

CENTRALIZER NEAR-RINGS OVER FREE RING MODULES

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Abstract

We treat centralizer near-rings over ring modules in general, with particular emphasis on the case of free modules. Questions like the following are answered. When is the near-ring a nonring? When is the near-ring simple? What are its maximal and minimal left ideals? What is its subgroup structure? What is the radical? The cases where the ring concerned is a PID or a field are treated in some detail.

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1. Introduction

Let $(G, +)$ be any group, and A a set of endomorphisms of G . The near-ring

$$M_A(G) := \{f : G \rightarrow G \mid f(g\alpha) = f(g)\alpha, \text{ for all } g \in G, \alpha \in A\}$$

has been the subject of investigation in numerous papers, as is evident from a near-ring bibliography as given, for example, in [4] or [5]. One particular case has received less attention than one might have expected, viz. the case where G is a right R -module, R a ring, and $A \subseteq R$. Apart from Maxson and Smith [2], where the case of a simple artinian ring is treated and [3] where finite semisimple rings are treated, there does not seem to exist a systematic study of the interplay between the properties of a ring R , a right R -module G and the near-ring $M_R(G)$. It is the object of this paper to lay some of the groundwork for such an investigation. In [2] it is shown that if R is simple artinian, not a field, then $M_R(G)$ is, in fact, the left endomorphism ring of

G_R . We therefore start by considering some general conditions for $M_R(G)$ to be a nonring (that is, not a ring). The rest of Section 2 is devoted to some general results for the case where G is a free module of finite rank, and, in fact, since it is easy to see that rank 2 shows all the salient features, we present our results only for the case $G = R^2$. In a number of our proofs we use the fact that $M_2(R)$, the ring of 2×2 matrices over R , is a subring of $M_R(R^2)$. Indeed, this fact accounts for much of the flavour of the results.

Section 3 treats the case $M_D(D^2)$, D a principal ideal domain, in some detail. In particular, we investigate the left ideal structure with regard to both minimal and maximal left ideals. In some of this work we use ultrafilters in much the same way that they were used by Fuchs [1]. Also, the invariant subgroups of $M_D(D^2)$ are determined. The section closes with a brief discussion of the case $M_F(F^2)$, where F is a field.

2. General results

Let R be a ring with identity. All modules will be unitary. For any subset S of a ring or a module, we let $S^* := S \setminus \{0\}$. In this section we collect several results concerning the structure of the near-ring $M_R(R^2)$. We start by presenting conditions which imply, in many very general situations, that $M_R(R^2)$ is a near-ring, but not a ring. In fact we give our conditions for any right R -module G .

LEMMA 2.1. *Let G be a right R -module and suppose $G = H \oplus K$, H and K R -submodules of G . Let $f \in M_R(G)$. If, for some non-zero $a \in H$, $b \in K$, $f(a+b) \neq f(a)+f(b)$ then f is not a distributive element of $M_R(G)$.*

PROOF. If e_H and e_K are the idempotents associated with the decomposition $G = H \oplus K$, then $e_H, e_K \in M_R(G)$ and $e_H + e_K = \text{id}$, the identity map in $M_R(G)$. Thus $f(e_H + e_K) = f$ whereas $(fe_H + fe_K)(a+b) = f(a) + f(b) \neq f(a+b)$ so $f(e_H + e_K) \neq fe_H + fe_K$.

THEOREM 2.2. *Let G be a right R -module with a non-trivial direct decomposition $G = H \oplus K$. If there exists a non-empty subset X of G such that*

- (i) $0 \notin X$, $X \neq G^*$,
- (ii) $x \in X$ implies $xr \in X \cup \{0\}$, for all $r \in R$,
- (iii) $xr \in X$, $r \in R$ imply $x \in X$,

then $M_R(G)$ is not a ring.

PROOF. Define $f : G \rightarrow G$ by

$$f(x) = \begin{cases} x, & x \in X, \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward, from the defining properties of X , to show that $f \in M_R(G)$. We next show that there exist $a \in H, b \in K$ such that $f(a+b) \neq f(a) + f(b)$. From the above lemma, f is not a distributive element of $M_R(G)$ so $M_R(G)$ is not a ring.

CASE (i). $H \cap X = \emptyset, K \cap X = \emptyset$. Then for $x \in X, x = a + b$, with $a \in H, b \in K$. From this we find $f(x) = x \neq 0 = f(a) + f(b)$.

CASE (ii). $H \cap X \neq \emptyset, K \cap X = \emptyset$. Let $a \in H \cap X$ and choose $0 \neq b \in K$. Then

$$f(a+b) = \begin{cases} a+b, & a+b \in X, \\ 0, & \text{otherwise,} \end{cases}$$

while $f(a) + f(b) = a$. Therefore $f(a+b) \neq f(a) + f(b)$.

CASE (iii). $H \cap X \neq \emptyset, K \cap X \neq \emptyset$. If $H \cup K \subseteq X \cup \{0\}$, then since $X \neq G^*$ there exists $y = a + b \in G^* \setminus X, a \in H, b \in K$. Hence $f(a+b) = 0 \neq a+b = f(a) + f(b)$. However, if (say) $H \not\subseteq X \cup \{0\}$, we take $a \in H^* \setminus X$ and $b \in K \cap X$. Then

$$f(a+b) = \begin{cases} a+b, & a+b \in X, \\ 0, & \text{otherwise,} \end{cases}$$

while $f(a) + f(b) = b$, which shows $f(a+b) \neq f(a) + f(b)$.

We next discuss a construction which will identify possible candidates for the role of X in the above theorem.

Define a relation \sim on G by $0 \sim 0$ and for $x, y \in G^*, x \sim y$ if there exist $c_i \in G^*, i = 1, 2, \dots, n$, with $x = c_1, y = c_n$ and $c_i R \cap c_{i+1} R \neq \{0\}, i = 1, 2, \dots, n-1$. Then \sim is an equivalence relation and the equivalence classes are called connected components or components of G . If G^* is a component, we say G is R -connected. We note that when G is not R -connected the non-zero components X satisfy the conditions (i), (ii), (iii) of Theorem 2.2. This gives the following result.

COROLLARY 2.3. *Let G be a right R -module with a non-trivial direct decomposition. If $M_R(G)$ is a ring then G is R -connected.*

The converse of this corollary is not true, as is shown by $R := \mathbb{Z}_6$ and $G := (\mathbb{Z}_6)^2$. One verifies that G is R -connected but as we see from the next result, $M_R(G)$ is not a ring. Recall that a proper ideal P in a ring R is called completely prime if $R \setminus P$ is a multiplicatively closed set.

THEOREM 2.4. *Let R be a ring with a non-zero proper completely prime ideal P such that $\text{Ann}_R(P) \neq \{0\}$. Then $M_R(R^2)$ is not a ring.*

PROOF. Let $d \in \text{Ann}_R(P)$, $d \neq 0$ and let $C := R \setminus P$, a multiplicatively closed set. Define $f : R^2 \rightarrow R^2$ by

$$f \begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \begin{bmatrix} da \\ 0 \end{bmatrix}, & \text{if } \begin{bmatrix} a \\ b \end{bmatrix} \in C^2, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

To see that $f \in M_R(R^2)$ observe that

- (i) $\begin{bmatrix} a \\ b \end{bmatrix} \in C^2$, $r \in C$ if and only if $\begin{bmatrix} ar \\ br \end{bmatrix} \in C^2$,
- (ii) $\begin{bmatrix} a \\ b \end{bmatrix} \in C^2$, $p \in P$ imply $\begin{bmatrix} dap \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} da \\ 0 \end{bmatrix} p$, and
- (iii) $\begin{bmatrix} a \\ b \end{bmatrix} \notin C^2$ implies $\begin{bmatrix} a \\ b \end{bmatrix} r \notin C^2$, for all $r \in R$.

Now $f \in M_R(R^2)$, and moreover,

$$f \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} d \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

while

$$f \begin{bmatrix} 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence from Lemma 2.1, f is not distributive so $M_R(R^2)$ is not a ring.

On the other hand, if D is an integral domain, not necessarily commutative, then $M_D(D^2)$ is never a ring. For in this situation, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} a \\ b \end{bmatrix}$ if and only if $a \neq 0$ and $b = 0$. Thus $X = \{\begin{bmatrix} a \\ 0 \end{bmatrix} | a \neq 0\}$ is a non-zero component and $X \neq D^2$.

COROLLARY 2.5. *For every $n > 1$, $M_{\mathbb{Z}_n}(\mathbb{Z}_n^2)$ is not a ring.*

PROOF. If n is a prime, \mathbb{Z}_n is a field and so the result follows from the above remarks. If $n = pt$, p a prime and $t > 1$, choose $P = \langle p \rangle$, the ideal generated by p and let $d = t \in \text{Ann}_{\mathbb{Z}_n}(\langle p \rangle)$. Now apply Theorem 2.4.

It should be pointed out, however, that rings do arise naturally as $M_R(G)$. In fact, as mentioned in the introduction, if R is any Artinian simple ring, not a field, and V any right R -module then $M_R(V) = \text{End}_R V$ is a ring. Further, if G is a cyclic R -module then $M_R(G)$ is a ring. As a specific

instance of this case let G be a finitely generated abelian group and let R denote the endomorphism ring of G . Then $M_R(G)$ is a ring. As a final example here, we let D be a commutative integral domain with field of quotients $Q(D)$. Then $M_D(Q(D))$ is a ring.

We next turn to a consideration of the structure of the near-ring $M_R(R^2)$. We first establish a very useful connection between this near-ring and the subring $M_2(R)$.

LEMMA 2.6. *Let $f \in M_R(R^2)$ and let A be a matrix with at most one non-zero column in $M_2(R)$, say*

$$A := \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \end{bmatrix}.$$

Then

$$f \circ A = \begin{bmatrix} 0 & c_1 \\ 0 & c_2 \end{bmatrix}, \quad \text{where } f \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

PROOF. let $\begin{bmatrix} x \\ y \end{bmatrix}$ be arbitrary in R^2 . We have

$$f \circ \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} b_1 y \\ b_2 y \end{bmatrix} = \left(f \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) y = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} y = \begin{bmatrix} c_1 y \\ c_2 y \end{bmatrix} = \begin{bmatrix} 0 & c_1 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and hence we have the result.

COROLLARY 2.7. *If C_k denotes the left ideal of $M_2(R)$ consisting of matrices which are zero except possibly in the k th column, then C_k is a (left) $M_R(R^2)$ -subgroup.*

THEOREM 2.8. *If R is a simple ring then $M_R(R^2)$ is a simple near-ring.*

PROOF. Let H be a non-zero ideal of $M_R(R^2)$ and let $f \in H$, $f \neq 0$. Thus there exists $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in R^2 such that $f \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore by Lemma 2.6,

$$f \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \in H \cap M_2(R).$$

But this means that $H \cap M_2(R)$ is a non-zero ideal in $M_2(R)$. Since R is simple, because of the correspondence between ideals in R and those in $M_2(R)$, we know $M_2(R)$ is simple. Hence $H \cap M_2(R) = M_2(R)$, and consequently $\text{id} \in H$ which means $H = M_R(R^2)$.

Actually, we have established a somewhat stronger result.

COROLLARY 2.9. *If R is a simple ring then $M_R(R^2)$ contains no non-trivial two-sided $M_R(R^2)$ -subgroups.*

We can also make use of Lemma 2.6 to show that the distributive elements in $M_R(R^2)$ are precisely the matrices in $M_R(R)$.

THEOREM 2.10. *The subnear-ring of $M_R(R^2)$ generated by the distributive elements of $M_R(R^2)$ is $M_2(R)$.*

PROOF. Let D denote the semigroup of distributive elements in $M_R(R^2)$. Then $\text{dg}D$ is the subnear-ring of $M_R(R^2)$ distributively generated by D . Clearly $M_2(R) \subseteq \text{dg}D$. For the reverse inclusion let $f \in D$. Then

$$\begin{aligned} f &= f \circ \text{id} = f \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = f \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= f \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(R). \end{aligned}$$

Thus $D \subseteq M_2(R)$ so $\text{dg}D = M_2(R)$.

We showed above that $M_R(R^2)$ is simple when R is simple. We investigate further when $M_R(R^2)$ is simple. Recall that a ring R is a prime ring if for each pair A, B of non-zero ideals of R , $AB \neq \{0\}$.

THEOREM 2.11. *If $M_R(R^2)$ is a simple near-ring then R is a prime ring.*

PROOF. We assume R is not prime and let A, B be non-zero ideals such that $AB = \{0\}$. Let $0 \neq b \in B$ and set $C := Rb \subseteq B$. We let $C^2 = \left\{ \begin{bmatrix} a \\ c \end{bmatrix} \in R^2 \mid a, c \in C \right\}$ and note that C^2 is an $M_R(R^2)$ -subgroup. In fact, if

$$\begin{bmatrix} r_1 b \\ r_2 b \end{bmatrix} \in C^2$$

then for each $f \in M_R(R^2)$,

$$f \begin{bmatrix} r_1 b \\ r_2 b \end{bmatrix} = f \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} b = \begin{bmatrix} s \\ t \end{bmatrix} b = \begin{bmatrix} sb \\ tb \end{bmatrix} \in C^2, \quad \text{where } \begin{bmatrix} s \\ t \end{bmatrix} := f \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

Therefore $K = (0 : C^2)$ is a proper ideal in $M_R(R^2)$. However $K \neq \{0\}$ because if $a \in A$, $a \neq 0$ then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in K$. This contradiction implies that R must be a prime ring.

As a partial converse we have the following.

THEOREM 2.12. *Let R be an integral domain, not necessarily commutative. Then $M_R(R^2)$ is a simple near-ring.*

PROOF. Suppose $M_R(R^2)$ has a non-trivial ideal J . Let $I := J \cap M_2(R)$. Since $J \neq \{0\}$, as in Lemma 2.6, if $0 \neq f \in J$, say $f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $f \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \in I$, so $I \neq \{0\}$. Thus, as is well known, there exists an ideal T of R such that $I = M_2(T)$. Let $0 \neq t \in T$. Then $t - 1 \notin T$, where 1 is the identity of R , otherwise $\text{id} \in I \subseteq J$, but $J \neq M_R(R^2)$. We shall show $\begin{bmatrix} t-1 & 0 \\ 0 & 0 \end{bmatrix} \in J$ and hence $\begin{bmatrix} t-1 & 0 \\ 0 & 0 \end{bmatrix} \in I$ which is a contradiction to $I = M_2(T)$ and so $M_R(R^2)$ must be simple. To this end, since $\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} \in J$, we must have $g := f(\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) - f \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in J$, for all $f \in M_R(R^2)$. We construct an f such that $g = \begin{bmatrix} t-1 & 0 \\ 0 & 0 \end{bmatrix}$. Since R is an integral domain, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} t \\ 1 \end{bmatrix}$ are in different connected components. (See the remark after Theorem 2.4.) Let $f : R^2 \rightarrow R^2$ be defined by

$$f \begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \begin{bmatrix} b \\ b \end{bmatrix}, & \text{if } a = 0, \\ \begin{bmatrix} a \\ b \end{bmatrix}, & \text{if } a \neq 0. \end{cases}$$

It is straightforward to verify that $f \in M_R(R^2)$. Using this f , for any $\begin{bmatrix} x \\ y \end{bmatrix} \in R^2$, we have

$$\begin{aligned} g \begin{bmatrix} x \\ y \end{bmatrix} &= \left(f \left(\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) - f \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} \\ &= f \left(\begin{bmatrix} tx \\ x \end{bmatrix} \right) - f \begin{bmatrix} 0 \\ x \end{bmatrix} = f \begin{bmatrix} t \\ 1 \end{bmatrix} x - \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix} x - \begin{bmatrix} 1 \\ 1 \end{bmatrix} x \\ &= \begin{bmatrix} t-1 \\ 0 \end{bmatrix} x = \begin{bmatrix} t-1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

Thus $g = \begin{bmatrix} t-1 & 0 \\ 0 & 0 \end{bmatrix}$, as required.

Recall that a commutative ring R is prime if and only if R is an integral domain. Thus from the above two theorems we have a nice characterization for $M_R(R^2)$ to be simple when R is commutative.

COROLLARY 2.13. *Let R be a commutative ring. Then $M_R(R^2)$ is a simple near-ring if and only if R is an integral domain.*

We next turn to the question of when $M_R(R^2)$ is 2-semisimple, that is, when $J_2(M_R(R^2)) = \{0\}$. For left artinian rings R we find a very tight connection between the structures of R and $M_R(R^2)$.

THEOREM 2.14. *Let R be a left artinian ring and let $N := M_R(R^2)$. Then $J_2(N) = \{0\}$ if and only if $J(R) = \{0\}$.*

PROOF. Suppose $J(R) \neq \{0\}$. Choose $0 \neq u \in J(R)$, let $U := Ru$ and let $A := \{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} | x, y \in U \}$. Clearly A is a subgroup of N . We show A is an N -subgroup. Let $f \in N$, $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \in A$. Then there exist $r_1, r_2 \in R$ such that $x = r_1u, y = r_2u$. If

$$f \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix},$$

then for any $\begin{bmatrix} a \\ b \end{bmatrix} \in R^2$,

$$f \begin{bmatrix} r_1ua \\ r_2ua \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} ua = \begin{bmatrix} cu & 0 \\ du & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Hence $f \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} = \begin{bmatrix} cu & 0 \\ du & 0 \end{bmatrix} \in A$. Since R is left artinian, $J(R)$ is nilpotent, say $(J(R))^n = \{0\}$. But then for $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \in A$, $x, y \in J(R)$ so $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore A is a nil N -subgroup of N and so $A \subseteq J_2(N)$. Hence $J_2(N) \neq \{0\}$.

Conversely let $J(R) = \{0\}$ and let L be any minimal left ideal of R . Thus, for all $c \in L, c \neq 0$, we have $Rc = L$. Consider $L^2 \subseteq R^2$. As above, if $\begin{bmatrix} x \\ y \end{bmatrix} \in L^2$ then for each $f \in N, f \begin{bmatrix} x \\ y \end{bmatrix} \in L^2$ so L^2 is an N -group. Further, since $Rc = L, L^2$ is monogenic by $\begin{bmatrix} c \\ 0 \end{bmatrix}$ (in fact using only elements from $M_2(R)$) and if $\{0\} \neq H$ is any N -subgroup of L^2 and $\begin{bmatrix} a \\ b \end{bmatrix} \in H, \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $N \begin{bmatrix} a \\ b \end{bmatrix} = L^2$. Thus L^2 is a type 2 N -group. We have shown that for any minimal left ideal L of R, L^2 is a type 2 N -group.

Assume $J_2(N) \neq \{0\}$. From Lemma 2.6 we get $J_2(N) \cap M_2(R) \neq \{0\}$. Thus $J_2(N) \cap M_2(R) = M_2(T)$, for some non-zero ideal T of R . Since R is left artinian, T contains a minimal left ideal of E , say L_0 . Thus, for each $a \in L_0, \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in M_2(T) \subseteq J_2(N)$. Let $b \in L_0$. Then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $M_2(T)$ annihilates every type 2 N -group, in particular L_0^2 . But this implies $L_0 \cdot L_0 = \{0\}$ which means that L_0 is a nilpotent left ideal of R and as such, $L_0 \subseteq J(R)$, contradicting the fact that $J(R) = \{0\}$. Thus we must have $J_2(N) = \{0\}$.

COROLLARY 2.15. *Let n be a positive integer, $n > 1$ and let $N := M_{\mathbb{Z}_n}(\mathbb{Z}_n^2)$. The following are equivalent:*

- (a) N is 2-semisimple;
- (b) $J(\mathbb{Z}_n) = \{0\}$;
- (c) n is square-free.

PROOF. As is well known, $J(\mathbb{Z}_n) = \{0\}$ if and only if n is square-free.

We have found several connections among the structures R , $M_R(R)$ and $M_R(R^2)$. In the next section we take R to be a principal ideal domain (PID) and determine various substructures of $M_R(R^2)$.

3. $M_D(D^2)$, D a PID

In this section we investigate the internal structure of the near-ring $M_D(D^2)$. We focus primarily on the case where D is a principal ideal domain (PID); however some of our results hold for more general integral domains.

To begin, we let D be an integral domain, not necessarily commutative. As above we consider only the case D^2 since the general situation, D^n , $n > 2$, is essentially the same. We say an element $\alpha \in D^2$ is a generator in D^2 if, for any $\beta \in D^2$, $\alpha r = \beta s \neq 0$ implies $\beta = \alpha t$ where $r, s, t \in D$. This is equivalent to $\alpha D \cap \beta D \neq \{0\}$ implies $\beta D \subseteq \alpha D$.

EXAMPLE 3.1. The following are generators in D^2 :

$$(1) \begin{bmatrix} d \\ 1 \end{bmatrix}; \quad (2) \begin{bmatrix} 1 \\ d \end{bmatrix}; \quad (3) \begin{bmatrix} d+1 \\ d \end{bmatrix}, \text{ for each } d \in D.$$

For, if $d = 0$ the result is clear. Let $d \neq 0$ and suppose $\begin{bmatrix} d \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} s \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $dr = xs$, $r = ys$ so $dys = xs$. But then $dy = x$ so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ 1 \end{bmatrix} y$. The situation for (2) is similar. For (3) suppose $\begin{bmatrix} d+1 \\ d \end{bmatrix} r = \begin{bmatrix} x \\ y \end{bmatrix} s$. Then $dr+r = xs$, $dr = ys$ and so $ys+r = xs$, that is, $r = (x-y)s$. Consequently, $(d+1)(x-t)s = xs$ and $d(x-y)s = ys$. Hence, $\begin{bmatrix} d+1 \\ d \end{bmatrix} (x-y) = \begin{bmatrix} x \\ y \end{bmatrix}$.

Let $S \subseteq D^2$. We say S is a set of generators in D^2 if each $\alpha \in S$ is a generator and for $\alpha, \beta \in S$, $\alpha D \subseteq \beta D$ implies $\alpha = \beta$. Using sets of generators, one can construct functions in $M_D(D^2)$. In fact, suppose S is a set of generators in D^2 , T is any subset of D^2 and $\varphi: S \rightarrow T$ is any function. Then it is straightforward to verify that the function $f: D^2 \rightarrow D^2$ defined by

$$f(\gamma) = \begin{cases} \varphi(\alpha)r, & \text{if } \gamma = \alpha r \text{ for some } \alpha \in S, r \in D, \\ 0, & \text{otherwise,} \end{cases}$$

is an element of $M_D(D^2)$. Thus functions in $M_D(D^2)$ can be defined by specifying their action on a set of generators. We shall use this frequently in the sequel.

Now let D be a unique factorization domain (UFD), let \sim be the equivalence relation defined after Theorem 2.2 and let E denote an equivalence class of \sim , $E \neq \{0\}$. We show that E contains a generator.

Let $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} a \\ b \end{bmatrix} \in E$ and let $d = \gcd(a, b)$ where $\gcd(x, 0) = x$. Let $da' = a$, $db' = b$. Since $\begin{bmatrix} a' \\ b' \end{bmatrix}d = \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a' \\ b' \end{bmatrix} \in E$. Suppose $\begin{bmatrix} a' \\ b' \end{bmatrix}r = \begin{bmatrix} x \\ y \end{bmatrix}s \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ where without loss of generality we take $\gcd(r, s)$ to be a unit. We have $a'r = xs$ and $b'r = ys$. If $\gcd(a', s) = d'$ is not a unit then from $b'r = ys$ we get $d'|b'$. But this is a contradiction since $\gcd(a', b')$ is a unit. Thus $\gcd(a', s)$ is a unit which means $a'|x$ and in the same manner, $b'|y$ say $x = a't_1$, $y = b't_2$. But then $a'r = xs$ implies $a'r = a't_1s$ so $r = t_1s$. In a like manner we get $r = t_2s$, and hence $t_1 = t_2$. But then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a' \\ b' \end{bmatrix}t_1$, and hence $\begin{bmatrix} a' \\ b' \end{bmatrix}$ is a generator for E .

THEOREM 3. *Let D be a UFD and let \sim be the equivalence relation defined previously. Then every non-zero equivalence class E has a generator, unique up to units. Further for each $0 \neq \alpha \in D^2$, there exists a generator β , unique up to units such that $\alpha = \beta r$ for some $r \in D$.*

PROOF. The existence of a generator has been established above. For uniqueness, suppose $\alpha, \beta \in D^2$ are generators contained in E . Since D is a UFD and $\alpha \sim \beta$, there exist $r, s \in D$ such that $\alpha r = \beta s \neq 0$. Hence, since α, β are generators, $\alpha = \beta t_1$, $\beta = \alpha t_2$, $t_1 t_2 \in D$. From this we find t_1 and t_2 are units in D . The final remark is immediate since for each $0 \neq \alpha \in D^2$, α is in some equivalence class and so there exists a generator β such that $\alpha = \beta r$, $r \in D$.

We next determine the minimal left ideals of $M_D(D^2)$, D a UFD. For each equivalence class E there is a generator α , unique up to units, such that the function $e_\alpha : D^2 \rightarrow D^2$ defined by

$$e_\alpha(x) = \begin{cases} x, & x \in E, \\ 0, & \text{otherwise,} \end{cases}$$

is an idempotent function in $M_D(D^2)$.

LEMMA 3.3. *Every non-zero left ideal L of $N := M_D(D^2)$ contains an idempotent of the form e_α for some equivalence class E with generator α .*

PROOF. Let $0 \neq l \in L$ with (say) $l \neq 0$ on E . Then $l(\alpha) \neq 0$ so $l(\alpha) = \beta r$ for some generator β and $r \in D$. The function $l' : D^2 \rightarrow D^2$ determined by $l'(\beta r) = \alpha r$ for βr in the equivalence class with generator β and $l'(x) = 0$, otherwise, is in N , thus $l'l \in N$. Moreover $l'l$ has the property $l'l(D^2) \subseteq \alpha D$. Thus we assume without loss of generality that l has this property. Define $g \in N$ by $g(\alpha) := 0$ and $g(x) := x$, $x \notin E$. Then $h := e_\alpha(l + g) - e_\alpha g$ is in L with $h(\alpha) = l(\alpha)$ and $h(x) = e_\alpha(l(x) + x)$,

$x \notin E$. Since $l(x) + x \notin E$, $h(x) = 0$, $x \notin E$. Define $\hat{g} \in N$ by $\hat{g}(\alpha) := \gamma$ for some generator $\gamma \neq \alpha$ and $\hat{g}(x) := 0$, $x \notin E$. Further, define $f \in N$ by $f(\gamma) := \alpha$ and $f(x) := 0$, x not in the equivalence class determined by γ . Now consider $k := f(h + \hat{g}) - f\hat{g}$ which is in L since $h \in L$. We have $k(\alpha) = f(l(\alpha) + \gamma) - f(\gamma) = -\alpha$ since $l(\alpha) + \gamma \notin E$ and $k(x) = f(0 + 0) - f(0) = 0$, $0, x \notin E$. But then $-k = e_\alpha \in L$ as was to be shown.

COROLLARY 3.4. *If L is a minimal left ideal for $N := M_D(D^2)$, D a UFD, then $L = Ne_\alpha$ for some generator α .*

PROOF. If $e_\alpha \in L$ and $k := f(ge_\alpha + h) - fh$ for $f, g, h \in N$, then $ke_\alpha = k$. So Ne_α is a non-zero left ideal contained in L .

THEOREM 3.5. *Let D be a UFD and let $N := M_D(D^2)$. Then N contains minimal left ideals and all minimal left ideals are of the form Ne_α , α a generator.*

PROOF. We only need to show that N contains minimal left ideals since the rest of the statement follows from the above corollary. We know Ne_α is a non-zero left ideal for each generator α . Suppose K is a non-zero left ideal contained in Ne_α . Then $K = Ne_\beta$ for some generator β , and hence $e_\beta = fe_\alpha$ for some $f \in N$. But this is possible if and only if $\alpha = \beta$.

For maximal left ideals we do not have a complete characterization. However, in several situations we can identify maximal left ideals.

THEOREM 3.6. *Let M be a maximal left ideal of $N := M_D(D^2)$, D a UFD. If there exists a generator α such that $e_\alpha \notin M$ then $M = \text{Ann}(E)$ where E is the equivalence class containing α .*

PROOF. Since $e_\alpha \notin M$, $M + Ne_\alpha = N$ which implies there exist $m \in M, n \in N$, with $m + ne_\alpha = \text{id}$. Thus, for $y \notin E$, $m(y) = y$ while for $y \in E$, $m(y) + n(y) = y$. Let $k := e_\alpha(m + e_\alpha) - e_\alpha e_\alpha \in M$. Thus $k(y) = 0$ for $y \notin E$ and $k(y) = e_\alpha(m(y) + y) - y$ for $y \in E$. Suppose $m(y) \neq 0$ for some $y \in E$. If $m(y) \in E$, then $m(y) + y \in E$, and hence in this case, $k(y) = m(y)$ which in turn gives $k(\alpha) = m(\alpha) = \alpha r$ for some $r \in D$. If $m(y) \notin E$, $m(y) + y \notin E$ so $k(\alpha) = -\alpha$. Thus k is a non-zero element in M with the property $k(D^2) \subseteq \alpha D$, and as in Lemma 3.3, we obtain $e_\alpha \in M$, a contradiction. Consequently, $m(y) = 0$ for all $y \in E$. Now let h be an arbitrary in $\text{Ann}(E)$ and let $l := h(m + e_\alpha)k - he_\alpha \in M$. For $y \in E$, $l(y) = 0$; for $y \notin E$, $l(y) = h(m(y)) = h(y)$, that is, $l = h$. This

shows $\text{Ann}(E) \subseteq M$. But from $N/\text{Ann}(E) \cong Ne_\alpha$ and the fact that Ne_α is a minimal left ideal we get $\text{Ann}(E)$ is a maximal left ideal, and hence $M = \text{Ann}(E)$.

From the above proof we obtain the following.

COROLLARY 3.7. *Let D be a UFD, $N := M_D(D^2)$ and let M be a maximal left ideal of N . Then $M = \text{Ann}(E)$ for some equivalence class E , or for each generator α , $e_\alpha \in M$.*

When D is finite and thus a field, the equivalence classes are the orbits of the action of D^* on D^2 . In this case we obtain the known result [5, 9.214] that the maximal left ideals are precisely the annihilators of single orbits. On the other hand, when D is infinite and M is a maximal ideal which is not an annihilator of an equivalence class then $M \supseteq \sum \bigoplus Ne_\alpha$ where α ranges over a set of generators, one from each equivalence class. As the next example points out, $\sum \bigoplus Ne_\alpha$ is in general not a maximal left ideal.

EXAMPLE 3.8. Let $D := \mathbb{Z}$, $N := M_D(D^2)$, H a set of generators, one chosen from each equivalence class. Further let $L := \sum_{\alpha \in H} \bigoplus Ne_\alpha$ and consider the subset $\{ \begin{bmatrix} 1 \\ m \end{bmatrix} | m \in \mathbb{Z}^+ \}$ of H . Define

$$T := \left\{ f \in N \mid \text{for some positive integer } k, f \left(\begin{bmatrix} 1 \\ m \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, m > k \right\}.$$

One verifies that T is a left ideal of N , and $L \subseteq T$ since $e_\alpha \in T$ for all $\alpha \in H$. But $L \neq T$, for if f is the function such that $f \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $m \in \mathbb{Z}^+$ but $f \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ for all generators $\begin{bmatrix} a \\ b \end{bmatrix} \in H \setminus \{ \begin{bmatrix} 1 \\ m \end{bmatrix} | m \in \mathbb{Z}^+ \}$ then $f \notin L$.

We now turn to another method for obtaining maximal left ideals. For this method we only require that D be an integral domain, not necessarily commutative and not a field. Again we let $N := M_D(D^2)$ and let H denote a collection of distinct generators in D^2 where two generators α, β are distinct if $\alpha D \cap \beta D = \{0\}$. Let \mathcal{F} be an ultrafilter on H . That is, \mathcal{F} is a non-empty collection of subsets of H with the properties (i) $\emptyset \notin \mathcal{F}$, (ii) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, (iii) if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$, and (iv) for each $A \subseteq H$, either $A \in \mathcal{F}$ or $H \setminus A \in \mathcal{F}$. Define $L(\mathcal{F}) := \{ f \in N \mid f(A) = \{0\} \text{ for some } A \in \mathcal{F} \}$. It is straightforward to verify that $L(\mathcal{F})$ is a left ideal of N . If \mathcal{F} is the principal filter generated by a single generator, say α , then we note that $L(\mathcal{F}) = \text{Ann}\{\alpha\}$. Of course when D is a UFD this agrees with $\text{Ann}(E)$ where D is the equivalence class with generator α . We now show that if H is countable, or more generally if every $A \in \mathcal{F}$ contains a countable subset which is also in \mathcal{F} , then $L(\mathcal{F})$ is a maximal left ideal of N .

To this end, let $f \notin L(\mathcal{F})$ and let $Z_f = \{x \in D \mid f(x) = 0\}$. Since $f \notin L(\mathcal{F})$, $Z_f \notin \mathcal{F}$ so we have $T := H \setminus Z_f \in \mathcal{F}$ and f is non-zero on T . Let S be a countable subset of T with $S \in \mathcal{F}$. Let $g : D^2 \rightarrow D^2$ be defined by $g(\alpha) = 0$, $\alpha \in S$, $g(\alpha) = f(\alpha)$, $\alpha \notin S$. Then $g \in N$ and in fact $g \in L(\mathcal{F})$. Consequently $f - g \notin L(\mathcal{F})$ and $(f - g)(\alpha) = f(\alpha)$, $\alpha \in S$, while $(f - g)(\alpha) = 0$, $\alpha \notin S$. So, without loss of generality we take f to be non-zero on S and zero off S , $S = \{\sigma_i\}$, a countable set of generators.

Let f' be the function in N defined by

$$f' \begin{bmatrix} a \\ b \end{bmatrix} k = \begin{cases} \begin{bmatrix} a \\ a \end{bmatrix} & \text{if } j \neq 0, \\ \begin{bmatrix} b \\ b \end{bmatrix} & \text{if } a = 0, \end{cases}$$

where $\begin{bmatrix} a \\ b \end{bmatrix} \in D^2$. Then $f'f$ is a function which is zero off S , non-zero on S with the $f'f(\alpha) = \begin{bmatrix} x \\ x \end{bmatrix}$ for some $x = x(\alpha)$, all $\alpha \in S$. We wish to construct a function $l \in N$ in the left ideal generated by f which will be the identity on S . We will then show that the ideal generated by $L(\mathcal{F}) \cup \{f\}$ is all of N which means that $L(\mathcal{F})$ is a maximal left ideal. Now $f'f$ is in the left ideal generated by f so again without loss of generality we call this function f . We have therefore

$$f(\sigma_i) = \begin{bmatrix} x_i \\ x_i \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \sigma_i \in S, \quad \text{and} \\ f(\alpha) = 0, \quad \alpha \neq \sigma_i r, \quad \sigma_i \in S, \quad r \in D.$$

Let $k := h(f + g) - hg$ where h is defined by $h \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $h \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}$ if $y \neq 0$ and g is defined recursively as follows.

Since D is not a field, D does not satisfy the descending chain condition for right ideals so we let A_1, A_2, \dots , be a strictly descending chain with the property $\bigcap A_i = \{0\}$. (For example one might take any non-unit $a \in D$ and let $A_i = a^i D$.) Note that for every $x \in A_1$ there is a unique integer $\varphi(x)$ such that $x \in A_{\varphi(x)} \setminus A_{\varphi(x)+1}$. Choose $a_1 a_2, \dots$, such that $a_i \in A_i \setminus A_{i+1}$. We define

$$g(\sigma_1) := \begin{bmatrix} (a_1 - 1)x_1 \\ 0 \end{bmatrix}$$

and set $j_0 = 0$, $j_1 = \varphi(a_1 x_1)$. Now suppose $g(\sigma_i)$ and j_i , $i = 1, 2, \dots, n$, have been defined. Then

$$g(\sigma_{n+1}) := \begin{bmatrix} (a_{j_n+1} - 1)x_{n+1} \\ 0 \end{bmatrix} \quad \text{and} \quad j_{n+1} := \varphi(a_{j_n+1} x_{n+1}).$$

From this we obtain

$$k(\sigma_{n+1}) = \begin{cases} \begin{bmatrix} a_1 x_1 \\ a_1 x_1 \end{bmatrix} & \text{if } n = 0, \\ \begin{bmatrix} a_{j_{n+1}} x_{n+1} \\ a_{j_{n+1}} x_{n+1} \end{bmatrix} & \text{if } n > 0. \end{cases}$$

Thus we have constructed a function k in the left ideal generated by f such that for $\sigma, \tau \in S$, if $\sigma \neq \tau$ then $k(\sigma) \neq k(\tau)$ and

$$k(\sigma_i) = \begin{bmatrix} z_i \\ z_i \end{bmatrix}.$$

Define

$$\delta_i := \begin{bmatrix} z_i + 1 \\ z_i \end{bmatrix}, \quad i = 1, 2, \dots$$

From Example 3.1, the δ_i are generators. We define g_1, h_1 in N by $g_1(\delta_i) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $g_1(x) = 0$ for $x \neq \delta_i r$, $r \in D$, $i = 1, 2, \dots$, and $h_1(\delta_i) = \sigma_i$, $h_1(x) = 0$, $x \neq \delta_i s$, $s \in D$, $i = 1, 2, \dots$. Now let $l := h_1(k + g_1) - h_1 g_1$ which is in the left ideal generated by f . For each $\sigma_i \in S$,

$$l(\sigma_i) = h_1 \left(\begin{bmatrix} z_i \\ z_i \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = h_1(\delta_i) = \sigma_i.$$

This is our desired function.

Now let l_0 be defined by $l_0(\sigma_i) = 0$, $\sigma_i \in S$, $l_0(x) = x - l(x)$, $x \neq \sigma_i r$, $r \in D$, $\sigma_i \in S$. Then $l_0 + l$ is in the left ideal generated by $L(\mathcal{F}) \cup \{f\}$. But $l_0 + l = \text{id}$. This establishes the next result. We note also that a similar result has been established by Fuchs in [1].

THEOREM 3.9. *Let $N := M_D(D^2)$ where D is an integral domain, not necessarily commutative, and D is not a field. Let H denote the set of distinct generators in D^2 and let \mathcal{F} be an ultrafilter on H with the property that for each $A \in \mathcal{F}$, there exists a countable set B , $B \subseteq A$, such that $B \in \mathcal{F}$. Then $L(\mathcal{F}) = \{f \in N \mid f(A) = \{0\} \text{ for some } A \in \mathcal{F}\}$ is a maximal left ideal of N .*

The case in which D is a field will be discussed later. We now turn to a characterization of the two-sided (or invariant) subgroups of $M_D(D^2)$.

We take D to be a PID and as usual we let $N := M_D(D^2)$. Let A be any ideal of D and define $A^+ = \{f \in N \mid f(\alpha) \in A^2, \text{ for all } \alpha \in D^2\}$. We note that A^+ is a subgroup of N and for any $h, k \in N$, $hfk \in A^+$, since A is principal. Thus A^+ is an (N, N) -subgroup, that is, an invariant subgroup of N . As above, we let H denote a set of generators, one chosen from each equivalence class.

LEMMA 3.10. *Let D be a PID, A, B ideals of D . If $A \neq B$ then $A^+ \neq B^+$.*

PROOF. Suppose $a \in A \setminus B$. Let $\alpha \in H$ and define $f(\alpha) = \begin{bmatrix} a \\ a \end{bmatrix}$ and $f(\beta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for each $\beta \in H, \beta \neq \alpha$. Then f is a function in N and $f \in A^+ \setminus B^+$.

LEMMA 3.11. *Every invariant subgroup T of N is of the form $T = A^+$ for some ideal A of D .*

PROOF. If $T = \{0\}$ the result is clear so we take $T \neq \{0\}$ and let

$$T_+ := \left\{ a \in D \mid \text{for some } \gamma \in D^2 \text{ and } f \in T, f(\gamma) = \begin{bmatrix} a \\ 0 \end{bmatrix} \right\}.$$

Clearly if $a \in T_+$ then $ar \in T_+$ for each $r \in D$. Suppose $a, b \in T_+$. Then there exist $f_1, f_2 \in T, \gamma_1, \gamma_2 \in D^2$ with $f_1(\gamma_1) = \begin{bmatrix} a \\ 0 \end{bmatrix}, f_2(\gamma_2) = \begin{bmatrix} b \\ 0 \end{bmatrix}$. If α is any generator then there exist $h, k \in N$ with $h(\alpha) = \gamma_1, k(\alpha) = \gamma_2$. Hence $(f_1h + f_2k)\alpha = \begin{bmatrix} a+b \\ 0 \end{bmatrix}$. Therefore T_+ is an ideal of D , say $T_+ = (m)$. We next show $T = (T_+)^+$. We have $T \subseteq (T_+)^+$ so we turn to the reverse inclusion.

From Lemma 2.6, $T \cap M_2(D) \neq \{0\}$. But then from the definition of T_+ we must have $T \cap M_2(D) = M_2(T_+)$. Let $f \in (T_+)^+$. For each $\alpha \in H, f(\alpha) \in (T_+)^2$, say $f(\alpha) = \begin{bmatrix} mr(\alpha) \\ ms(\alpha) \end{bmatrix}, r(\alpha), s(\alpha) \in D$. Define f' as the function obtained from $f, f'(\alpha) := \begin{bmatrix} r(\alpha) \\ s(\alpha) \end{bmatrix}, \alpha \in H$. Clearly $f' \in N$. From the definition of T_+ we know there exist $g \in T, \begin{bmatrix} c \\ d \end{bmatrix} \in D^2$, with $g \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} m \\ 0 \end{bmatrix}$. As in Lemma 2.6, $g \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in T$. But then $\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \in T$, and consequently $f = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} f' \in T$. Therefore $(T_+)^+ \subseteq T$ and we have the result.

From the above proof we obtain the next result.

COROLLARY 3.12. *Let T_1 and T_2 be invariant subgroups of $M_D(D^2), D$ a PID. If $T_1 \neq T_2$ then $(T_1)_+ \neq (T_2)_+$.*

COROLLARY 3.13. *Let D be a PID. If A is an ideal of D then $(A^+)_+ = A$.*

PROOF. We know $(A^+)_+$ is an ideal of D , say $B := (A^+)_+$. As in the above proof, $B^+ = ((A^+)_+)^+ = A^+$. But then from Lemma 3.10, $A = B$.

From the above results it is straightforward to verify that the map $A \rightarrow A^+, A$ an ideal of D , is an order preserving bijection between the lattice of ideals of D and the lattice of invariant subgroups of N . We summarize these results in the following theorem.

THEOREM 3.14. *Let D be a PID and $N := M_D(D^2)$. There is a lattice isomorphism between the lattice of ideals of D and the lattice of invariant subgroups of N .*

We conclude this section and the paper with a discussion of the case in which $N := M_F(F^2)$, F a field. Thus F^* is a group of fixed point free automorphisms acting on a group F^2 and some of our results follow from the general theory of such actions.

From Theorem 2.8, $M_F(F^2)$ is a simple near-ring. We now show that for left ideals one obtains a nice relationship with filters. Thus we let F denote a field and let $H := \{\sigma_\alpha := \begin{bmatrix} a & \\ & 1 \end{bmatrix} | a \in F\} \cup \{\sigma_\infty := \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\}$, a set of generators, one from each equivalence class.

LEMMA 3.15. *Let $S \subseteq H$ be a non-empty set of generators. If $f \in M_F(F^2)$ is non-zero on S and zero off S , then there exists a function k in the left ideal generated by f such that $k(\sigma) = \sigma$, $\sigma \in S$, and $k(\sigma) = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$, $\sigma \in H \setminus S$.*

PROOF. Let $S = \{\sigma_i\}$, $i \in I$. Without loss of generality we take

$$f(\sigma_i) = \begin{bmatrix} x_i \\ x_i \end{bmatrix}.$$

Let $T := \{\delta_i\}$, $i \in I$, be a subset of H such that $\sigma_\infty \notin T$. We take

$$\delta_i = \begin{bmatrix} a_i \\ & 1 \end{bmatrix}, \quad i \in I.$$

Define $h, g \in M_F(F^2)$ by $h(\sigma_\infty) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $h(\delta_i) = \delta_i x_i^{-1}$, $i \in I$, $h(\alpha) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ otherwise and

$$g(\sigma_i) = \begin{bmatrix} (a_i - 1)x_i \\ & 0 \end{bmatrix}, \quad g(\alpha) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{otherwise.}$$

Then $k := h(f + g) - hg$ is in the left ideal generated by f and

$$k(\sigma_i) = h \left(\begin{bmatrix} x_i \\ x_i \end{bmatrix} + \begin{bmatrix} (a_i - 1)x_i \\ & 0 \end{bmatrix} \right) - h \begin{bmatrix} (a_i - 1)x_i \\ & 0 \end{bmatrix} = h \begin{bmatrix} a_i \\ & 1 \end{bmatrix} x_i = \sigma_i$$

and k is zero off S , and hence we have the result.

For any $f \in M_F(F^2)$ we define the zero set Z_f of f by $Z_f := \{\alpha \in H | f(\alpha) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$.

COROLLARY 3.16. *If I is a proper left ideal of $M_F(F^2)$ and $f \in I$ then $Z_f \neq \emptyset$.*

LEMMA 3.17. *Let I be a proper left ideal of $M_F(F^2)$, $0 \neq f \in I$ and let $\alpha \in F^2$. There exists a function $k \in I$ such that*

$$k(\sigma) = \begin{cases} 0, & \sigma \in Z_f, \\ \alpha, & \sigma \in H \setminus Z_f. \end{cases}$$

PROOF. From Lemma 3.15, we may assume without loss of generality that f is the identity on Z_f . Let h be defined by $h(\sigma) = \alpha$ for each $\sigma \in H$. Then $k := hf$ is the desired function.

LEMMA 3.18. *Let I be a proper left ideal of $M_F(F^2)$ and set $\mathcal{F}(I) := \{Z_f | f \in I\}$. Then $\mathcal{F}(I)$ is a filter on H .*

PROOF. Clearly $\emptyset \notin \mathcal{F}(I)$. Suppose $Z_f, Z_g \in \mathcal{F}(I)$. Let $\alpha, \beta \in F^2$ be such that $\alpha \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \beta$ and $\alpha + \beta \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. From Lemma 3.17, there exist functions $f', g' \in M_F(F^2)$ such that $f'(\sigma) = \alpha$ off Z_f , $g'(\sigma) = \beta$ off Z_g and $Z_{f'} = Z_f, Z_{g'} = Z_g$. Then $Z_f \cap Z_g = Z_{f'+g'}$. But $f' + g' \in I$ so $Z_f \cap Z_g \in \mathcal{F}(I)$. Finally, let $Z_f \subseteq S$. Then $f \in I$ is zero on Z_f , non-zero off Z_f . From Lemma 3.15, there exists $k \in I$ such that k is the identity off Z_f and zero on Z_f . Define $h \in M_F(F^2)$ by $h(\sigma) = \sigma, \sigma \in H \setminus S, h(\sigma) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma \in S$. Then $hk \in I$ and $Z_{hk} = S$, so $S \in \mathcal{F}(I)$.

LEMMA 3.19. *Let I be a proper left ideal of $M_F(F^2)$. Then $I^* := \{f \in M_F(F^2) | Z_f \in \mathcal{F}(I)\} = I$.*

PROOF. We have $I \subseteq I^*$. For the reverse inclusion, let $f^* \in I^*$. Since $Z_{f^*} \in \mathcal{F}(I)$, there exists a function $g \in I$ with $Z_g = Z_{f^*}$. As in the proof of Lemma 3.17 we may take $g(\sigma) = \sigma, \sigma \in H \setminus Z_{f^*}$. But then $f^* = f^*g \in I$.

Let \mathcal{F} be a filter on H and define $I(\mathcal{F}) := \{f \in M_F(F^2) | Z_f \in \mathcal{F}\}$. Using the properties of \mathcal{F} it is straightforward to verify that $I(\mathcal{F})$ is a left ideal of $M_F(F^2)$. Our final theorem follows from this remark and the above sequence of lemmas. Recall that a filter \mathcal{F} on H is said to be a minimal filter if \mathcal{F} consists only of H and just one other set S such that $H \setminus S$ is a singleton.

THEOREM 3.20. *There is an order preserving bijection between the proper left ideals of $M_F(F^2)$, F a field, and filters on H . An ultrafilter corresponds to a maximal left ideal and a minimal filter corresponds to a minimal left ideal.*

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