0} \tau(Yg(T+sX)^q) = \lim_{s\to 0} \sum_{i=0}^{n} \frac{\sin(r\pi)}{\pi} \times \tau\Big(\int_0^\infty \lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}\, d\lambda \Big).$$

We would like to conclude that this limit evaluates to

$$\sum_{i=0}^{n} \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{-r} \tau\Big(Y g(T)^{r(n-i)} R_\lambda g(T) d_{g,0}(T,X) R_\lambda g(T) g(T)^{ri} \Big)\, d\lambda,$$

the expression obtained by first exchanging the integral and the trace and then plugging in $s=0$.

We consider one term at a time; that is, in most of the following (until we return to the derivative calculation), i is fixed, and we recall that r and n are determined by the exponent q in the formula for α, and hence are also fixed. For each $s \in [-1,1]$, define

$$J_s(\lambda) = \lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}.$$

In our last limit expression, $J_s(\lambda)$ is the integrand; we need to show that

$$\lim_{s\to 0} \tau\Big(\int_0^\infty J_s(\lambda)\, d\lambda \Big) = \int_0^\infty \tau(J_0(\lambda))\, d\lambda.$$

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This is accomplished in three steps: from Lemma 3.3 and Lemma 3.4 we conclude that we can apply Fubini's theorem to switch the order of the trace and the integral, and Lemma 3.5 allows us to apply the Lebesgue Convergence Theorem and evaluate the limit. We can then use the trace property and the restrictions on $d_{g,0}(T,X)$ to show that each term that appears in the formula for $\frac{d}{ds}\big|_{s=0}\tau(Xg(T+sY)^q)$, also appears in the formula for $\frac{d}{ds}\big|_{s=0}\tau(Yg(T+sX)^q)$.

Step 1 of the proof is the continuity of the integrand as a function of λ. As the result follows easily from the triangle inequality and Hölder's inequality, along with the conditions on g and the continuity of the spectral calculus, the proof is omitted. We note, however, that in this proof we need the fact that at least one of $g(T+sX)^{r(n-i)}$ and $g(T)^{ri}$ is trace class, which follows from condition (ii) of Proposition 3.1 and the fact that $r(n-i)+ri = rn = q-r > q-1$.

Lemma 3.3 *Fix $s \in [-1,1]$. Recall that t is a fixed positive real number, and we used the notation $n = \lfloor t \rfloor$ and $r = \frac{t}{n+1}$ (so $0 < r < 1$); g is a function whose properties are given in Proposition 3.1. For each fixed integer i, $0 \le i \le n$, we defined*

$$J_s(\lambda) = \lambda^{-r} Yg(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}.$$

The map $\lambda \mapsto J_s(\lambda)$ is continuous (in $\mathcal{L}^1$ norm) on the interval $(0,\infty)$.

The following lemma accomplishes step two in our proof (finding a function $h(\lambda)$ that can be used as a bound for $J_s(\lambda)$ for all $s \in [-1,1]$ simultaneously).

Lemma 3.4 *Assume $s \in [-1,1]$. Recall that t is a fixed positive real number, and we used the notation $n = \lfloor t \rfloor$ and $r = \frac{t}{n+1}$ (so $0 < r < 1$); g is a function whose properties are given in Proposition 3.1. For each fixed integer i, $0 \le i \le n$, we defined*

$$J_s(\lambda) = \lambda^{-r} Yg(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}.$$

There exists a constant k that does not depend on either s or λ such that, for each $\lambda \in (0,\infty)$, $\|J_s(\lambda)\|_1 \le \min\{\lambda^{-r}, \lambda^{-r-1}\}k$. It follows that

$$\int_0^\infty \|J_s(\lambda)\|_1 \, d\lambda < k \int_0^\infty \min\{\lambda^{-r}, \lambda^{-r-1}\} \, d\lambda < \infty.$$

Proof Recall that $s \mapsto d_{g,s}(T,X)$ is continuous by definition, so we can define the constant $k_0 = \sup_{s\in[-1,1]} \|d_{g,s}(T,X)\|$. As the value of $\min\{\lambda^{-r}, \lambda^{-r-1}\}$ depends on how λ compares to 1, we will divide the proof into cases according to the value of λ.

Case 1: $\lambda \in (0,1)$; then $\lambda^{-r} < \lambda^{-r-1}$. We have

$$\begin{aligned}
&\|\lambda^{-r} Yg(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}\|_1 \\
&\qquad \le \lambda^{-r}\big(\|Y\| \|g(T+sX)^{r(n-i)}\|_\star \|R_\lambda g(T+sX)\| \|d_{g,s}(T,X)\| \\
&\qquad\qquad \times \|R_\lambda g(T)\| \|g(T)^{ri}\|_\star \big).
\end{aligned}$$

Using the properties of g and functional calculus, it is easy to argue that

$$k_1 = k_0 \|Y\| \|g(T)^{ri}\|_\star \sup_{s\in[-1,1]} (\|g(T+sX)^{r(n-i)}\|_\star)$$

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defines a constant (independent of s and λ) satisfying

$$\|\lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}\|_1 \le \lambda^{-r} k_1.$$

Case 2: Now consider $\lambda \in [1,\infty)$. In this case $\lambda^{-r-1} \le \lambda^{-r}$, so the estimates from the previous case are not sufficient. However, by design, either $r(n-i)-1 \ge \frac{q-3}{2}$ or $ri - 1 \ge \frac{q-3}{2}$, enabling us to get a better upper bound. Since $r(n-i) + ri = rn = q - r > q - 1$, it follows that at least one of $r(n-i)$ and ri must be greater than or equal to $\frac{q-1}{2}$. Consider the two cases $r(n-i) \ge \frac{q-1}{2}$ and $r(n-i) < \frac{q-1}{2}$, exactly one of which holds for our fixed r, n, and i, and in each case define a constant k_2 such that $\|J_s(\lambda)\| \le \lambda^{-r-1} k_2$.

Suppose first that $r(n-i) \ge \frac{q-1}{2} = \frac{q-3}{2} + 1$; so $r(n-i) - 1 \ge \frac{q-3}{2}$, whence $g(T+sX)^{r(n-i)-1}$ is still in $\mathcal{L}^1$ by condition (ii) of Proposition 3.1 and, using Hölder's inequality, we can write

$$\begin{aligned}
&\|\lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}\|_1 \\
&\quad \le \lambda^{-r} \|Y\| \|g(T+sX)^{r(n-i)-1}\|_1 \|g(T+sX) R_\lambda g(T+sX)\| \|d_{g,s}(T,X)\| \\
&\qquad \times \|R_\lambda g(T)\| \|g(T)^{ri}\|.
\end{aligned}$$

Recall that $g(T+sX)$ is a positive operator; hence, by the Spectral Theorem, $\|g(T+sX) R_\lambda g(T+sX)\| \le \frac{1}{\lambda}$. Moreover, $\|R_\lambda g(T)\| \le 1$ and $\|d_{g,s}(T,X)\|$ is bounded above by the constant k_0 defined at the beginning of the proof. As $\|g(T)^{ri}\|$ does not depend on s, we only need to show that $\|g(T+sX)^{r(n-i)-1}\|_1$ is uniformly bounded for $s \in [-1,1]$. However, this follows immediately from condition (iii) of Proposition 3.1 so define the constant

$$k_2 = k_0 \|Y\| \|g(T)^{ri}\| \big(\sup_{s \in [-1,1]} \|g(T+sX)^{r(n-i)-1}\|_1 \big)$$

independent of s and λ for which

$$\|\lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}\|_1 \le \lambda^{-r-1} k_2.$$

This gives us the desired inequality in the case when $r(n-i) \ge \frac{q-1}{2}$.

On the other hand, if $r(n-i) < \frac{q-1}{2}$, then we must have $ri \ge \frac{q-1}{2} = \frac{q-3}{2} + 1$, so we can perform a similar calculation by writing

$$(1 + \lambda g(T))^{-1} g(T)^{ri} = g(T)(1 + \lambda g(T))^{-1} g(T)^{ri-1}.$$

In this case we need to let

$$k_2 = k_0 \|Y\| \|g(T)^{ri-1}\|_1 \big(\sup_{s \in [-1,1]} \|g(T+sX)^{r(n-i)}\| \big),$$

to obtain

$$\|\lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}\|_1 \le \lambda^{-r-1} k_2.$$

Therefore, for $k = \max\{k_1, k_2\}$, we obtain the desired result

$$\|J_s(\lambda)\|_1 \le k \min\{\lambda^{-r}, \lambda^{-r-1}\}.$$

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As $\int_0^1 \lambda^{-r}\,d\lambda$ and $\int_1^\infty \lambda^{-r-1}\,d\lambda$ are both finite (recall $0 < r < 1$),

$$\int_0^\infty \min\{\lambda^{-r}, \lambda^{-r-1}\}\,d\lambda$$

converges. Using the inequality shown above, we can thus conclude that

$$\int_0^\infty \|J_s(\lambda)\|_1\,d\lambda < \infty. \qquad \blacksquare$$

Finally, we need to show that $J_s(\lambda)$ converges pointwise to $J_0(\lambda)$ (step 3 of the proof). This is again accomplished by judicious use of the triangle inequality and Hölder's inequality; as the proof is quite straight forward, we omit it, noting only that it relies on conditions (iii) and (iv) of Proposition 3.1.

Lemma 3.5 *Recall that t is a fixed positive real number, and we used the notation $n = \lfloor t \rfloor$ and $r = \frac{t}{n+1}$ (so $0 < r < 1$); g is a function whose properties are given in Propositon 3.1. For each fixed integer i, $0 \le i \le n$, we defined*

$$J_s(\lambda) = \lambda^{-r} Y g(T+sX)^{r(n-i)} R_\lambda g(T+sX) d_{g,s}(T,X) R_\lambda g(T) g(T)^{ri}.$$

Fix $\lambda \in (0,\infty)$. Then $\tau(J_s(\lambda)) \to \tau(J_0(\lambda))$ as $s \to 0$.

By Lemmas 3.3 and 3.4, $\lambda \mapsto J_s(\lambda)$ is continuous in $\mathcal{L}^1$ norm, and we can find a constant k such that for all $s \in [-1,1]$ we have

$$\int_0^\infty \|J_s(\lambda)\|_1\,d\lambda \le k \int_0^\infty \min\{\lambda^{-r}, \lambda^{-r-1}\}\,d\lambda < \infty,$$

where k is a constant which does not depend on s. Moreover, by Lemma 3.5, $\tau(J_s(\lambda))$ converges pointwise to $\tau(J_0(\lambda))$ as $s \to 0$. We can thus use the Lebesgue Dominated Convergence Theorem to conclude that, for any sequence $\{r_n\}$ converging to 0, we have $\int_0^\infty \tau(J_{r_n}(\lambda))\,d\lambda \to \int_0^\infty \tau(J_0(\lambda))\,d\lambda$. Since $\mathbb{R}$ is first-countable, this is sufficient to ensure that $\int_0^\infty \tau(J_s(\lambda))\,d\lambda$ converges to $\int_0^\infty \tau(J_0(\lambda))\,d\lambda$ as $s \to 0$. This concludes the proof that

$$\frac{d}{ds}\Big|_{s=0} \tau(Yg(T+sX)^q) = \sum_{i=0}^n \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{-r} \tau\big(Yg(T)^{r(n-i)} R_\lambda g(T) d_{g,0}(T,X) R_\lambda g(T) g(T)^{ri}\big)\,d\lambda.$$

Since $d_{g,0}(T,X) = \lim_{s\to 0} \frac{1}{s}(g(T+sX) - g(T))$ can be written as sums of the form $g_1(T)Xg_2(T)$ (condition (v) of Proposition 3.1), we can break up each trace into a sum of terms that look like

$$\tau(Yg(T)^{r(n-i)} R_\lambda g(T) g_1(T) X g_2(T) R_\lambda g(T) g(T)^{ri}).$$

By the trace property

$$\tau(Yg(T)^{r(n-i)} R_\lambda g(T) g_1(T) X g_2(T) R_\lambda g(T) g(T)^{ri}) = \tau(Xg_2(T) R_\lambda g(T) g(T)^{ri} Y g(T)^{r(n-i)} R_\lambda g(T) g_1(T)).$$

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Since the various functions of T commute with each other, the right-hand side can be rewritten as

$$\tau(Xg(T)^{ri}R_\lambda g(T)g_2(T)Yg_1(T)R_\lambda g(T)g(T)^{r(n-i)}).$$

However, by symmetry, $\frac{d}{ds}\big|_{s=0}\tau(Xg(T+sY)^q)$ is equal to

$$\sum_{i=0}^{n}\frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}\tau(Xg(T)^{r(n-i)}R_\lambda g(T)d_{g,0}(T,Y)R_\lambda g(T)g(T)^{ri})\,d\lambda,$$

so it should be clear that changing the index from i to $n-i$ and expanding $d_{g,0}(T,Y)$ will show that the expression we obtained for the derivative in the X direction appears in the derivative for the Y direction. Here we rely on condition (v) of Proposition 3.1 which ensures that either $g_1 = g_2$ or, if $g_1 \neq g_2$, then the term $g_2(T)Yg_1(T)$ also appears in $d_{g,0}(T,Y)$. This concludes the proof that the two limits are equal, and hence that α is closed.

Since $\mathcal{S}$ is a Banach space, and the manifold under consideration is simply $T_0+\mathcal{S}$, the fact that α is exact follows as in the Poincaré Lemma [Lan62, Theorem V.4.1]. Using properties of integrals, it follows that α is independent of the path over which it is integrated [CP98, Remark 1.4], concluding the proof of Proposition 3.1.

We will use this result for both $\tau(X|1-F^2|^q)$ in the bounded case, which we can re-write as $\tau(X((1-F^2)^2)^{\frac{q}{2}})$ to get rid of the absolute value sign, giving us $g(F)=(1-F^2)^2$, and $\tau(X(1+D^2)^{-m})$ in the unbounded case, giving us $g(D)=(1+D^2)^{-1}$. The fact that these two functions satisfy the desired properties will be shown in Section 4.

3.1 A Different Restriction on the Function g

The purpose of this section is to replace one of the requirements imposed on g in the definition of our one-form by a (possibly) stronger one which will be easier to prove for some choices of g. We will use this alternate description in the bounded case in Section 6.1. See Proposition 3.1 for a list of the current restrictions placed on g; recall in particular the definition of $d_{g,s}$:

$$d_{g,s}(T,X)=\begin{cases}\dfrac{g(T+sX)-g(T)}{s} & \text{for } s\neq 0,\\[2ex] \lim\limits_{s\to 0}\dfrac{g(T+sX)-g(T)}{s} & \text{for } s=0,\end{cases}$$

where the limit is calculated with respect to the operator norm.

Lemma 3.6 *Suppose g is a function that satisfies conditions* (i), (ii), *and* (iv) *of Proposition 3.1. Consider $t\in\mathbb{R}_+$ with $t\geq\frac{q-3}{2}$, and fix $T\in T_0+\mathcal{S}$, $X\in\mathcal{S}$; we know that $g(T+sX)^t\in\mathcal{L}^1$ by condition* (ii) *of Proposition 3.1. Suppose in addition that g satisfies the following property:*

(iii′) *For each fixed $t\geq\frac{q-3}{2}$, and $s\in[-1,1]$ we have $d_{g,s}(T,X)\in\mathcal{L}^t$. Moreover, $s\mapsto\|d_{g,s}(T,X)\|_t$ is uniformly bounded for $s\in[-1,1]$.*

It follows that the function from $\mathbb{R}$ to $\mathcal{L}^1$ defined by $s\mapsto g(T+sX)^t$ is continuous at $s=0$, i.e., g also satisfies (iii).

Proof By assumption (iii′), we know $\{\|d_{g,s}(T,X)\|_t : s \in [-1,1]\}$ is bounded, say by M (chosen such that $M \neq 0$). Rearrange the definition of $d_{g,s}$ to $g(T+sX) = sd_{g,s}(T,X) + g(T)$, and note that the formula holds even for $s = 0$. It follows that $\{\|g(T+sX)\|_t : s \in [-1,1]\}$ is also bounded by $L := M + \|g(T)\|_t$.

Using (3.1), we can write

$$g(T+sX)^t - g(T)^t = \sum_{i=0}^{n} g(T+sX)^{ri}(g(T+sX)^r - g(T)^r)g(T)^{r(n-i)},$$

where $n = \lfloor t \rfloor$ and $r = \frac{t}{n+1}$. Apply the triangle and Hölder inequalities, respectively, (with the understanding that the norm $\|\cdot\|_{\frac{t}{u}}$ represents the operator norm when the denominator u is 0) to get

$$\begin{aligned}
&\|g(T+sX)^t - g(T)^t\|_1 \\
&\quad\le \sum_{i=0}^{n} \|g(T+sX)^{ri}(g(T+sX)^r - g(T)^r)g(T)^{r(n-i)}\| \\
&\quad\le \sum_{i=0}^{n} \|g(T+sX)^{ri}\|_{\frac{t}{ri}} \|g(T+sX)^r - g(T)^r\|_{\frac{t}{r}} \|g(T)^{r(n-i)}\|_{\frac{t}{r(n-i)}}.
\end{aligned}$$

In order to justify the use of Hölder's inequality, we must check that $g(T+sX)^{ri} \in L^{\frac{t}{ri}}$ and so on for the other factors, but this should be clear from the properties of g. Algebraically manipulate the right-hand side of the last inequality and use the upper bound on $\|g(T+sX)\|_t$ to write

$$\begin{aligned}
&\|g(T+sX)^t - g(T)^t\|_1 \\
&\qquad\le \|g(T+sX)^r - g(T)^r\|_{\frac{t}{r}} \Big(\sum_{i=0}^{n} \|g(T+sX)^{ri}\|_{\frac{t}{ri}} \|g(T)^{r(n-i)}\|_{\frac{t}{r(n-i)}}\Big) \\
&\qquad\le \|g(T+sX)^r - g(T)^r\|_{\frac{t}{r}} \Big(\sum_{i=0}^{n} (\|g(T+sX)\|_t)^{ri}(\|g(T)\|_t)^{r(n-i)}\Big) \\
&\qquad\le \|g(T+sX)^r - g(T)^r\|_{\frac{t}{r}} \Big(\sum_{i=0}^{n} (L^{ri}L^{r(n-i)})\Big) \\
&\qquad\le \|g(T+sX)^r - g(T)^r\|_{\frac{t}{r}} ((n+1)L^{rn}).
\end{aligned}$$

However, note that $\frac{t}{r} = n+1 = \lfloor t \rfloor + 1 \ge 1$, since $t \ge 0$, and that $0 < r < 1$ by definition; use the well-known inequality $\|A^r - B^r\|_p \le \||A-B|^r\|_p$, which holds for $0 < r < 1$, $p \ge 1$, and A, B positive operators in $\mathcal{L}^{pr}$, to conclude that

$$\|g(T+sX)^r - g(T)^r\|_{\frac{t}{r}} \le \||g(T+sX) - g(T)|^r\|_{\frac{t}{r}} = \|g(T+sX) - g(T)\|_t^r.$$

Using again $g(T+sX) - g(T) = sd_{g,s}(T,X)$, for all s (even $s = 0$), we get

$$\|g(T+sX) - g(T)\|_t^r = (\|sd_{g,s}(T,X)\|_t)^r \le |s|^r M^r.$$

Combining this with our earlier calculations,

$$\|g(T+sX)^t - g(T)^t\|_1 \le ((n+1)L^{rn})|s|^r M^r.$$

Recall that $n = \lfloor t \rfloor$ and L, M are norm bounds that do not depend on s; it follows that $g(T+sX)^r \to g(T)^t$ in $\mathcal{L}^1$-norm as $s \to 0$. ■

Therefore, Condition (iii) of Proposition 3.1 can be replaced by (iii′). As an aside, (iii′) is satisfied by both functions g that we will have occasion to consider, but it is only used in the bounded case, as (iii) is easy to show in the unbounded case.

4 Integral Formulas for Spectral Flow

We now consider restrictions under which we can provide integral formulas for spectral flow. The approach below is the same as the one used in [CP04, Theorem 4.1] and [Phi97, Theorem 3.1] . In the unbounded case we consider paths in the manifold $D_0 + \mathcal{N}_{sa}$, where D_0 is self-adjoint Breuer–Fredholm, and satisfies an additional summability condition, usually stated as $(1+D_0^2)^{-1} \in \mathcal{J}$ for some operator ideal $\mathcal{J}$. In our case, D_0 is p-summable, so $(1+D_0^2)^{-1} \in \mathcal{L}^{p/2}$. As a different example, the requirement that D_0 is θ-summable implies $(1+D_0^2)^{-1}$ belongs to an ideal Li_0 [CP04, Corollary B.6]. Once we apply the Riesz transform $D \mapsto D(1+D^2)^{-\frac{1}{2}}$, we end up with paths in some manifold $F_0 + \mathcal{S}$, where $\mathcal{S}$ is a real Banach space, as well as an operator ideal related to $\mathcal{J}$. The equality that allows us to flip between the two pictures (bounded and unbounded) is $(1+D^2)^{-1} = 1 - F_D^2$, where F_D is the image of D under the Riesz transform.

See Section 2 for an overview of how the bounded case and unbounded case formulas are related. Let it suffice to reiterate that in the bounded case we consider one-forms $X \mapsto \frac{1}{C}\tau(Xh(1-F^2))$ for some suitable constant C and function h. We show that integrating this one-form gives spectral flow for straight-line paths whose endpoints are of the form $2P-1$ and $2Q-1$, respectively, for two projections P and Q. The general formula is then obtained from this case.

In the unbounded case, for $(\mathcal{N}, D_0)$ an unbounded Breuer–Fredholm module, we will consider paths $\{D_0 + A_t\}$, where the $A_t \in \mathcal{N}$ define a continuous path, and $(1+ D_0^2)^{-1} \in \mathcal{J}$, for some ideal of operators $\mathcal{J}$ satisfying the properties laid out in Definition 1.15. We need to describe first the manifold $F_0 + \mathcal{S}$ in which we can find the image of the path $\{D_t\}$ under the Riesz transform. With $F_D = D(1+D^2)^{-\frac{1}{2}}$, the equality $(1+D^2)^{-1} = 1-F_D^2$ makes it obvious that $(1+D^2)^{-1} \in \mathcal{J}$ means $1-F_D^2 \in \mathcal{J}$. The spaces that we are compelled to consider in the bounded case are $F_0 + \mathcal{S}\mathcal{J}_{F_0}$, where F_0 is self-adjoint Breuer–Fredholm, and $\mathcal{S}\mathcal{J}_{F_0} = \{X \in \mathcal{J}_{sa}^{1/2} : 1-(F_0+X)^2 \in \mathcal{J}\}$. This space was introduced in [CP04], where many of the properties we are going to use were also established. With the norm $\|X\|_{\mathcal{S}\mathcal{J}_{F_0}} := \|X\|_{\mathcal{J}^{\frac{1}{2}}} + \|XF_0 + F_0X\|_{\mathcal{J}}$, $\mathcal{S}\mathcal{J}_{F_0}$ is a Banach space; moreover, choosing any other base point instead of F_0 will define the same space, and an equivalent norm [CP04, Lemma B.12]. Suppose that $F_0 = 2P-1$ for some projection P. Then $\mathcal{S}\mathcal{J}_{F_0}$ consists of all operators in $\mathcal{J}^{\frac{1}{2}}$ that, with respect to the decomposition $P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, have the form

$$\begin{bmatrix} P\mathcal{J}P & P\mathcal{J}^{\frac{1}{2}}P^{\perp} \\ P^{\perp}\mathcal{J}^{\frac{1}{2}}P & P^{\perp}\mathcal{J}P^{\perp} \end{bmatrix}.$$

If F_0 is not equal to $2P-1$ for some projection P, then the above still holds, but with $P = \chi_{[0,\infty)}(F_0)$, in which case $2P-1 \in F_0 + \mathcal{S}\mathcal{J}_{F_0}$. For these properties of $\mathcal{S}\mathcal{J}_{F_0}$ see [CP04, pp. 143–144, Appendix B]. Note that $\mathcal{S}\mathcal{J}_{F_0}$ is denoted in [CP04] by $\mathcal{J}_{F_0}$; be warned that a typo on page 143 of the article suggests that $\mathcal{I}_{F_0}$ denotes the space $F_0 + \mathcal{S}\mathcal{I}_{F_0}$ instead, which was clearly not intended.

We further mention that, while [CP04] introduced these spaces for general $\mathcal{I}$, the proof for the spectral flow formula in these spaces is not verified there in full generality; however, our work is reduced to checking that the main steps go through. In particular, Lemma 4.2 and Lemma 4.5 are slightly more general statements of already existing results, culminating in the integral formulas stated in Theorem 4.3 for the bounded case and Theorem 4.7 for the unbounded case; the proofs, which are exactly as in [CP04], are omitted.

We will show that with $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$ for some $0 < \varepsilon < 1$, if $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{\mathrm{sa}}$ with $(1 + D_0^2)^{-1} \in \mathcal{I}$, then $\{F_t\}$ is C^1 in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Hence, these are the kind of spaces we consider for the bounded case formula.

The final goal is, for a path $\{D_t\}$ in $D_0 + \mathcal{N}_{\mathrm{sa}}$, to determine conditions we can put on the function k to ensure that we get a spectral flow formula of the type

$$\mathrm{sf}(\{D_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\Big)\,dt + \beta(D_1) - \beta(D_0),$$

where the appropriate choices for constant C and correction terms $\beta(D)$ will come out of the proof. We have divided the proof into three steps.

Step 1. Suppose first that $\{F_t\}$ is a *straight line path* from $F_0 = 2P - 1$ to $F_1 = 2Q - 1$, with $\{F_t\} \subset F_0 + \mathcal{S}\mathcal{J}_{F_0}$ for some suitable operator ideal $\mathcal{J}$. Then Lemma 4.2 states that, for judiciously chosen functions h,

$$\mathrm{sf}(\{F_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(F_t)h(1 - F_t^2)\Big)\,dt.$$

Step 2. Next, consider a general path $\{F_t\}$ in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Use concatenation and homotopy to obtain a path of the type considered in the previous step, allowing us to extend the formula. The correction terms enter the proof at this step.

Step 3. If $\{D_t\}$ is a path of unbounded operators, let $F_t = D_t(1+D_t^2)^{-\frac{1}{2}}$. We know $\mathrm{sf}(\{D_t\}) = \mathrm{sf}(\{F_t\})$, and $(1 + D_t^2)^{-1} = 1 - F_t^2$. We show that $\{F_t\}$ is a C^1 path in an appropriate affine space, allowing us to use the bounded case. So it immediately follows that, if h is suitably chosen,

$$\mathrm{sf}(\{D_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(F_t)h((1+D_t^2)^{-1})\Big)\,dt + \beta(D_1) - \beta(D_0).$$

Lemma 4.6 will allow us to get rid of the $\frac{d}{dt}(F_t)$ in this formula, by showing that

$$\tau\Big(\frac{d}{dt}(F_t)h((1+D_t^2)^{-1})\Big) = \tau\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-\frac{3}{2}}h((1+D_t^2)^{-1})\Big).$$

It is this equality that determines the relationship between h and k:

$$k((1+D_t^2)^{-1}) = (1+D_t^2)^{-\frac{3}{2}}h((1+D_t^2)^{-1}).$$

We now proceed to carry out this program.

4.1 Integral Formulas for Straight-line Paths With Special Endpoints (Step 1)

We need to show that the integral formula calculates spectral flow in the special case when the endpoints of the path are of the form $2P-1$ and $2Q-1$ for P, Q projections. Note that, if P and Q are projections for which $Q-P \in \mathcal{K}_{\mathcal{N}}$, then, since $\pi(P)=\pi(Q)$, the spectral flow of the straight line path from $2P-1$ to $2Q-1$ is $\operatorname{ind}(PQ)$ (directly from Definition 1.4). The following theorem gives us a formula for calculating this index.

Theorem 4.1 ([CP04, Theorem 3.1]) *Let $f\colon[-1,1]\to\mathbb{R}$ be a continuous odd function with $f(1)\neq 0$. Let P and Q be projections with $Q-P\in\mathcal{K}_{\mathcal{N}}$ and $f(Q-P)$ trace class. Then $\operatorname{ind}(PQ)=\frac{1}{f(1)}\tau(f(Q-P))$, where $\operatorname{ind}(PQ)$ is the index of PQ as an operator from $Q\mathcal{H}$ to $P\mathcal{H}$.*

The above result applied to a suitable family of functions allows us to get the spectral flow as an integral, but only in the case when our paths are linear and have special endpoints. For such a path $\{F_t\}$ one can calculate that

$$\tau\Big(\frac{d}{dt}(F_t)h(1-F_t^2)\Big)=\tau\Big(2(Q-P)h\Big(4t(1-t)(Q-P)^2\Big)\Big),$$

and use Theorem 4.1 to relate this value (for each t) to the spectral flow. In [CP04], this is done for the function $h(x)=x^{-r}e^{-x^{-1/q}}$; since the proof is essentially the same, we omit it.

Lemma 4.2 ([CP04, Theorem 4.1]) *For P and Q projections, let $F_0=2P-1$ and $F_1=2Q-1$, and suppose $F_1\in F_0+\mathcal{S}\mathcal{J}_{F_0}$, where $\mathcal{J}$ is a small power invariant operator ideal (Definition 1.15). Denote by $\{F_t\}$ the straight-line path from F_0 to F_1,* i.e., *$F_t=F_0+t(F_1-F_0)$ for $t\in[0,1]$. Suppose, moreover, that $h\colon\mathbb{R}\to\mathbb{R}$ is a continuous function, non-zero on $(0,1]$, and for which $h(T)$ is trace class for all $T\in\mathcal{J}_{sa}$. Then*

$$\operatorname{sf}(\{F_t\})=\frac{1}{C}\int_0^1\tau\Big(\frac{d}{dt}(F_t)h(1-F_t^2)\Big)\,dt,$$

with $C=\int_{-1}^1 h(1-s^2)\,ds$.

4.2 Integral Formulas in the Bounded Setting (Step 2)

In this step, we relate the calculation of spectral flow for a general path in $F_0+\mathcal{S}\mathcal{J}_{F_0}$ to a straight-line path of the type for which we already have a formula; to this end, we follow the blueprint laid out in [CP98].

Consider the function

$$\operatorname{sign}(x)=\begin{cases}1 & \text{if } x\geq 0,\\ -1 & \text{if } x<0,\end{cases}$$

and for any F self-adjoint write $\widetilde{F}$ for $\operatorname{sign}(F)$. Note that $\widetilde{F}=2\chi_{[0,\infty)}(F)-1$ and that $\widetilde{F}^2=1$. Suppose that additionally $F\in F_0+\mathcal{S}\mathcal{J}_{F_0}$ for some appropriate F_0; we want to show that $\widetilde{F}\in F_0+\mathcal{S}\mathcal{J}_{F_0}$. Since $F+\mathcal{S}\mathcal{J}_F=F_0+\mathcal{S}\mathcal{J}_{F_0}$, it is sufficient to check that $\widetilde{F}\in F+\mathcal{S}\mathcal{J}_F$. Going back to the definition of $F+\mathcal{S}\mathcal{J}_F$, it is easy to see $1-\widetilde{F}^2=0\in\mathcal{J}$, but

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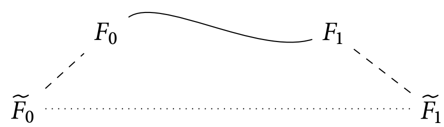

Figure 1: Extend the original path F_t by straight lines F_0–$\widetilde{F}_0$ and F_1–$\widetilde{F}_1$ (indicated by dashes in the figure). Integrating our one-form along either path from $\widetilde{F}_0$ to $\widetilde{F}_1$ should give the same value, *i.e.*, the spectral flow from F_0 to F_1.

we also need to check that $\widetilde{F} - F \in \mathcal{J}_{sa}^{1/2}$. However, $1 - F^2 = \widetilde{F}^2 - F^2 = (\widetilde{F} - F)(\widetilde{F} + F)$ (here we used the fact that F and $\widetilde{F}$ commute); $\widetilde{F} + F$ is invertible, so we have

$$\widetilde{F} - F = (1 - F^2)(\widetilde{F} + F)^{-1},$$

which gives us that $\widetilde{F} - F \in \mathcal{J}$, since by assumption $1 - F^2 \in \mathcal{J}$ and $\mathcal{J}$ is an ideal. Finally, since $\mathcal{J} \subset \mathcal{J}^{1/2}$, we have $\widetilde{F} - F \in \mathcal{J}^{1/2}$. Hence $\widetilde{F}$ is indeed in $F + \mathcal{S}\mathcal{J}_F$; note that the straight line path from F to $\widetilde{F}$ is then also necessarily contained in $F + \mathcal{S}\mathcal{J}_F$.

If $\{F_t\}$ is any path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$, then extend the path by connecting F_0 to $\widetilde{F}_0$ and F_1 to $\widetilde{F}_1$ via straight lines (Figure 1). As there is no spectral flow from F_0 to $\widetilde{F}_0$ or from F_1 to $\widetilde{F}_1$ (Remark 1.5), the additivity property of spectral flow allows us to conclude that the spectral flow along the path $\widetilde{F}_0$ - - - F_0 —— F_1 - - - $\widetilde{F}_1$ is the same as the spectral flow of $\{F_t\}$. On the other hand, we can join $\widetilde{F}_0$ to $\widetilde{F}_1$ by a straight line (indicated in the figure by a dotted line), which also lies in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Under the assumption that $\alpha_F(X) = \tau(Xh(1 - F_t^2))$ defines an exact one-form on $F_0 + \mathcal{S}\mathcal{J}_{F_0}$, integrating it along either path from $\widetilde{F}_0$ to $\widetilde{F}_1$ will give us the same answer. Finally, we know that integrating the one-form along the straight line path from $\widetilde{F}_0$ to $\widetilde{F}_1$ gives us the spectral flow from $\widetilde{F}_0$ to $\widetilde{F}_1$ (this is Lemma 4.2). Hence, in order to get the spectral flow it is not sufficient to integrate the one-form along the original path; we need to adjust our formula to include correction terms, consisting of the integral of the one-form from $\widetilde{F}_0$ to F_0 and from F_1 to $\widetilde{F}_1$, *i.e.*, along the dashed lines in Figure 1. In summary, we obtain the following result.

Theorem 4.3 *Let $\{F_t\}$ be a C^1 path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Suppose that $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that h is non-zero on $(0,1]$ and $h(T)$ is trace-class for all $T \in \mathcal{J}_{sa}$. Moreover, suppose that $\alpha_F\colon X \mapsto \tau(Xh(1 - F^2))$ is a one-form on $F_0 + \mathcal{S}\mathcal{J}_{F_0}$ whose integral is independent of the path of integration. Let $C = \int_{-1}^{1} h(1 - s^2)\, ds$, and define a function $\gamma\colon F_0 + \mathcal{S}\mathcal{J}_{F_0} \to \mathbb{R}$ by $\gamma(F) = \frac{1}{C}\int_0^1 \tau\big(\frac{d}{dt}(G_t)h(1 - G_t^2)\big)\, dt$, where $\{G_t\}$ is the straight line path from F to $\widetilde{F} = \operatorname{sign}(F)$. Then*

$$\operatorname{sf}(\{F_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(F_t)h(1 - F_t^2)\Big)\, dt + \gamma(F_1) - \gamma(F_0).$$

Remark 4.4 Note that, if F_0 is unitarily equivalent to F_1, then $\gamma(F_0) = \gamma(F_1)$ (since the expressions whose traces we are calculating are also unitarily equivalent); that is,

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the correction terms in the Theorem 4.3 formula cancel, and we are left with

$$\mathrm{sf}(\{F_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(F_t)h(1-F_t^2)\Big)\,dt.$$

4.3 Integral Formulas in the Unbounded Setting (Step 3)

We would now like to reduce the unbounded case to the bounded case. Consider $(\mathcal{N}, D_0)$ an unbounded Breuer–Fredholm module satisfying the condition

$$(1+D_0^2)^{-1} \in \mathcal{I}$$

for some small power invariant operator ideal $\mathcal{I}$ (Definition 1.15). Suppose that $\{D_t\}$ is a path in $D_0 + \mathcal{N}_{\mathrm{sa}}$ and $\{F_t\}$ is its image under the Riesz transform. Recall the notation $\mathcal{S}\mathcal{I}_{F_0} = \{X \in \mathcal{I}^{\frac{1}{2}} : 1-(F_0+X)^2 \in \mathcal{I}\}$. Clearly, $1-F_t^2 \in \mathcal{I}$ for every t; however, in order to use Theorem 4.3, we would need $\{F_t\}$ to be a C^1 path in $F_0 + \mathcal{S}\mathcal{I}_{F_0}$. The various norm inequalities used in the proof are not strong enough to prove this (assuming it is even true); in order for this approach to work, we will have to replace $\mathcal{I}$ by $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$ for some $0 < \varepsilon < 1$ (recall that $\mathcal{I} \subset \mathcal{I}^{1-\varepsilon}$), and show $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Hence, if h is a function that satisfies the hypotheses of Theorem 4.3 for $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$, we could easily conclude that

$$\mathrm{sf}(\{D_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(F_t)h((1+D_t^2)^{-1})\Big)\,dt \\ + \gamma(D_1(1+D_1^2)^{-\frac{1}{2}}) - \gamma(D_0(1+D_0^2)^{-\frac{1}{2}}).$$

A second goal of this section is to get rid of $\frac{d}{dt}(F_t)$ and replace it by $\frac{d}{dt}(D_t)$. The price we pay for this is an extra factor of $(1+D_t^2)^{-\frac{3}{2}}$ in the trace argument (Lemma 4.6).

Fix $0 < \varepsilon < 1$ and let $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$. We want to show that applying the Riesz transform to a C^1 path $\{D_t\}$ in $D_0 + \mathcal{N}_{\mathrm{sa}}$ gives us a C^1 path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Following the main steps of [CP98, CP04], we produce a formula for $\frac{d}{dt}(F_t)$ (in fact, we use the formula given in the aforementioned references), and check the continuity.

Lemma 4.5 ([CP98, Proposition 2.10], [CP04, Proposition 6.5]) *Suppose that $\{D_t = D_0 + A_t\}$ is a path in $D_0 + \mathcal{N}_{\mathrm{sa}}$, with $(D_0^2+1)^{-1} \in \mathcal{I}$, and $\{A_t\}$ a C^1 path in $\mathcal{N}_{sa}$. If $F_t = D_t(1+D_t^2)^{-\frac{1}{2}}$ and $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$ for any $0<\varepsilon<1$, then*

$$\frac{d}{dt}(F_t) = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big[(1+\lambda)(1+D_t^2+\lambda)^{-1}\frac{d}{dt}(A_t)(1+D_t^2+\lambda)^{-1} \\ - D_t(1+D_t^2+\lambda)^{-1}\frac{d}{dt}(A_t)D_t(1+D_t^2+\lambda)^{-1}\Big]\,d\lambda,$$

where the integral converges in $\mathcal{J}^{\frac{1}{2}}$-norm. It follows that $\{F_t\}$ is C^1 in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$.

Proof By [CP04, Lemma B.15], $\{F_t\} \subset F_0 + \mathcal{S}\mathcal{J}_{F_0}$ is a C^1 path if and only if $\{F_t\}$ is C^1 in $\mathcal{J}^{\frac{1}{2}}$-norm and $\{1-F_t^2\}$ is C^1 in $\mathcal{J}$-norm. Additionally, [CP04, Proposition 6.4] gives us that $\{1-F_t^2\}$ is C^1 in $\mathcal{I}$-norm, and hence in $\mathcal{J}$-norm. The only part we still need to check is that $\{F_t\}$ is C^1 in $\mathcal{J}^{\frac{1}{2}}$ norm. In order to simplify the exposition, we introduce notation for two of the expressions appearing in our purported formula for $\frac{d}{dt}(F_t)$. Namely, let

$$L_t(\lambda) = (1+\lambda)(1+D_t^2+\lambda)^{-1}\frac{d}{dt}(A_t)(1+D_t^2+\lambda)^{-1},$$
$$R_t(\lambda) = D_t(1+D_t^2+\lambda)^{-1}\frac{d}{dt}(A_t)D_t(1+D_t^2+\lambda)^{-1}.$$

First, let us ensure that, for each fixed $t \in [0,1]$, the integral $\int_0^\infty \lambda^{-\frac{1}{2}}[L_t(\lambda)-R_t(\lambda)]\,d\lambda$ converges. Let $\sigma = \frac{1-\varepsilon}{2}$; note $0 < \sigma < \frac{1}{2}$. Apply Lemma A.5 to show that L_t and R_t are continuous as functions of λ from $\mathbb{R}_+$ to $\mathcal{J}^{1/2}$ (part v), and $\|L_t(\lambda)\|_{\mathcal{J}^{\frac{1}{2}}}$ and $\|R_t(\lambda)\|_{\mathcal{J}^{\frac{1}{2}}}$ are each bounded by a constant multiple of $(1+\lambda)^{-(1-\sigma)}$ (Lemma A.5 (ii), along with Lemma A.6 (i)). Since $\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)^{-(1-\sigma)}\,d\lambda$ converges for $0<\sigma<\frac{1}{2}$, this is sufficient to prove that $\int_0^\infty \lambda^{-\frac{1}{2}}[L_t(\lambda)-R_t(\lambda)]\,d\lambda$ converges in $\mathcal{J}^{\frac{1}{2}}$-norm.

We next want to show that the integral calculates $\frac{d}{dt}(F_t)$. We concentrate on the derivative at $t=0$, as other values of t are similar. By Corollary A.2,

$$F_t - F_0 = B_{0,t}(1+D_t^2)^{-\sigma},$$

where $\{B_{0,t}\}$ is differentiable in norm at $t=0$. Note that the expression we have for $\frac{d}{dt}\big|_{t=0}(F_t)$ is equal to $\frac{d}{dt}\big|_{t=0}(B_{0,t})(1+D_0^2)^{-\sigma}$, where the operator norm derivative $\frac{d}{dt}\big|_{t=0}(B_{0,t})$ is as in Corollary A.2. The fact that $\mathcal{I}$ is a small power invariant ideal gives us $(1+D_t^2)^{-\sigma} \in \mathcal{J}^{1/2}$ (see Corollary A.4). Since $\mathcal{J}^{1/2}$ is an invariant operator ideal, this shows that

$$\begin{aligned}\Big\|\frac{F_t-F_0}{t} - \frac{d}{dt}\Big|_{t=0}(F_t)\Big\|_{\mathcal{J}^{\frac{1}{2}}} &= \Big\|\frac{B_{0,t}}{t}(1+D_t^2)^{-\sigma} - \frac{d}{dt}\Big|_{t=0}(B_{0,t})(1+D_t^2)^{-\sigma}\Big\|_{\mathcal{J}^{\frac{1}{2}}}\\ &\le \Big\|\frac{B_{0,t}}{t} - \frac{d}{dt}\Big|_{t=0}(B_{0,t})\Big\|\,\|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}.\end{aligned}$$

Finally, we must show that $\frac{d}{dt}(F_t)$ is continuous in $\mathcal{J}^{\frac{1}{2}}$ norm. Fix $t_0 \in [0,1]$. Using the integral formula just established for $\frac{d}{dt}(F_t)$, along with Lemma A.6 (ii) and (iii) for $s \in [0,1]$, we get

$$\begin{aligned}&\Big\|\frac{d}{dt}\Big|_{t=t_0}(F_t) - \frac{d}{dt}\Big|_{t=s}(F_t)\Big\|_{\mathcal{J}^{\frac{1}{2}}}\\ &\qquad\le 2\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)^{-(1-\sigma)}K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\nu_{s,t_0}\,d\lambda\\ &\qquad= 2K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\nu_{s,t_0}\Big(\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)^{-(1-\sigma)}\,d\lambda\Big).\end{aligned}$$

Since the integral is finite, and $\nu_{s,t_0}\to 0$ as $s\to t_0$, continuity of $\frac{d}{dt}(F_t)$ at t_0 follows.

This concludes the proof that $\{F_t\}$ is C^1 in $\mathcal{J}^{\frac{1}{2}}$-norm, and hence that $\{F_t\}$ is C^1 as a path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Note that, since $\|\cdot\|_{\mathcal{S}\mathcal{J}_{F_0}} \ge \|\cdot\|_{\mathcal{J}^{\frac{1}{2}}}$ (for those elements that are in both spaces), and the derivative of $\{F_t\}$ exists with respect to both the $\mathcal{S}\mathcal{J}_{F_0}$ norm and the $\mathcal{J}^{\frac{1}{2}}$ norm, the two derivatives must be equal. ∎

The key idea that allows us to rewrite the spectral flow formula in terms of $\{D_t\}$ instead of $\{F_t\}$ is the equality of traces stated in the following lemma. Proposition 2.12

of [CP98] proves this result for $h(x) = x^q$ with q a positive integer large enough so that $(1-F_t^2)^q$ is trace class. In spite of the result being stated in slightly more generality below, the proof goes through in exactly the same manner, so we omit it.

Lemma 4.6 ([CP98, Proposition 2.12]) *Let $\{D_t\}$ be a C^1 path in $D_0 + \mathcal{N}_{\text{sa}}$, with $(1 + D_0^2)^{-1} \in \mathcal{I}$. Let $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$, for $0 < \varepsilon < 1$. If $\{F_t\}$ is the image of $\{D_t\}$ under the Riesz transform, then $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Moreover, if h is a continuous function such that $h(1 - F_t^2) \in \mathcal{L}^1$, then*

$$\tau\Big(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{3}{2}} h((1 + D_t^2)^{-1})\Big) = \tau\Big(\frac{d}{dt}(F_t)h(1 - F_t^2)\Big).$$

Theorem 4.7 *Suppose that $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{\text{sa}}$ such that $(1 + D_0^2)^{-1} \in \mathcal{I}$ for some small power invariant operator ideal $\mathcal{I}$ (Definition 1.15) and that k is a continuous function on $\mathbb{R} \setminus \{0\}$ which is non-zero on $(0, 1]$ and for which $\lim_{x\to 0} \frac{k(x)}{x^{3/2}} = 0$. We can define*

$$h(x) = \begin{cases} x^{-\frac{3}{2}} k(x) & \text{for } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is in turn continuous on $\mathbb{R}$ and non-zero on $(0, 1]$. Suppose h satisfies the remaining conditions of Theorem 4.3 for some $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$, where $0 < \varepsilon < 1$, i.e., $h(T)$ is trace class for all $T \in \mathcal{J}_{sa}$ and $\alpha_F: X \mapsto \tau(Xh(1 - F^2))$ is an exact one-form on $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. Then

$$\operatorname{sf}(\{D_t\}) = \frac{1}{\widetilde{C}} \int_0^1 \tau\Big(\frac{d}{dt}(D_t)k((1 + D_t^2)^{-1})\Big)\, dt + \beta(D_1) - \beta(D_0),$$

where $\widetilde{C} = \int_{-1}^1 h(1 - s^2)\, ds$ and $\beta(D) = \gamma(D(1 + D^2)^{-\frac{1}{2}})$ (with γ as defined in Theorem 4.3). Note that if k is defined on $\mathbb{R}_+ \setminus \{0\}$ instead, we can replace x by $|x|$ in the definition of h in order to define h on all of $\mathbb{R}$. The rest of the conditions remain unchanged.

Proof Lemma 4.5 tells us that $\{F_t\}$ is C^1 in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. By hypothesis, h satisfies all the requirements of Theorem 4.3, which gives us the formula

$$\operatorname{sf}(\{F_t\}) = \frac{1}{C} \int_0^1 \tau\Big(\frac{d}{dt}(F_t)h(1 - F_t^2)\Big)\, dt + \gamma(F_1) - \gamma(F_0),$$

where $C = \int_{-1}^1 h(1 - s^2)\, ds$ and $\gamma(F)$ is the integral of the one-form α on the straight-line path from F to $\widetilde{F}$.

Since $(1 + D_t^2)^{-1}$ is a positive operator and for positive values of x we have $h(x) = x^{-\frac{3}{2}} k(x)$, Lemma 4.6 gives us that

$$\begin{aligned}
\tau\Big(\frac{d}{dt}(F_t)h(1 - F_t^2)\Big) &= \tau\Big(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{3}{2}} h(1 - F_t^2)\Big) \\
&= \tau\Big(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{3}{2}} h((1 + D_t^2)^{-1})\Big) \\
&= \tau\Big(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{3}{2}} (1 + D_t^2)^{\frac{3}{2}} k((1 + D_t^2)^{-1})\Big) \\
&= \tau\Big(\frac{d}{dt}(D_t)k((1 + D_t^2)^{-1})\Big).
\end{aligned}$$

Since $(1+D_t^2)^{\frac{3}{2}}$ is unbounded, we stop for a second to worry about the above calculation; however, $h((1+D_t^2)^{-1}) = (1+D_t^2)^{\frac{3}{2}}k((1+D_t^2)^{-1})$ is a bounded operator, so the domain issues we might have expected do not materialize.

Finally, $\mathrm{sf}(\{D_t\}) = \mathrm{sf}(\{F_t\})$, so we can conclude

$$\mathrm{sf}(\{D_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\Big)\,dt \\ + \gamma(D_1(1+D_1^2)^{-\frac{1}{2}}) - \gamma(D_0(1+D_0^2)^{-\frac{1}{2}}),$$

as desired. Note that $\widetilde{C} = \int_{-1}^1 h(1-s^2)\,ds = \int_{-1}^1 (1-s^2)^{-\frac{3}{2}}k(1-s^2)\,ds$ and $\beta(D) = \gamma(D(1+D^2)^{-\frac{1}{2}})$. ∎

Remark 4.8 Note that, if D_1 is unitarily equivalent to D_0, then $D_1(1+D_1^2)^{-\frac{1}{2}}$ and $D_0(1+D_0^2)^{-\frac{1}{2}}$ are likewise unitarily equivalent; so the correction terms cancel, as observed in Remark 4.4. In this case, the spectral flow formula simplifies to

$$\mathrm{sf}(\{D_t\}) = \frac{1}{C}\int_0^1 \tau\Big(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\Big)\,dt.$$

On the other hand, for some choices of k it might be possible, following the same steps as in [CP04], to re-write the correction terms $\gamma(D)$ in terms of a path dependent on D, *i.e.*, $\gamma(D) = \frac{1}{2}\int_1^\infty t^{-1/2}\tau(Dk((1+tD^2)^{-1})\,dt - \frac{1}{2}[\ker(D)]$ (where $[\ker(D)]$ is the projection onto the kernel of D). As the conditions on k would get onerous, and the proof is reasonably involved, we simply outline the main steps that would need to be taken (see [CP04, §8]). The current correction terms are calculated by letting $F = D(1+D^2)^{-1/2}$ and integrating the bounded one-form along the straight-line path from $\mathrm{sign}(F)$ to F. Instead (using the fact that the bounded one-form is exact), one would consider the operators $F_s = D(s+D^2)^{-1/2}$ defined for $s > 0$, and a new path obtained by concatenating the straight line path from $\mathrm{sign}(F)$ to F_δ for some small $\delta > 0$ with the path $\{F_s\}_{s\in[\delta,1]}$. By taking the limit as $\delta \to 0$, the integral along the first path goes to zero if D is invertible, similar to the proof of [CP04, Lemma 8.9]. This would require the extra condition that h is increasing, which would be needed in order to apply [FK86, Lemma 2.5] and obtain that, for each t,

$$\mu_t(h(1-F_\delta^2)) = h(\mu_t(1-F_\delta^2)) \to 0.$$

In order to deal with the integral over the second path, the calculation of $\frac{d}{ds}F_s$ should carry over [CP04, Proposition 8.6], and the change of variables $s = \frac{1}{t}$ would give the final formula. One would also need to ensure that

$$\int_1^\infty t^{-1/2}\tau(Dk((1+tD^2)^{-1}))\,dt$$

onverges [CP04, Lemma 8.3, Corollary 8.4].

5 Analytic Continuation

We briefly outline the idea of the analytic continuation approach, as we do not try to address it in full generality, and it might apply in other situations not covered by this

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section. For a specific path $\{D_t\}$, usually with unitarily equivalent endpoints, we have a family of formulas for spectral flow, say of the form

$$\operatorname{sf}(\{D_t\}) = \frac{1}{C(m)} \int_0^1 \tau\Big(\frac{d}{dt}(D_t) g(D_t)^m \Big)\, dt.$$

We know that the equality holds for all $m \geq N$ for some number N, but we also know that we can calculate the integral on the right-hand side for values smaller than N. Let q_0 be the infimum of all values m for which $\tau\big(\frac{d}{dt}(D_t) g(D_t)^m \big)$ is finite (and note that usually $\tau\big(\frac{d}{dt}(D_t) g(D_t)^{q_0} \big)$ is not finite). We want to conclude that the spectral flow equality continues to hold for the numbers between q_0 and N. Rearranging the spectral flow formula, we get

$$C(m) \operatorname{sf}(\{D_t\}) = \int_0^1 \tau\Big(\frac{d}{dt}(D_t) g(D_t)^m \Big)\, dt.$$

The two sides of the equality can be thought of as functions in m, where m is a real number; we would like to show that the two functions make sense if instead m is in some subset of the complex numbers, and use properties of complex functions to show that the spectral flow formula holds for all values of m for which the right-hand side integral makes sense. For this, we need $z \mapsto C(z)$ and $z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t) g(D_t)^z)\, dt$ to be analytic on the set $\{z \in \mathbb{C} : \operatorname{Re}(z) > q_0\}$. In this section, we address how one might go about proving this for the latter function. In Section 6, we apply the results from this section to the case when D_0 is p-summable, $g = (1 + x^2)^{-1}$, and m is much larger than $\frac{p}{2}$.

Remark 5.1 One can easily replace $\frac{d}{dt}(D_t)$ by $\frac{d}{dt}(D_t) f(D_t)$ where $f(D_t)$ is bounded for all t and $t \mapsto f(D_t)$ is continuous. It should be clear that the proofs of Lemma 5.2 and Lemma 5.5 go through the same way (with $B_t = \frac{d}{dt}(D_t) f(D_t)$ instead), allowing us to prove the result for a more complicated integrand. To generalize this scenario even further, one can replace $g(D_t)^m$ by a function $f(m, D_t)$. Theorem 5.3 provides a glimmer of how to proceed in that direction; however, such a level of abstraction seemed unnecessary for our situation.

In the beginning, we need only assume that D_0 is an unbounded self-adjoint operator, though, of course, additional properties will be added when we talk about spectral flow. Let $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ be a fixed C^1 path. We wish to consider the complex function $\varphi\colon z \mapsto \int_0^1 \tau\big(\frac{d}{dt}(D_t) g(D_t)^z \big)\, dt$. Sufficient restrictions on g to ensure that φ is well defined and analytic are covered in Lemma 5.2 and Lemma 5.5, respectively. In the following lemma we also establish the domain of φ; as the proof follows quite naturally from the conditions imposed on g, we omit it.

Lemma 5.2 *Let $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ be a fixed C^1 path. Suppose g is a bounded continuous function $\mathbb{R} \to \mathbb{R}_+$ (ensuring that $g(D_t)$ is a bounded operator for all $t \in [0,1]$) for which there exists a real number $q_0 > 0$, such that for any $m > q_0$, we have $g(D_t)^m \in \mathcal{L}^1$, and $t \mapsto g(D_t)^m$ is continuous in $\|\cdot\|_1$-norm. Let $\{B_t\} \subset \mathcal{N}_{sa}$ be any norm-continuous path, such as $B_t = \frac{d}{dt}(D_t)$. Then for any fixed $z_0 \in \mathbb{C}$ with $\operatorname{Re}(z_0) > q_0$, the function $\varphi\colon z \mapsto \int_0^1 \tau(B_t g(D_t)^z)\, dt$ is defined at z_0.*

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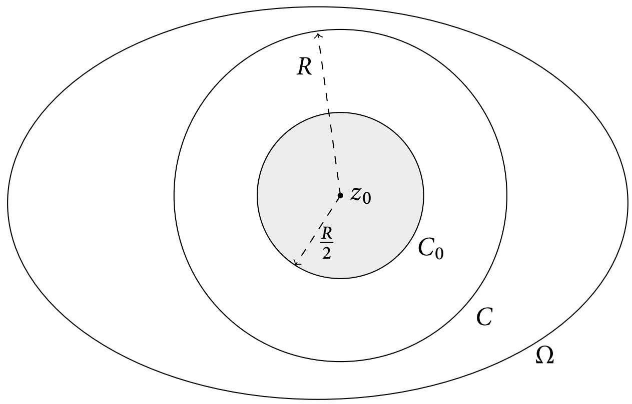

Figure 2: Setup for the proof of Lemma 5.3.

We would next like to show that φ is analytic. In the proof of this fact we will need the derivative of the function $z \mapsto T^z$ for fixed T appropriately chosen. In Corollary 5.4, we use Taylor's Theorem to show that for T positive and of norm at most 1, the difference quotient at z_0 converges to $T^{z_0}\log(T)$ at a rate which is independent of T.

Lemma 5.3 *Consider $S \subset \mathbb{R}$ and an open set $\Omega \subset \mathbb{C}$. Suppose $f: S \times \Omega \to \mathbb{C}$ is such that*

(i) *for each fixed $t_0 \in S$, $z \mapsto f(t_0, z)$ is analytic in Ω, in which case let $l(t,z) = \frac{\partial}{\partial z} f$ for $t, z \in S \times \Omega$;*
(ii) *for each fixed $z_0 \in \Omega$, $t \mapsto f(t, z_0)$ is continuous on S;*
(iii) *for each fixed $z_0 \in \Omega$, $t \mapsto l(t, z_0)$ is continuous on S;*
(iv) *$\{f(t,z) : t \in S, z \in \Omega\}$ is a bounded subset of $\mathbb{C}$.*

Finally, suppose that T is a self-adjoint operator with $\sigma(T) \subset S$. Then, for each fixed $z_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ (depending on z_0, but not T) such that $|z - z_0| < \delta$, $z \neq z_0$ implies $z \in \Omega$, and $\left\| \frac{f(T,z) - f(T,z_0)}{z - z_0} - l(T, z_0) \right\| < \varepsilon$.

Proof Given z_0 in the open set Ω, we can find a circle C around z_0 of some radius R such that C and its interior are contained in Ω. Let C_0 be the circle around z_0 of radius $\frac{R}{2}$, see Figure 2. Fix $t_0 \in S$, and let $h(z) = f(t_0, z)$. Then $h(z)$ is analytic in Ω (hypothesis (i)) and by Taylor's Theorem for analytic functions ([Ahl79, §3.1, Theorem 8], applied with $n = 2$ and $a = z_0$), $h(z) = h(z_0) + h'(z_0)(z - z_0) + h_2(z)(z - z_0)^2$, where $h_2(z) = \frac{1}{2\pi i}\int_C h(w)(w - z_0)^{-2}(w - z)^{-1}\,dw$ for z in the interior of C. If $z \neq z_0$, we can rearrange this equality to

$$\frac{h(z) - h(z_0)}{z - z_0} - h'(z_0) = h_2(z)(z - z_0).$$

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However,

$$|h_2(z)| = \Big|\frac{1}{2\pi i}\int_C \frac{h(w)}{(w-z_0)^2(w-z)}\,dw\Big| \le \Big|\frac{1}{2\pi i}\Big| \max_{w\in C}\Big|\frac{h(w)}{(w-z_0)^2(w-z)}\Big| \int_C |dw|.$$

Using $|w-z_0| = R$ (since w is on the circle C) and the fact that $\int_C |dw| = 2\pi R$, we can continue from the last inequality proven to

$$|h_2(z)| \le \frac{1}{2\pi}\frac{1}{R^2}\frac{\max_{w\in C}|h(w)|}{\min_{w\in C}|w-z|}(2\pi R) = \frac{1}{R}\frac{\max_{w\in C}|h(w)|}{\min_{w\in C}|w-z|}.$$

Going back to the notation used in the statement of the theorem, $h(z) = f(t_0, z)$ and $h'(z_0) = l(t_0, z_0)$; so we have shown that, for all z in the interior of C,

$$\frac{f(t_0,z) - f(t_0,z_0)}{z-z_0} - l(t_0,z_0) = h_2(z)(z-z_0),$$

where $|h_2(z)| \le \frac{1}{R}\max_{w\in C}|f(t_0,w)|/\min_{w\in C}|w-z|$.

By (iv), $\{f(t,z) : t \in S, z \in \Omega\}$ is bounded, say $|f(t,z)| < M$ for all $t \in S$ and $z \in \Omega$. If in addition we consider z in the interior of C_0, so that $|z-z_0| < \frac{R}{2}$, then the reverse triangle inequality gives us $|w-z| \ge \frac{R}{2}$ for w on the circle C, which implies $|h_2(z)| \le \frac{1}{R}\frac{M}{R/2} = \frac{2M}{R^2}$. Hence, for any z in the interior of C_0 with $z \ne z_0$,

$$\Big|\frac{f(t_0,z) - f(t_0,z_0)}{z-z_0} - l(t_0,z_0)\Big| = |h_2(z)||z-z_0| \le \frac{2M}{R^2}|z-z_0|.$$

So, given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta < \min\{\frac{R^2}{2M}, \frac{R}{2}\}\min\{\varepsilon, 1\}$; note that the ball centered at z_0 of radius δ is contained in Ω. If $|z-z_0| < \delta$ with $z \ne z_0$, then z is in the interior of C_0 and for all $t \in S$ we have

$$\Big|\frac{f(t,z) - f(t,z_0)}{z-z_0} - l(t,z_0)\Big| \le \frac{2M}{R^2}|z-z_0| < \frac{2M}{R^2}\delta < \varepsilon.$$

Finally, since $\sigma(T) \subset S$, the functional calculus gives us that, if $|z-z_0| < \delta$ with $z \ne z_0$, then $\|\frac{f(T,z)-f(T,z_0)}{z-z_0} - l(T,z_0)\| < \varepsilon$. Note in particular that δ depends on z_0 (for the choice of R) and the bound M on f, but does not depend on T. ■

Applying the above result with $S = [0,1]$, $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and

$$f(t,z) = \begin{cases} t^z & \text{if } t \ne 0, \\ 0 & \text{if } t = 0, \end{cases}$$

we get immediately the following.

Corollary 5.4 *For T a positive operator with $\|T\| \le 1$ and $z_0 \in \mathbb{C}$ with $\mathrm{Re}(z_0) > 0$, for any $\varepsilon > 0$ there exists a δ (dependent on z_0, but not on T) such that if $|z-z_0| < \delta$ (but $z \ne z_0$), then $\|\frac{T^z - T^{z_0}}{z-z_0} - T^{z_0}\log T\| < \varepsilon$.*

Lemma 5.5 *Let $\{D_t\} \subset D_0 + \mathcal{N}_{\mathrm{sa}}$ be a fixed C^1-path. Suppose g is a continuous function $\mathbb{R} \to \mathbb{R}_+$ such that $\|g(D_t)\| \le 1$ for all $t \in [0,1]$. Moreover, suppose that there exists a real number $q_0 > 0$ such that $g(D_t)^m \in \mathcal{L}^1$ for all $m > q_0$, and $t \mapsto g(D_t)^m$ is continuous in trace norm for $m > q_0$. Then the function $\varphi: z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t)g(D_t)^z)\,dt$ is analytic in the open half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > q_0\}$.*

Proof From Lemma 5.2, we know that φ is defined in the half-plane $\{\mathrm{Re}(z) > q_0\}$, so we need to show that $\frac{d}{dz}\int_0^1 \tau\big(\frac{d}{dt}(D_t)g(D_t)^z\big)\,dt$ exists at all z_0 with $\mathrm{Re}(z_0) > q$. Write $B_t = \frac{d}{dt}(D_t)$. Note that

$$\frac{d}{dz}\Big|_{z=z_0}\int_0^1 \tau(B_t g(D_t)^z)\,dt$$
$$= \lim_{z\to z_0} \frac{\int_0^1 \tau(B_t g(D_t)^z)\,dt - \int_0^1 \tau(B_t g(D_t)^{z_0})\,dt}{z-z_0}$$
$$= \lim_{z\to z_0}\int_0^1 (z-z_0)^{-1}(\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0}))\,dt.$$

We claim that

$$\lim_{z\to z_0}\int_0^1 (z-z_0)^{-1}(\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0}))\,dt = \int_0^1 \tau(B_t g(D_t)^{z_0}\log(g(D_t)))\,dt.$$

Since the interval over which we are integrating has finite measure, it is sufficient to show that the difference quotient converges uniformly to its limit; that is, given $\varepsilon > 0$, find a $\delta > 0$ such that $|z - z_0| < \delta$ implies that

$$\Big|\frac{\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0})}{z-z_0} - \tau(B_t g(D_t)^{z_0}\log(g(D_t))\Big| < \varepsilon \quad \text{for all } t\in[0,1].$$

To this end, use Corollary 5.4. Fix t; by assumption, $g(D_t)$ is positive and has norm less than or equal to 1. Also fix $q\in\mathbb{R}$ such that $q_0 < q < \mathrm{Re}(z_0)$. Then

$$\Big|\frac{\tau(B_t g(D_t)^z - \tau(B_t g(D_t)^{z_0})}{z-z_0} - \tau(B_t g(D_t)^{z_0}\log(g(D_t)))\Big|$$
$$= \Big|\tau\Big(B_t\frac{(g(D_t)^z - g(D_t)^{z_0})}{z-z_0} - B_t g(D_t)^{z_0}\log(g(D_t))\Big)\Big|$$
$$= \Big|\tau(B_t g(D_t)^q\Big[\frac{g(D_t)^{z-q} - g(D_t)^{z_0-q}}{z-z_0} - g(D_t)^{z_0-q}\log(g(D_t))\Big]\Big|$$
$$\le \|B_t\|\,\|g(D_t)^q\|_1\Big\|\frac{g(D_t)^{z-q} - g(D_t)^{z_0-q}}{z-z_0} - g(D_t)^{z_0-q}\log(g(D_t))\Big\|.$$

Use Corollary 5.4 with $z_0 - q$ instead of z_0 (recall that $\mathrm{Re}(z_0) > q$ by choice of q) to conclude that there exists $\delta > 0$ (independent of t) such that if

$$|z-z_0| = |(z-q)-(z_0-q)| < \delta,$$

then

$$\Big\|\frac{g(D_t)^{z-q} - g(D_t)^{z_0-q}}{z-z_0} - g(D_t)^{z_0-q}\log(g(D_t))\Big\| < \varepsilon.$$

Since $t\mapsto B_t$ is operator-norm continuous (recall that $\{D_t\}$ is a C^1-path) and $t\mapsto g(D_t)^q$ is trace-norm continuous, we know that $\|B_t\|$ and $\|g(D_t)^q\|_1$ are both uniformly bounded; hence, the difference quotient of the function $z\mapsto\tau(B_t g(D_t)^z)$

at z_0 converges uniformly for $t \in [0,1]$ to $\tau(B_t g(D_t)^{z_0} \log(g(D_t)))$. Therefore,

$$\frac{d}{dz}\int_0^1 \tau(B_t g(D_t)^z)\,dt = \int_0^1 \tau(B_t g(D_t)^z \log(g(D_t)))\,dt.$$

This concludes the proof that $\varphi(z)$ is analytic. ∎

Finally, we use these results to extend a family of spectral flow formulas by analytic continuation.

Theorem 5.6 *Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$ for which the equality*

$$C(m)\,\mathrm{sf}(\{D_t\}) = \int_0^1 \tau\Big(\frac{d}{dt}(D_t) g(D_t)^m\Big)\,dt$$

holds for all $m \geq N$, where $N \in \mathbb{R}_+$ is fixed. Let $q_0 = \inf\{m : \tau(g(D_t)^m) < \infty\}$. Suppose that $C(m)$ extends to a complex function $C(z)$ that is analytic in the half-plane $\{\mathrm{Re}(z) > q_0\}$. Suppose further that $g\colon \mathbb{R} \to \mathbb{R}_+$ is continuous, that $\|g(D_t)\| \leq 1$ for all t, and that $t \mapsto g(D_t)^m$ is continuous in trace norm for all $m > q_0$. Then for any m between q_0 and N (not including q_0) we also have

$$C(m)\,\mathrm{sf}(\{D_t\}) = \int_0^1 \tau\Big(\frac{d}{dt}(D_t) g(D_t)^m\Big)\,dt.$$

Proof By Lemma 5.5, $z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t) g(D_t)^z)\,dt$ is analytic in the half-plane $\{\mathrm{Re}(z) > q_0\}$. The desired result follows immediately since the complex functions $z \mapsto C(z)\,\mathrm{sf}(\{D_t\})$ and $z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t) g(D_t)^z)\,dt$ are both analytic on the open connected set $\{\mathrm{Re}(z) > q_0\}$ (see Lemma 5.5) and agree on the set $\{\mathrm{Re}(z) \geq N\}$, so they must be equal on the whole half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > q_0\}$. It follows that the desired formula holds for any $m > q_0$. ∎

6 Proof of the Integral Formula for p-summable Unbounded Operators

Our final goal (namely, the proof of Theorem 1.1) is to show that, if $(\mathcal{N}, D_0)$ is a p-summable unbounded Breuer–Fredholm module, then we can use the integral

$$\int \tau\Big(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{p}{2}}\Big)\,dt$$

(appropriately modified) to calculate the spectral flow of a suitable path $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$. We follow the program set out in Section 2: for each formula we want to reframe the problem in terms of bounded operators and show that one can calculate the spectral flow in the bounded setting. At various points in the proof we will need extra maneouvering room, so we replace $\frac{p}{2}$ by an m sufficiently larger than $\frac{p}{2}$, and use analytic continuation to eventually get the result with $\frac{p}{2}$.

The formula for the corresponding bounded operator is

$$\int \tau\Big(\frac{d}{dt}(F_t)|1 - F_t^2|^q\Big)\,dt.$$

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To start with, we prove that our one-forms are closed. Then we go through the process of modifying them, so they calculate spectral flow. And, finally, we use analytic continuation to prove Theorem 1.1.

6.1 Bounded Case Manifold and One-form

As this is the bounded setting we arrive at when D_0 satisfies $(1+D_0^2)^{-1} \in \mathcal{L}^{\frac{p}{2}}$, we need to consider ideals $\mathcal{J} = (\mathcal{L}^{\frac{p}{2}})^{1-\varepsilon}$ for some $0 < \varepsilon < 1$, and the corresponding manifolds $F_0 + \mathcal{S}\mathcal{J}_{F_0}$, where $D_0(1+D_0^2)^{-1/2}$ will determine our choice of F_0. As $\mathcal{J}$ is also an ideal of finitely summable operators, we suppose $\mathcal{J} = \mathcal{L}^{\frac{p}{2}}$ and worry about the relation between the p used in this case and the p used in the unbounded case when it is time to put the various results together. At the beginning of Section 4 we described the Banach space $\mathcal{S}\mathcal{J}_{F_0}$ and its properties (the corresponding specific results about $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$ can also be found in [CP98]); we recall that $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0} = \{X \in \mathcal{L}^{p}_{sa} \mid 1-(F_0+X)^2 \in \mathcal{L}^{\frac{p}{2}}\}$, and the norm on $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$ is given by $\|X\|_{\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}} = \|X\|_p + \|F_0X + XF_0\|_{\frac{p}{2}}$.

We now tackle the task of defining a one-form on $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$. Choose $r \in \mathbb{R}_+$ such that $\frac{r}{2} - 3 \geq p$, and define the one-form $\alpha_F(X) = \frac{1}{C}\tau(X|1-F^2|^r)$ for $X \in \mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$ and C a constant. Let $g(F) = (1-F^2)^2$; then $|1-F^2|^r = g(F)^{r/2}$, and we can write the one-form as $\alpha_F(X) = \frac{1}{C}\tau(Xg(F)^{r/2})$. We need to check that g satisfies the hypotheses of Proposition 3.1 (with $q = \frac{r}{2}$) in order to conclude that α_F is a closed one-form on $F_0 + \mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$.

- For every F self-adjoint, $(1-F^2)^2$ is positive and bounded, so $g(F)$ is positive and bounded for all $F \in F_0 + \mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$.
- By definition of $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$, $1-F^2 \in \mathcal{L}^{\frac{p}{2}}$ for any $F \in F_0 + \mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$. For $t \geq \frac{q-3}{2}$, we have $t \geq \frac{p}{2}$, so it follows that $\tau(|1-F^2|^{2t}) < \infty$. Since $g(F)^t = |1-F^2|^{2t}$, the desired property is satisfied.
- Fix $F \in F_0 + \mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$ and $X \in \mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$. To show that $s \mapsto g(F+sX)^t$ is continuous from $\mathbb{R}$ to $\mathcal{L}^1$ for large enough t, show that g satisfies the alternate requirement set out in Lemma 3.6. Fix $t \geq \frac{q-3}{2}$ (which implies $t \geq \frac{p}{2}$). Straightforward calculation gives us

$$d_{g,s}(F,X) = \frac{1}{s}(g(F+sX) - g(F)) = -\frac{1}{2}\{\{X,F\} + sX^2, 2-(F+sX)^2 - F^2\},$$

where $\{\cdot,\cdot\}$ denotes the anti-commutator. We need to show that, for F and X fixed, $\{\|d_{g,s}(F,X)\|_t : s \in [-1,1]\}$ is bounded. Consider first $t = \frac{p}{2}$. From the definition of $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$ (which guarantees that $X \in \mathcal{L}^p$ and $1-(F+X)^2 \in \mathcal{L}^{\frac{p}{2}}$) it follows easily that X^2, $\{X,F\}$, $1-(F+sX)^2$, and $1-F^2$ are all in $\mathcal{L}^{\frac{p}{2}}$, so certainly $d_{g,s}(F,X) \in \mathcal{L}^{\frac{p}{2}}$ as well. Moreover, if we assume $s \in [-1,1]$, the triangle inequality yields an upper bound on $\|d_{g,s}(F,X)\|_{\frac{p}{2}}$ that depends on the norms of $1-F^2$, X^2, and $\{X,F\}$, but not on s. This gives us that $\{\|d_{g,s}(F,X)\|_{\frac{p}{2}} : s \in [-1,1]\}$ is uniformly bounded. Finally, since $\mathcal{L}^{\frac{p}{2}} \hookrightarrow \mathcal{L}^t$ for $t > \frac{p}{2}$, the same conclusion can be reached for $t > \frac{p}{2}$.

- Using our earlier calculation we get $d_{g,0} = \lim_{s\to 0} d_{g,s} = (-1)\{\{X,F\}, 1-F^2\}$, where the limit is calculated with respect to the operator norm. Hence $s \mapsto g(T+sX)$ is differentiable at 0.
- Expanding the expression obtained for $d_{g,0}$, we get

$$d_{g,0} = (-1)XF(1-F^2) + F(1-F^2)X(-1) + (-F)X(1-F^2) + (1-F^2)X(-F).$$

Recall that we need to be able to pair up terms of the form $g_1(F)Xg_2(F)$ with terms of the form $g_2(F)Xg_1(F)$; but it is clear we can achieve this by pairing up the first two terms and the last two terms. Moreover, $F \mapsto F$, $F \mapsto 1-F^2$, and $F \mapsto F(1-F^2)$ are easily seen to be continuous as functions from $\mathcal{S}\mathcal{L}^{\frac{p}{2}}{}_{F_0}$ to $\mathcal{N}_{\mathrm{sa}}$.

So, with g as above, as a consequence of Proposition 3.1 we get the following.

Corollary 6.1 *Let $\mathcal{J} = \mathcal{L}^{\frac{p}{2}}$, and F_0 a self-adjoint Breuer–Fredholm operator such that $1 - F_0^2 \in \mathcal{J}$. Consider $r \in \mathbb{R}_+$ such that $r \geq 2p + 6$, and let C be a non-zero constant. The one-form $\alpha_F(X) = \frac{1}{C}\tau(X|1-F^2|^r)$ is a closed one-form on $F_0 + \mathcal{S}\mathcal{J}_{F_0}$; hence, α is independent of the path over which it is integrated.*

6.2 Unbounded Case Manifold and One-form

Assume that $(\mathcal{N}, D_0)$ is a p-summable unbounded Breuer–Fredholm module, so $(1+D_0^2)^{-1} \in \mathcal{L}^{\frac{p}{2}}$. In this case, our manifold is $D_0 + \mathcal{N}_{sa}$. For any fixed $m \geq p+3$, consider the one-form $\alpha_D: X \mapsto \frac{1}{C}\tau(X(1+D^2)^{-m})$; we want to show this is closed and exact. To this end, we need to show that the function $g(D) = (1+D^2)^{-1}$ defined on $D_0 + \mathcal{N}_{\mathrm{sa}}$ satisfies the hypotheses of Proposition 3.1 (note that in this case we are using $q = m$, so by choice of m we get $\frac{q-3}{2} \geq \frac{p}{2}$):

- $(1+D^2)^{-1}$ is a positive operator of norm at most one for any self-adjoint unbounded operator D, so certainly $g(D)$ is positive and bounded for all $D \in D_0 + \mathcal{N}_{sa}$.
- By assumption, D_0 is an operator for which $\tau((1+D_0^2)^{-t}) < \infty$ for $t \geq \frac{q-3}{2}$. By [CP98, Corollary B.8], $\tau((1+D^2)^{-t}) < \infty$, for all $D \in D_0 + \mathcal{N}_{\mathrm{sa}}$
- Fix $D \in D_0 + \mathcal{N}_{\mathrm{sa}}$ and $A \in \mathcal{N}_{\mathrm{sa}}$. The continuity of $s \mapsto g(D+sA)^{-t}$ for $t \geq \frac{q-3}{2}$ at $s = 0$, as a map from $\mathcal{N}_{\mathrm{sa}}$ to $\mathcal{L}^1$, follows immediately from [CP98, Proposition B.11].
- Fix $D \in D_0 + \mathcal{N}_{\mathrm{sa}}$ and $A \in \mathcal{N}_{\mathrm{sa}}$, and start out by calculating the directional derivative $d_{g,s}(D,A)$. Using the resolvent formula [CP98, Lemma 2.9], we have

$$\begin{aligned} d_{g,s}(D,A) &= \frac{(1+(D+sA)^2)^{-1} - (1+D^2)^{-1}}{s} \\ &= -D(1+D^2)^{-1}A(1+(D+sA)^2)^{-1} \\ &\quad - (1+D^2)^{-1}A(D+sA)(1+(D+sA)^2)^{-1}. \end{aligned}$$

By Lemma A.1 (v), $(1+(D+sA)^2)^{-1} \to (1+D^2)^{-1}$ and

$$(D+sA)(1+(D+sA)^2)^{-1} \to D(1+D^2)^{-1}$$

in operator norm as $s \to 0$. This allows us to conclude

$$\begin{aligned} d_{g,0}(D,A) &= \lim_{s\to 0} d_{g,s}(D,A) \\ &= -D(1+D^2)^{-1}A(1+D^2)^{-1} - (1+D^2)^{-1}AD(1+D^2)^{-1}. \end{aligned}$$

• With $g_1(x) = -x(1+x^2)^{-1}$ and $g_2(x) = (1+x^2)^{-1}$, the derivative $d_{g,0}$ (calculated above) has the form $g_1(D)Ag_2(D) + g_2(D)Ag_1(D)$, as desired (the continuity of g_1 and g_2 are given by Lemma A.1, (v)).

Therefore, g satisfies the desired conditions and, by Proposition 3.1, α is a closed form. Again, we state the result for future reference.

Corollary 6.2 *Let $\mathcal{I} = \mathcal{L}^{\frac{p}{2}}$. Let D_0 be a self-adjoint unbounded operator such that $(1+D_0^2)^{-1} \in \mathcal{I}$, and fix $m \geq p+3$. For any constant C, $\alpha_D(X) = \frac{1}{C}\tau(X(1+D_2)^{-m})$ is a closed one-form on $D_0 + \mathcal{N}_{\mathrm{sa}}$. In this context, it means that α is also exact, so integrating α is independent of path.*

6.3 Spectral Flow Formula for p-summable Operators

If $(\mathcal{N}, D_0)$ is a p-summable unbounded Breuer–Fredholm module, then we have $(1+D_0^2)^{-1} \in \mathcal{L}^{\frac{p}{2}}$. Hence, we want to consider the invariant operator ideal $\mathcal{I} = \mathcal{L}^{\frac{p}{2}}$ with the function $k(x) = x^{q/2}$ for some q (the end goal is to have $q = p$, but our method of proof will require us to start with much larger values of q). For the corresponding bounded operator's picture, pick any $0 < \varepsilon < 1$ and let $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$ (so $\mathcal{J}$ is still an ideal of finitely summable operators); the function that we need to use to calculate spectral flow for paths in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$ is then $h(x) = |x|^{\frac{q-3}{2}}$ (recall that the connection between the functions h and k was explained in Section 2).

We obtain the following as an immediate consequence of Theorem 4.3 and Corollary 6.1. Note that here, in order for our proof that the one-form is closed to go through, we must ensure that the power used in the integral formula is large enough. This explains why the power is not optimal in this result.

Corollary 6.3 *Let $\mathcal{J} = \mathcal{L}^{q/2}$ for some $q > 0$. If $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}\mathcal{J}_{F_0}$ and $\frac{r}{2} \geq q+3$,*

$$\operatorname{sf}(\{F_t\}) = \frac{1}{C_{r/2}} \int_0^1 \tau\Big(\frac{d}{dt}(F_t)|1-F_t^2|^{\frac{r}{2}}\Big)\, dt + \gamma_r(F_1) - \gamma_r(F_0).$$

Here $C_{r/2} = \int_{-1}^1 (1-s^2)^{\frac{r}{2}}\, ds$ and $\gamma(F) = \frac{1}{C_{r/2}} \int_0^1 \tau\big(\frac{d}{dt}(G_t)|1-G_t^2|^{\frac{r}{2}}\big)\, dt$, where $\{G_t\}$ is the straight-line path from F to $\operatorname{sign}(F)$.

We can similarly use Theorem 4.7 to obtain an unbounded formula. If, additionally, the endpoints are unitarily equivalent, we can use analytic continuation to improve our formula, which we will proceed to do in the following.

Theorem 6.4 *Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{\mathrm{sa}}$. Define*

$$q_0 = \inf\{p \in \mathbb{R}_+ : D_0 \text{ is } p\text{-summable}\}.$$

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If there exists u a unitary operator such that $uD_0 - D_0u$ is bounded and $D_1 = uD_0u^*$, then, for any $p > q_0$,

$$\mathrm{sf}(\{D_t\}) = \frac{1}{\widetilde{C}_{\frac{p}{2}}} \int_0^1 \tau\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-\frac{p}{2}} \Big)\, dt.$$

The constant $\widetilde{C}_{p/2}$ is equal to $\int_{-\infty}^{\infty}(1+x^2)^{-\frac{p}{2}}\,dx$. Note that the formula works regardless of the path $\{D_t\}$ chosen from D_0 to D_1.

If the endpoints of $\{D_t\}$ are not unitarily equivalent, the formula is weaker. If $m > 2p + \frac{15}{2}$, then

$$\mathrm{sf}(\{D_t\}) = \frac{1}{\widetilde{C}_m} \int_0^1 \tau\Big(\frac{d}{dt}(D_t)(1+D_t)^{-m} \Big)\, dt + \beta(D_1) - \beta(D_0).$$

Here $\beta(D) = \frac{1}{C}\int_0^1 \tau\big(\frac{d}{dt}(G_t)(1-G_t^2)^{m-\frac{3}{2}} \big)\, dt$ with $\{G_t\}$ the straight line path from $F = D(1+D^2)^{-\frac{1}{2}}$ to $\mathrm{sign}(F)$.

Proof We will start by proving the formula when the endpoints are not necessarily unitarily equivalent, and use analytic continuation in the case when the β terms cancel to obtain the formula with the better choice of exponent. Denote by $\mathcal{I}$ the ideal $\mathcal{L}^{\frac{p}{2}}$. For our given $m > 2p + \frac{15}{2}$, choose a q such that $m > 2q + \frac{15}{2} > 2p + \frac{15}{2}$. Then $\frac{p}{q} < 1$, so we can let $\varepsilon = 1 - \frac{p}{q}$; we will need $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$, which is really $\mathcal{L}^{q/2}$.

Let $k(x) = x^m$ for $x \in \mathbb{R}_+$, which is continuous on $\mathbb{R}_+$ and non-zero on $(0,1]$, and let $h(x) = |x|^{m-\frac{3}{2}}$. Note that if $T \in \mathcal{J}_{sa}$, then $h(T) = |T|^{m-\frac{3}{2}} \in \mathcal{L}^1$, since

$$m - \frac{3}{2} > 2q + 6 > \frac{q}{2}.$$

On the other hand, since $m - \frac{3}{2} \geq 2q + 6$, by Corollary 6.1, $X \mapsto \tau(Xh(1-F^2))$ is a closed one-form on $F_0 + \mathcal{S}\mathcal{J}_{F_0}$. This gives us (as a consequence of Theorem 4.7) an integral formula with exponent m; namely,

$$\mathrm{sf}(\{D_t\}) = \frac{1}{\widetilde{C}_m} \int_0^1 \tau\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m} \Big)\, dt + \beta(D_1) - \beta(D_0),$$

where $\widetilde{C}_m = \int_{-1}^1 (1-t^2)^{m-\frac{3}{2}}\,dt$.

We are now interested in upgrading the exponent to $\frac{p}{2}$. Let us examine $\widetilde{C}_m$. The change of variables $t = x(1+x^2)^{-\frac{1}{2}}$ gives us $\widetilde{C}_m = \int_{-\infty}^{\infty}(1+x^2)^{-m}\,dx$, as stated in the theorem statement. With the further change of variables $u = \frac{1}{1+x^2}$, we get $\widetilde{C}_m = \int_0^1 u^{m-\frac{3}{2}}(1-u)^{-\frac{1}{2}}\,du$, which is now recognizable as an instance of the beta function [Rud53, Theorem 8.20]:

$$\widetilde{C}_m = B\Big(m - \frac{1}{2}, \frac{1}{2}\Big) = \frac{\Gamma(m-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(m)}$$

(note that $m - \frac{1}{2} > 0$). It is known [Rud53] that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Using properties of the Γ function [BN82, §8.2], the function $m \mapsto C_m$ can be extended to a complex function $z \mapsto C(z)$, *i.e.*, $C(z) = \frac{\Gamma(z-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(z)}$, which is in particular analytic for all z in the half-plane $\mathrm{Re}(z) > \frac{q_0}{2}$.

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Now suppose that the endpoints are unitarily equivalent. Then, the β correction terms cancel (see Remark 4.8). We want to use analytic continuation with $g(D_t) = (1+D_t^2)^{-1}$ and $N = 2p+8$. We have shown above that, for all $m > 2p + \frac{15}{2}$,

$$\mathrm{sf}(\{D_t\}) = \frac{1}{\widetilde{C}_m}\int_0^1 \tau\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\,dt,$$

so the formula holds for $m \geq N$. The function $g(x) = (1+x^2)^{-1}$ is continuous from $\mathbb{R}$ to $\mathbb{R}_+$, and $\|g(D_t)\| \leq 1$ for all t. The fact that $g(D_t)^m$ is trace norm continuous for any $m > \frac{q_0}{2}$ is exactly the content of [CP98, Proposition B.11]. Therefore, all the conditions of Theorem 5.6 are satisfied, and we can conclude that

$$\mathrm{sf}(\{D_t\}) = \frac{1}{\widetilde{C}_m}\int_0^1 \tau\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\,dt$$

for any $m > q_0$, and in particular for $m = \frac{p}{2}$. ■

The first part of the above theorem was our final goal for this article: the p-summable formula from Theorem 1.1, proved using the steps outlined in Section 2.

A Continuity and Bounds in a Small Power Invariant Ideal

The purpose of this appendix is to collect some of the more tedious details in the proof of the integral formula for $\frac{d}{dt}(B_{0,t})$ (Corollary A.2) and $\frac{d}{dt}(F_t)$ (Lemma 4.5). Namely, since we are dealing with integrals converging to other integrals, we must prove continuity and find bounds on the rate of convergence for the various expressions involved.

Most of the norm bounds that we use in the proof below are results from [CP98]. The bounds in the following lemma that do not have a reference quoted are easily provable from the Spectral Theorem.

Lemma A.1 *Let D be a self-adjoint unbounded operator.*

(i) *For* $0 < r \leq 1$ *(with* $r \in \mathbb{R}$*), and* $\lambda \in [0,\infty)$, $\|(1+D^2+\lambda)^{-r}\| < (1+\lambda)^{-r}$.

(ii) *For* $r \in \mathbb{R}$ *such that* $\frac{1}{2} < r \leq 1$*, and* $\lambda \in [0,\infty)$, $\|D(1+D^2+\lambda)^{-r}\| \leq (1+\lambda)^{\frac{1}{2}-r}$.

(iii) $\lambda \mapsto (1+D^2+\lambda)^{-1}$ *is a continuous function from* $[0,\infty)$ *into* $\mathcal{N}$ [CP98, Remark A.5].

(iv) $\lambda \mapsto D(1+D^2+\lambda)^{-1}$ *is a continuous function from* $[0,\infty)$ *into* $\mathcal{N}$[CP98, Remark A.5].

(v) *If* $D = D_0 + A$ *for some self-adjoint bounded operator A and* $\lambda \in [0,\infty)$ *is fixed, then* $\|(1+D^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}\| \leq (1+\lambda)^{-\frac{3}{2}}\|A\|$*, and* $\|D(1+D^2+\lambda)^{-1} - D_0(1+D_0^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1}\|A\|$. [CP98, Lemma A.6]

(vi) *For* $\lambda \geq 0$ *and* $0 \leq \sigma \leq \frac{1}{2}$, $\|(1+D^2)^{\sigma}(1+D^2+\lambda)^{-1}\| \leq (1+\lambda)^{\sigma-1}$ [CP98, Lemma 2.6].

(vii) *For* $\lambda \geq 0$ *and* $0 \leq \sigma \leq \frac{1}{2}$, $\|D(1+D^2)^{\sigma}(1+D^2+\lambda)^{-1}\| \leq (1+\lambda)^{\sigma-\frac{1}{2}}$ [CP98, Lemma 2.6].

Corollary A.2 ([CP98, Proposition 2.10]) *Suppose* D_0 *is an unbounded self-adjoint operator affiliated with a von Neumann algebra* $\mathcal{N}$*, and* $\{D_t = D_0 + A_t\}$ *is a* C^1 *path in* $D_0 + \mathcal{N}_{\mathrm{sa}}$*. Denote by* $\{F_t\}$ *the image of this path under the Riesz transform,* i.e., $F_t =$

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$D_t(1+D_t^2)^{-\frac{1}{2}}$ for each $t \in [0,1]$. Fix $\sigma \in (0, \frac{1}{2})$. We can write $F_t - F_0 = B_{0,t}(1+D_0^2)^{-\sigma}$, where $B_{0,t} \in \mathcal{N}$. Moreover, $\{B_{0,t}\}$ is differentiable at 0 in operator norm; writing A_0' for $\frac{d}{dt}\big|_{t=0} A_t$ we have

$$\frac{d}{dt}\Big|_{t=0}(B_{0,t}) = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\big[(1+\lambda)(1+D_0^2+\lambda)^{-1}A_0'(1+D_0^2)^\sigma(1+D_0^2+\lambda)^{-1} - D_0(1+D_0^2+\lambda)^{-1}A_0'(1+D_0^2)^\sigma D_0(1+D_0^2+\lambda)^{-1}\big]\,d\lambda.$$

Note We use $B_{0,t}$ to remind ourselves of the dependence of the formula on D_0. Indeed, using $F_t - F_s = B_{s,t}(1+D_s^2)^{-\sigma}$, we could get a similar formula for $\frac{d}{dt}B_{s,t}$. Note that we do not go to the extent of writing $A_{0,t}$ since $\frac{d}{dt}A_t$ is the only related expression appearing in the formula, and it is the same regardless of whether $A_t = D_t - D_0$ or $A_t = D_t - D_s$ for some other fixed $s \in [0,1]$.

Proof This is indirectly shown as part of the proof of [CP98, Proposition 2.10], but we collect some of the details here. Note that, for $0 \le \sigma < \frac{1}{2}$, the integral

$$\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)^{\sigma-1}\,d\lambda \le \int_0^1 \lambda^{-\frac{1}{2}}\,d\lambda + \int_1^\infty \lambda^{-(\frac{3}{2}-\sigma)}\,d\lambda$$

converges. We use the formula for $B_{0,t}$ [CP98, Lemma 2.7]:

$$B_{0,t} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big[(1+\lambda)(1+D^2+\lambda)^{-1}A(1+D_0^2)^\sigma(1+D_0^2+\lambda)^{-1} - D(1+D^2+\lambda)^{-1}A(1+D_0^2)^\sigma D_0(1+D_0^2+\lambda)^{-1}\Big]\,d\lambda,$$

with the integral converging in operator norm. Using the norm bounds from Lemma A.1 and the triangle inequality, it is straightforward to show that the integrand in the formula for $\frac{d}{dt}\big|_{t=0}B_{0,t}$ is continuous, that the integral converges, and finally that the integral for $\frac{B_{0,t}-B_{0,0}}{t}$ converges to the integral formula given above for $\frac{d}{dt}\big|_{t=0}B_{0,t}$. We only show part of the proof that $\lim_{t\to 0}\frac{B_{0,t}-B_{0,0}}{t} = \frac{d}{dt}\big|_{t=0}B_{0,t}$. Splitting up the formula into two terms, if we consider the first term without the $\lambda^{-\frac{1}{2}}(1+\lambda)$ factor, we have that

$$\begin{aligned}
&\Big\|(1+D_t^2+\lambda)^{-1}A_t(1+D_0^2)^\sigma(1+D_0^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}A_0'(1+D_0^2)^\sigma(1+D_0^2+\lambda)^{-1}\Big\| \\
&\le \|(1+D_t^2+\lambda)^{-1}A_t - (1+D_0^2+\lambda)^{-1}A_0'\|\,\|(1+D_0^2)^\sigma(1+D_0^2+\lambda)^{-1}\| \\
&\le \Big(\|(1+D_t^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}\|\,\|A_t\| + \|(1+D_0^2+\lambda)^{-1}\|\,\|A_t - A_0'\|\Big)(1+\lambda)^{\sigma-1} \\
&\le \Big((1+\lambda)^{-\frac{3}{2}}\|A_t\|\,\|A_t\| + (1+\lambda)^{-1}\|A_t - A_0'\|\Big)(1+\lambda)^{\sigma-1} \\
&\le (1+\lambda)^{-1}(1\|A_t\|^2 + \|A_t - A_0'\|)(1+\lambda)^{\sigma-1}
\end{aligned}$$

Multiplying by $\lambda^{-\frac{1}{2}}(1+\lambda)$, we get an upper bound of $\lambda^{-\frac{1}{2}}(1+\lambda)^{\sigma-1}$ multiplied by $(\|A_t\|^2 + \|A_t - A_0'\|)$, which is enough to prove the desired convergence for the first term (since $\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)^{\sigma-1}\,d\lambda$ converges). The remaining calculations proceed similarly, and we omit them. ∎

Recall that our ideals $\mathcal{I}$ are small power invariant ideals (Definition 1.15), and hence have the property that, if $A \in \mathcal{I}$ is a positive operator and $B \le A$, then $B \in \mathcal{I}$ and moreover $\|B\|_{\mathcal{I}} \le \|A\|_{\mathcal{I}}$. We will need this in order to conclude that $(1+D^2)^{-1}$ is in $\mathcal{I}$ whenever D is a bounded perturbation of D_0. To that end, we make note of the following result.

Lemma A.3 ([CP04, Lemma 6.1]) *For D_0 an unbounded self-adjoint operator and A bounded and self-adjoint, let $D = D_0 + A$. Then*

$$-(f(\|A\|)-1)(1+D_0^2)^{-1} \le (1+D^2)^{-1} - (1+D_0^2)^{-1} \le (f(\|A\|)-1)(1+D_0^2)^{-1},$$

where $f(a) = 1 + \frac{1}{2}(a^2 + a\sqrt{a^2+4})$.

In particular, since for A and B unbounded and self-adjoint operators with common domain $0 < c1 \le A \le B$ implies $0 \le B^{-1} \le A^{-1} \le \frac{1}{c}1$ on all of $\mathcal{H}$ [CP98, Lemma B.1], we easily get the following.

Corollary A.4 *Suppose that D_0 is an unbounded self-adjoint operator, and $\mathcal{I}$ is a small power invariant operator ideal for which $(1+D_0^2)^{-1} \in \mathcal{I}$. For any $0 \le \varepsilon < 1$, let $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$ and $\sigma = \frac{1}{2} - \frac{\varepsilon}{2}$. If A is any self-adjoint bounded operator and $D = D_0 + A$, then for any $\lambda \in \mathbb{R}_+$, $(1+D^2+\lambda)^{-\sigma} \in \mathcal{J}^{\frac{1}{2}}$. Moreover, $\|(1+D^2+\lambda)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}} \le \|(1+D^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}$.*

Lemma A.5 *Suppose $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ is continuous, with $(1+D_0^2)^{-1} \in \mathcal{I}$, and $\{E_t\}$ is a path of bounded operators. Fix $0 < \varepsilon < 1$; let $\mathcal{J} := \mathcal{I}^{1-\varepsilon}$ and $\sigma = \frac{1}{2} - \frac{\varepsilon}{2}$ (note that $0 < \sigma < \frac{1}{2}$).*

(i) *Fix $t \in [0,1]$. Then $\|(1+D_t^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \le (1+\lambda)^{-(1-\sigma)}\|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}$, and $\|D_t(1+D_t^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \le (1+\lambda)^{\sigma-\frac{1}{2}}\|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}$.*

(ii) *For fixed $s, t \in [0,1]$,*

$$\|(1+\lambda)(1+D_t^2+\lambda)^{-1}E_t(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \le (1+\lambda)^{-(1-\sigma)}\|E_t\|\|(1+D_s^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}},$$

$$\|D_t(1+D_t^2+\lambda)^{-1}E_tD_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \le (1+\lambda)^{-(1-\sigma)}\|E_t\|\|(1+D_s^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}.$$

(iii) *Fix $t \in [0,1]$. The function $\lambda \mapsto (1+D_t^2+\lambda)^{-1}$ is uniformly continuous as a function from $\mathbb{R}_+$ to $\mathcal{J}^{\frac{1}{2}}$.*

(iv) *Fix $t \in [0,1]$. The function $\lambda \mapsto D_t(1+D_t^2+\lambda)^{-1}$ is uniformly continuous as a function from $\mathbb{R}_+$ to $\mathcal{J}^{\frac{1}{2}}$.*

(v) *Fix $t \in [0,1]$. The functions from $\mathbb{R}_+$ to $\mathcal{J}^{\frac{1}{2}}$ given by*

$$\lambda \longmapsto (1+\lambda)(1+D_t^2+\lambda)^{-1}E_t(1+D_t^2+\lambda)^{-1},$$

and

$$\lambda \longmapsto D_t(1+D_t^2+\lambda)^{-1}E_tD_t(1+D_t^2+\lambda)^{-1}$$

are both continuous.

Proof The idea for most of the proofs will be to split up $(1+D_t^2+\lambda)^{-1}$ into the product of $(1+D_t^2+\lambda)^{-(1-\sigma)}$ and $(1+D_t^2+\lambda)^{-\sigma}$, the second of which is in $\mathcal{J}^{\frac{1}{2}}$ by Corollary A.4, and the first of which can be handled using operator norms (see Lemma A.1). By the above discussion, $(1+D_t^2+\lambda)^{-\sigma} \in \mathcal{J}^{\frac{1}{2}}$, so we can write

$$\|(1+D_t^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \le \|(1+D_t^2+\lambda)^{-(1-\sigma)}\| \|(1+D_t^2+\lambda)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}$$
$$\le (1+\lambda)^{-(1-\sigma)} \|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}},$$

and similarly

$$\|D_t(1+D_t^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \le \cdots \le (1+\lambda)^{\sigma-\frac{1}{2}} \|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}.$$

(ii)–(v) are shown in the same straightforward manner, so the proofs are omitted. ■

We now tackle the continuity of the same type of expressions as in the previous lemma, except from the point of view of continuity as functions in t rather than functions in λ. As we will be using these expressions to prove convergence of various integral expressions, it is not sufficient to prove continuity, but we must additionally get λ-dependent bounds for the differences.

Lemma A.6 *Suppose $\{D_t = D_0 + A_t\} \subset D_0 + \mathcal{N}_{\mathrm{sa}}$ is continuous with $(1+D_0^2)^{-1} \in \mathcal{I}$, and $\{E_t\}$ is a path of bounded operators. Fix $0 < \varepsilon < 1$; let $\mathcal{J} := \mathcal{I}^{1-\varepsilon}$ and $\sigma = \frac{1}{2} - \frac{\varepsilon}{2}$.*

(i) *There exists a $K \in \mathbb{R}$ such that $\|(1+D_t^2+\lambda)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}} \le K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}$, for all $t \in [0,1]$ and $\lambda \in \mathbb{R}_+$. Here K depends on $\max_t \|A_t\|$ and σ, but not on D_0.*

(ii) *The map from $[0,1]$ to $\mathcal{J}^{\frac{1}{2}}$ given by $t \mapsto (1+\lambda)(1+D_t^2+\lambda)^{-1}E_t(1+D_t^2+\lambda)^{-1}$ is continuous. In fact, for $t,s \in [0,1]$, we have*

$$\|(1+\lambda)(1+D_t^2+\lambda)^{-1}E_t(1+D_t^2+\lambda)^{-1} - (1+\lambda)(1+D_s^2+\lambda)^{-1}E_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}$$
$$\le K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}(1+\lambda)^{-(1-\sigma)}\nu_{s,t},$$

where K is the same constant as in (i), $\nu_{s,t}$ do not depend on λ, and $\nu_{s,t} \to 0$ as $t \to s$.

(iii) *The map from $[0,1]$ to $\mathcal{J}^{\frac{1}{2}}$ given by $t \mapsto D_t(1+D_t^2+\lambda)^{-1}E_tD_t(1+D_t^2+\lambda)^{-1}$ is continuous. In fact, for $t,s \in [0,1]$, we have*

$$\|D_t(1+D_t^2+\lambda)^{-1}E_tD_t(1+D_t^2+\lambda)^{-1} - D_s(1+D_s^2+\lambda)^{-1}E_sD_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}$$
$$\le K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}(1+\lambda)^{-(1-\sigma)}\nu_{s,t},$$

where K and $\nu_{s,t}$ are as in (ii).

Proof (i) By Corollary A.4, for any $\lambda \in \mathbb{R}_+$ we have

$$\|(1+D_t^2+\lambda)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}} \le \|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}.$$

So we just need to relate this latter norm to $\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}$. By Lemma A.3, we have $(1+D_t^2)^{-1} \le f(\|A_t\|)(1+D_0^2)^{-1}$ for each $t \in [0,1]$, where

$$f(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2}x\sqrt{x^2+4}.$$

Since $\{A_t\}$ is continuous on $[0,1]$, there exists an $M \geq 0$ such that $\|A_t\| \leq M$ for all t, and since f is increasing on $[0,M]$, it follows that $f(\|A_t\|) \leq f(M)$ for all t. So we have $(1+D_t^2)^{-1} \leq f(M)(1+D_0^2)^{-1}$. Since the function $x \mapsto x^r$ is operator monotone for $r \leq 1$, $(1+D_t^2)^{-\sigma} \leq f(M)^\sigma(1+D_0^2)^{-\sigma}$. From Theorem 1.13 it follows that

$$\|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}} \leq f(M)^\sigma \|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}.$$

Hence $K = f(M)^\sigma$ satisfies the requirements; in particular, note that the value of K depends on the path $\{D_t\}$ and on σ (and hence on ε), but does not depend on either λ or t.

(ii) From Lemma A.1 we will need the operator norm bounds

$$\|(1+D_t^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1}, \quad \text{and} \quad \|D_t(1+D_t^2+\lambda)^{-1}\| \leq (1+\lambda)^{-\frac{1}{2}}.$$

We also refer to Lemma A.1 (v) for the additional norm inequalities

$$\|(1+D_t^2+\lambda)^{-1} - (1+D_s^2+\lambda)^{-1}\| \leq (1+\lambda)^{-\frac{3}{2}}\|A_t - A_s\|,$$

and

$$\|D_t(1+D_t^2+\lambda)^{-1} - D_s(1+D_s^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1}\|A_t - A_s\|.$$

For $s,t \in [0,1]$, using these bounds, the triangle inequality, Lemma A.5 (i), and Lemma A.6 (i), we have

$$\begin{aligned}\|(1+\lambda)(1+D_t^2+\lambda)^{-1}E_t(1+D_t^2+\lambda)^{-1} - (1+\lambda)(1+D_s^2+\lambda)^{-1}E_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\ \leq (1+\lambda)^{-(1-\sigma)}K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\\ \times(\|A_s - A_t\|\|E_t\| + \|E_t - E_s\| + \|E_s\|\|A_s - A_t\|).\end{aligned}$$

Let $\nu_{s,t} = \|A_s - A_t\|\|E_t\| + \|E_t - E_s\| + \|E_s\|\|A_s - A_t\|$. Fix $s \in [0,1]$. Since $\{E_t\}_{t\in[0,1]}$ and $\{A_s - A_t\}_{t\in[0,1]}$ are both continuous,

$$\{\|E_t\| : t \in [0,1]\} \quad \text{and} \quad \{\|A_s - A_t\| : t \in [0,1]\}$$

are bounded sets; so $A_s - A_t \to 0$ and $E_t - E_s \to 0$ as $t \to s$ is sufficient to ensure $\nu_{s,t} \to 0$ as $t \to s$. Hence

$$\begin{aligned}\|(1+\lambda)(1+D_t^2+\lambda)^{-1}E_t(1+D_t^2+\lambda)^{-1} - (1+\lambda)(1+D_s^2+\lambda)^{-1}E_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\ \leq K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}(1+\lambda)^{-(1-\sigma)}\nu_{s,t},\end{aligned}$$

where $\nu_{s,t} \to 0$ as $s \to t$ and $\nu_{s,t}$ does not depend on λ.

(iii) Use the triangle inequality and the various norm estimates established. We must be careful how we break up the expressions involved, as we did not establish a bound that depends on λ for $\|D_t(1+D_t^2+\lambda)^{-1} - D_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}$. For $s,t \in [0,1]$

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we have

$$\begin{aligned}
&\|D_t(1+D_t^2+\lambda)^{-1}E_tD_t(1+D_t^2+\lambda)^{-1}-D_s(1+D_s^2+\lambda)^{-1}E_sD_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\quad\le \|D_t(1+D_t^2+\lambda)^{-1}-D_s(1+D_s^2+\lambda)^{-1}\|\|E_t\|\|D_t(1+D_t^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad+\|D_s(1+D_s^2+\lambda)^{-1}\|\|E_t-E_s\|\|D_t(1+D_t^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad+\|D_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\|E_s\|\|D_t(1+D_t^2+\lambda)^{-1}-D_s(1+D_s^2+\lambda)^{-1}\|\\
&\quad\le (1+\lambda)^{-1}\|A_s-A_t\|\|E_t\|(1+\lambda)^{\sigma-\frac{1}{2}}\|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad+(1+\lambda)^{-\frac{1}{2}}\|E_t-E_s\|(1+\lambda)^{\sigma-\frac{1}{2}}\|(1+D_t^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad+(1+\lambda)^{\sigma-\frac{1}{2}}\|(1+D_s^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\|E_s\|(1+\lambda)^{-1}\|A_s-A_t\|.
\end{aligned}$$

Use the bound in (i) and simplify, keeping in mind that $(1+\lambda)^{-\frac{1}{2}}\le 1$, to get

$$\begin{aligned}
&\|D_t(1+D_t^2+\lambda)^{-1}E_tD_t(1+D_t^2+\lambda)^{-1}-D_s(1+D_s^2+\lambda)^{-1}E_sD_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad\le (1+\lambda)^{-(1-\sigma)}K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad\qquad\times(\|A_s-A_t\|\|E_t\|+\|E_t-E_s\|+\|E_s\|\|A_s-A_t\|).
\end{aligned}$$

This is the same expression as obtained in (ii), giving us

$$\begin{aligned}
&\|D_t(1+D_t^2+\lambda)^{-1}E_tD_t(1+D_t^2+\lambda)^{-1}-D_s(1+D_s^2+\lambda)^{-1}E_sD_s(1+D_s^2+\lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}\\
&\qquad\le K\|(1+D_0^2)^{-\sigma}\|_{\mathcal{J}^{\frac{1}{2}}}(1+\lambda)^{-(1-\sigma)}\nu_{s,t},
\end{aligned}$$

as desired. ■

The above bounds and continuity results are all that is needed to fill out the details in the proof of Lemma 4.5.

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Department of Mathematics, Ben Gurion University, 8410501 Be'er Sheva, Israel
e-mail: magda@uvic.ca