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ISOMETRY OF RIEMANNIAN MANIFOLDS TO SPHERES, II

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1. Introduction. Let M^n be a Riemannian manifold of dimension $n \geqq 2$ and class C^3, (g_{ij}) the symmetric matrix of the positive definite metric of M^n, and (g^{ij}) the inverse matrix of (g_{ij}), and denote by ∇_i, R_{hijk}, $R_{ij} = R^k{}_{ijk}$ and $R = g^{ij}R_{ij}$ the operator of covariant differentiation with respect to g_{ij}, the Riemann tensor, the Ricci tensor and the scalar curvature of M^n respectively. Let d be the operator of exterior differentiation, δ the operator of codifferentiation, and $\Delta = d\delta + \delta d$ the Laplace-Beltrami operator. Throughout the paper all indices take the values $1, \ldots, n$ unless stated otherwise and can be raised and lowered by using g^{ij} and g_{ij} respectively, and repeated indices indicate summation.

Let v be a vector field defining an infinitesimal conformal transformation of M^n, and L_v the Lie derivative with respect to v. Then we have

$$(1.1) \qquad L_v g_{ij} = \nabla_i v_j + \nabla_j v_i = 2\rho g_{ij}.$$

The infinitesimal transformation v is said to be homothetic or an infinitesimal isometry according as the scalar function ρ is constant or zero. We also denote by $L_{d\rho}$ the Lie derivative with respect to the vector field ρ^i defined by

$$(1.2) \qquad \rho^i = g^{ij}\rho_j, \quad \rho_j = \nabla_j \rho.$$

Let $\xi_{I(p)}$ and $\eta_{I(p)}$ be two tensor fields of the same order $p \leqq n$ on a compact orientable manifold M^n, where $I_{(p)}$ denotes an ordered subset $\{i_1, \ldots, i_p\}$ of the set $\{1, \ldots, n\}$ of positive integers less than or equal to n. Then the local and global scalar products $\langle \xi, \eta \rangle$ and (ξ, η) of the tensor fields ξ and η are defined by

$$(1.3) \qquad \langle \xi, \eta \rangle = \frac{1}{p!} \xi^{I(p)} \eta_{I(p)},$$

$$(1.4) \qquad (\xi, \eta) = \int_{M^n} \langle \xi, \eta \rangle dV,$$

where dV is the element of volume of the manifold M^n at a point. We also define

$$(1.5) \qquad ||\xi|| = p! \langle \xi, \xi \rangle.$$

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From (1.3) and (1.4) it follows that (ξ, ξ) is nonnegative, and that $(\xi, \xi) = 0$ implies $\xi = 0$ on the whole manifold M^n.

In the last decade or so many authors have studied the conditions for a Riemannian manifold M^n of dimension $n > 2$ to be either conformal or isometric to an n-sphere. Very recently K. Amur and V. S. Hedge [**2**] weakened one of the two conditions $L_vR = 0$ and $L_{d\rho}R = 0$ studied jointly by Hsiung and Stern [**6**], and Yano and Hiramatu [**11**] removed the condition from some of these results of Hsiung and Stern and some other known results. The purpose of this paper is to continue the joint work of Yano and Hiramatu to obtain the following theorems by removing both conditions $L_vR = 0$ and $L_{d\rho}R = 0$ from the joint results of Hsiung with Stern [**6**] and Ackler [**1**].

In the following Theorems 1 and 2, M^n will denote a compact Riemannian manifold of dimension $n > 2$ with metric g_{ij}, which admits an infinitesimal nonisometric conformal transformation v satisfying (1.1) with $\rho \neq 0$.

THEOREM 1. *An oriented manifold M^n is isometric to an n-sphere if it satisfies one of the following three equivalent conditions:*

$$(1.6)\qquad \begin{aligned} &\left(P + \frac{c}{n}\,[nR\rho_i\rho^i - (L_vR + nR\rho)\Delta\rho], 1\right) \geqq 0, \\ &\left(P - \frac{c}{n}\,\rho(nL_{d\rho}R + \Delta L_vR), 1\right) \geqq 0, \\ &\left(P + \frac{c}{n}\,[L_v, L_{d\rho}]R, 1\right) \geqq 0, \end{aligned}$$

where

$$(1.7)\qquad \begin{aligned} &P = \rho L_v\left[a^2A + \frac{c - 4a^2}{n-2}B - \frac{1}{n}\left(\frac{2a^2}{n-1} + \frac{c-4a^2}{n-2}\right)R^2\right], \\ &[L_v, L_{d\rho}] = L_vL_{d\rho} - L_{d\rho}L_v, \end{aligned}$$

A and B are defined by

$$(1.8)\qquad A = R^{hijk}R_{hijk}, \quad B = R^{ij}R_{ij},$$

and a, c are constants such that

$$(1.9)\qquad \begin{aligned} c \equiv 4a^2 + (n-2)\Bigg[2a\sum_{i=1}^{4} b_i + \left(\sum_{i=1}^{6}(-1)^{i-1}b_i\right)^2 \\ - 2(b_1b_3 + b_2b_4 - b_5b_6) + (n-1)\sum_{i=1}^{6} b_i^2\Bigg] > 0, \end{aligned}$$

b's being arbitrary constants.

An elementary calculation shows that $c \geqq 0$ where equality holds if and only if $b_1 = \ldots = b_4$, $b_5 = b_6 = 0$, $a = -(n-2)b_1$.

For $L_vR = 0$, Theorem 1 (referred to the first inequality of (1.6) for $P = 0$ and $(nR\rho_i\rho^i - (L_vR + nR\rho)\Delta\rho, 1) \geqq 0$) with "isometric" replaced by "conformal" is due to Yano [10] for either $a \neq 0, c - 4a^2 = 0$ or $a = 0, c - 4a^2 \neq 0$, and due to Hsiung and Stern [6] for general a and b's. Theorem 1 (referred to the first inequality of (1.6) for $P = 0$ and $(nR\rho_i\rho^i - (L_vR + nR\rho)\Delta\rho, 1) \geqq 0)$ is due to Yano and Hiramatu [11] for $a \neq 0, c - 4a^2 = 0$ or $a = 0, c - 4a^2 \neq 0$.

For constant R, Theorem 1 (referred to the second inequality of (1.6) for $P = 0$) is due to Lichnerowicz [8] for $a = 0$, $c \neq 0$, $B =$ constant, due to Hsiung [3] for $a \neq 0$, $c - 4a^2 = 0$, $A =$ constant, due to Yano [9] for either $a = 0, c \neq 0$, or $a \neq 0, c - 4a^2 = 0$, due to Hsiung [5] for $b_2 = \ldots = b_6 = 0$, due to Yano and Sawaki [13] for $b_1 = \ldots = b_4 = b/(n-2)$, $b_5 = b_6 = 0$, and due to Hsiung [5] for general a and b's. For $L_vR = 0$, $L_{d\rho}R = 0$, Theorem 1 (referred to the second inequality of (1.6) for $P = 0$) is due to Ackler and Hsiung [1].

THEOREM 2. *A manifold M^n is isometric to an n-sphere if it satisfies*

$$(1.10) \quad L_v(A^aB^b) = 0,$$

$$(1.11) \quad c\left(\frac{2a}{A} + \frac{(n-1)b}{B}\right) = \frac{2^a(a+b)R^{2(a+b-1)}}{n^{a+b-1}(n-1)^{a-1}},$$

$$(1.12) \quad \left(\frac{b}{2(n-1)} A^aB^{b-1}RL_vR - A^aB^b\left(\frac{4a}{A} + \frac{(n-2)b}{B}\right) \times \left(R^{ij}\nabla_i\nabla_j\rho + \frac{R^2\rho}{n(n-1)}\right), \rho\right) \leqq 0,$$

where A, B are given by (1.8), *and a, b are nonnegative integers and not both zero.*

For constant R and A^aB^b, Theorem 2 is due to Lichnerowicz [7] for $a = 0$, $b = 1$, and due to Hsiung [3] for general a and b. For constant A^aB^b and $L_vR = 0$, $L_{d\rho}R = 0$, Theorem 2 is due to Hsiung and Stern [6]; in this case condition (1.12) is satisfied automatically since

$$(1.13) \quad \left(R^{ij}\nabla_i\nabla_j\rho + \frac{R^2\rho}{n(n-1)}, \rho\right) \geqq 0,$$

which is due to Hsiung and Stern [6], and due to Lichnerowicz [8] for constant R.

In the proofs of the above theorems we need the following theorems.

THEOREM A (Yano and Nagano [12]). *If a complete Einstein space M^n of dimension $n > 2$ admits an infinitesimal nonisometric conformal transformation, then M^n is isometric to an n-sphere.*

THEOREM B (Tashiro [8]). *If a complete Riemannian manifold M^n of dimension $n > 2$ admits a complete vector field v satisfying* (1.1) *with $\rho \neq$ const. and*

$$(1.14) \quad \nabla_i\nabla_j\rho = -g_{ij}\Delta\rho/n,$$

then M^n is isometric to an n-sphere.

2. Notation and formulas. In this section we shall list some well known formulas which will be needed in the proofs to follow.

Let v be a vector field defining an infinitesimal conformal transformation on a Riemannian manifold M^n of dimension $n \geqq 2$ so that (1.1) holds. Then we have

$$(2.1) \qquad \rho = \nabla_i v^i / n,$$

$$(2.2) \qquad L_v R^h{}_{ijk} = -\epsilon_k{}^h \nabla_i \rho_j + \epsilon_j{}^h \nabla_i \rho_k - g_{ij} \nabla_k \rho^h + g_{ik} \nabla_j \rho^h,$$

where $\rho^h = \nabla^h \rho$, and $\epsilon_k{}^h = 1$ for $h = k$ and $\epsilon_k{}^h = 0$ for $h \neq k$. From (1.1) and (2.2) it follows immediately that

$$(2.3) \qquad L_v R_{hijk} = 2\rho R_{hijk} - g_{hk} \nabla_i \rho_j + g_{hj} \nabla_i \rho_k - g_{ij} \nabla_h \rho_k + g_{ik} \nabla_h \rho_j,$$

$$(2.4) \qquad L_v R_{ij} = g_{ij} \Delta\rho - (n-2) \nabla_i \rho_j,$$

$$(2.5) \qquad L_v R = 2(n-1)\Delta\rho - 2R\rho.$$

For any scalar field f on M^n, we have

$$(2.6) \qquad \Delta f = -\nabla^i \nabla_i f.$$

On the manifold M^n consider the following tensors:

$$(2.7) \qquad T_{ij} = R_{ij} - \frac{1}{n} R g_{ij},$$

$$(2.8) \qquad T_{hijk} = R_{hijk} - \frac{1}{n(n-1)} R(g_{hk} g_{ij} - g_{hj} g_{ik}),$$

$$(2.9) \qquad W_{hijk} = a T_{hijk} + b_1 g_{hk} T_{ij} - b_2 g_{hj} T_{ik} + b_3 g_{ij} T_{hk} - b_4 g_{ik} T_{hj} + b_5 g_{hi} T_{jk} - b_6 g_{jk} T_{hi},$$

where a and b's are constants satisfying (1.9). From (2.7) and (2.8) it follows immediately that

$$(2.10) \qquad g^{ij} T_{ij} = 0, \; g^{hk} T_{hijk} = T_{ij},$$

which, together with (2.9), imply that

$$(2.11) \qquad g^{hk} g^{ij} W_{hijk} = 0, \quad g^{hj} g^{ik} W_{hijk} = 0, \quad g^{hi} g^{jk} W_{hijk} = 0.$$

Moreover by (1.3), (1.5) and (2.9) we have

$$(2.12) \qquad ||W|| = a^2 A + \frac{c - 4a^2}{n-2} B - \frac{1}{n} \left(\frac{2a^2}{n-1} + \frac{c^2 - 4a^2}{n-2} \right) R^2,$$

where A, B, and c are defined by (1.8) and (1.9).

3. Lemmas. Throughout this section M^n will always denote a compact oriented Riemannian manifold of dimension $n > 2$.

LEMMA 3.1 (Yano [**5**, (2.11), (2.12); or **11**, Lemma 4]). *If ρ is a scalar field on M^n, then*

$$(3.1)\qquad \left(R_{ij}\rho^i\rho^j - \frac{n-1}{n}(\Delta\rho)^2, 1\right) + 2\left(\nabla_i\rho_j + \frac{1}{n}g_{ij}\Delta\rho, \nabla_i\rho_j + \frac{1}{n}g_{ij}\Delta\rho\right) = 0,$$

or equivalently

$$(3.2)\qquad \left(R_{ij}\rho^i\rho^j - \frac{n-1}{n}\rho^i\nabla_i\Delta\rho, 1\right) + 2\left(\nabla_i\rho_j + \frac{1}{n}g_{ij}\Delta\rho, \nabla_i\rho_j + \frac{1}{n}g_{ij}\Delta\rho\right) = 0.$$

For a proof of Lemma 3.1 one may also see [**1**, p. 58].

LEMMA 3.2 (Yano [**10**]). *If M^n of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v satisfying* (1.1) *with $\rho \neq$ const. and either one of the following two conditions:*

$$(3.3)\qquad \left(R_{ij}\rho^i\rho^j - \frac{n-1}{n}(\Delta\rho)^2, 1\right) \geqq 0,$$

$$(3.4)\qquad \left(R_{ij}\rho^i\rho^j - \frac{n-1}{n}\rho^i\nabla_i\Delta\rho, 1\right) \geqq 0,$$

then M^n is isometric to an n-sphere.

Proof. This follows from Lemma 3.1 and Theorem B.

Substitution of (2.5) in (3.3), (3.4) and use of

$$(3.5)\qquad L_{d\rho}R = \rho^i\nabla_i R$$

and Lemma 3.2 yield

LEMMA 3.3. *If M^n of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v satisfying* (1.1) *with $\rho \neq$ const. and either one of the following two conditions:*

$$(3.6)\qquad \left(R_{ij}\rho^i\rho^j - \frac{1}{4n(n-1)}(L_vR + 2R\rho)^2, 1\right) \geqq 0,$$

$$(3.7)\qquad \left(R_{ij}\rho^i\rho^j - \frac{1}{n}R\rho_i\rho^i - \frac{1}{n}\rho L_{d\rho}R - \frac{1}{2n}\rho^i\nabla_i L_vR, 1\right) \geqq 0,$$

then M^n is isometric to an n-sphere.

Condition (3.7) is due to Yano and Hiramatu [**11**]. In particular, when $L_vR = 0$ and $L_{d\rho}R = 0$, conditions (3.3) and (3.4) are reduced to

$$(3.8)\qquad \left(R_{ij}\rho^i\rho^i - \frac{1}{n(n-1)}R^2\rho^2, 1\right) \geqq 0,$$

$$(3.9)\qquad (T_{ij}, \rho_i\rho_j) \geqq 0,$$

so that in this special case Lemma 3.3 is due to Ackler and Hsiung [**1**].

LEMMA 3.4. *If M^n admits an infinitesimal nonisometric conformal transformation v satisfying* (1.1) *with $\rho \neq 0$, then for any scalar field f on M^n*

$$(3.10)\quad (L_v f, 1) = -n(f\rho, 1).$$

Proof. From the definition of L_v and (2.1) we have

$$(3.11)\quad \nabla_i(fv^i) = L_v f + nf\rho.$$

Integration of (3.11) over M^n and use of the well-known Green's formula

$$(3.12)\quad (\nabla^i \xi_i, 1) = 0,$$

where ξ_i is any vector field on M^n, give (3.10) immediately.

LEMMA 3.5. *For any scalar fields f and h on M^n,*

$$(3.13)\quad (L_{df}h, 1) = (L_{dh}f, 1) = (\nabla_i f \nabla^i h, 1) = (f\Delta h, 1) = (h\Delta f, 1).$$

Proof. (3.13) follows from

$$(\nabla_i(f\nabla^i h), 1) = (\nabla_i f \nabla^i h, 1) - (f\Delta h, 1) = 0,$$
$$(\nabla_i(h\nabla^i f), 1) = (\nabla_i h \nabla^i f, 1) - (h\Delta f, 1) = 0.$$

LEMMA 3.6. *If M^n admits an infinitesimal conformal transformation v satisfying* (1.1), *then*

$$(3.14)\quad (\rho^2 \Delta R, 1) = (2\rho L_{d\rho} R, 1).$$

Proof. (3.14) follows from (3.13) and (3.5) by putting $f = R$ and $h = \rho^2$ in (3.13).

LEMMA 3.7. *If M^n admits an infinitesimal conformal transformation v satisfying* (1.1), *then*

$$(3.15)\quad (L_v L_{d\rho} R, 1) = -\frac{n}{2}(\rho^2 \Delta R, 1),$$

$$(3.16)\quad (L_{d\rho} L_v R, 1) = ((L_v R)\Delta\rho, 1),$$

$$(3.17)\quad ([L_v, L_{d\rho}]R, 1) = -\frac{n}{2}(\rho^2 \Delta R, 1) - ((L_v R)\Delta\rho, 1).$$

Proof. By Lemmas 3.4 and 3.6 we have

$$(L_v L_{d\rho} R, 1) = -n(\rho L_{d\rho} R, 1) = -\frac{n}{2}(\rho^2 \Delta R, 1),$$

which proves (3.15). By putting $f = \rho$ and $h = L_v R$ in (3.13) we readily obtain (3.16), and (3.17) follows from (3.15) and (3.16).

LEMMA 3.8. *For any scalar field* ρ *on* M^n,

$$(3.18)\quad \frac{1}{2}(\rho^2\Delta R, 1) - (R\rho\Delta\rho, 1) + (R\rho_i\rho^i, 1) = 0,$$

$$(3.19)\quad \frac{1}{n}(L_v L_{d\rho} R, 1) + (R\rho\Delta\rho, 1) - (R\rho_i\rho^i, 1) = 0.$$

Proof. Integration of

$$(3.20)\quad \nabla_i(R\rho\rho^i) = \rho\rho^i\nabla_i R + R\rho_i\rho^i - R\rho\Delta\rho$$

over M^n and use of (3.12), (3.14) and (3.5) give (3.18). (3.19) follows from (3.15) and (3.18).

LEMMA 3.9. *If* M^n *admits an infinitesimal conformal transformation* v *satisfying* (1.1), *then*

$$(3.21)\quad \mathscr{A} = -\mathscr{B} = ([L_v, L_{d\rho}]R, 1),$$

where

$$(3.22)\quad \mathscr{A} = (nR\rho_i\rho^i - (L_vR + nR\rho)\Delta\rho, 1),\quad \mathscr{B} = (nL_{d\rho}R + \Delta L_vR, \rho).$$

Proof. This proof is due to H. Hiramatu. From (3.19) we have $\mathscr{A} = (L_vL_{d\rho}R - (L_vR)\Delta\rho, 1)$ which together with (3.16) gives immediately $\mathscr{A} = ([L_v, L_{d\rho}]R, 1)$.

On the other hand, by putting $f = L_{d\rho}R$ in Lemma 3.4 and $f = \rho$, $h = L_vR$ in Lemma 3.5 we obtain $\mathscr{B} = -([L_v, L_{d\rho}]R, 1)$.

4. Proof of the theorems.

Proof of Theorem 1. By means of (2.9), (2.8), (2.7), (1.1), (2.3), (2.4) and (2.5) we can easily obtain

$$\begin{aligned} L_vW_{hijk} = {} & 2a\rho R_{hijk} - [a + (n-2)b_1]g_{hk}\nabla_i\rho_j + [a + (n-2)b_2]g_{hj}\nabla_i\rho_k \\ & - [a + (n-2)b_3]g_{ij}\nabla_h\rho_k + [a + (n-2)b_4]g_{ik}\nabla_h\rho \\ & - (n-2)b_5g_{hi}\nabla_j\rho_k + (n-2)b_6g_{jk}\nabla_h\rho_i \\ (4.1)\qquad & + \frac{1}{n(n-1)}\{2a[\rho R + (n-1)\Delta\rho] + (n-1)[2\rho R + (n-2)\Delta\rho]\} \\ & \cdot [-g_{ij}g_{hk}(b_1 + b_3) + g_{ik}g_{hj}(b_2 + b_4)] \\ & + \frac{1}{n}g_{hi}g_{jk}[2\rho R + (n-2)\Delta\rho](-b_5 + b_6) + 2\rho(b_1g_{hk}R_{ij} \\ & - b_2g_{hj}R_{ik} + b_3g_{ij}R_{hk} - b_4g_{ik}g_{hj} + b_5g_{hi}R_{jk} - b_6g_{jk}R_{hi}). \end{aligned}$$

Multiplying both sides of (4.1) by W^{hijk} and making use of (2.7), . . . , (2.11), (1.9) and $R_i{}^{ijk} = 0$ we have, by an elementary but lengthy calculation,

$$(4.2)\qquad W^{hijk}L_vW_{hijk} = 2\rho||W|| - cT^{ij}\nabla_i\rho_j.$$

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Substitution of (4.2) in the well known formula

$$L_v||W|| = 2W^{hijk}L_vW_{hijk} - 8\rho||W|| \tag{4.3}$$

thus gives

$$\rho L_v||W|| = -4\rho^2||W|| - 2c\rho T^{ij}\nabla_i\rho_j. \tag{4.4}$$

Integrating (4.4) over M^n we obtain

$$-2c(\rho T^{ij}\nabla_i\rho_j, 1) = (L_v||W||, \rho) + 4(||W||, \rho^2). \tag{4.5}$$

The equivalence of the three conditions given by (1.6) is obvious from Lemma 3.9. For proving Theorem 1 we assume that the second inequality of (1.6) holds. Applying covariant differentiation and using (2.6), (3.6), (4.5) we obtain

$$\begin{aligned}\nabla^i\left(R_{ij}\rho\rho^j - \frac{1}{n}R\rho\rho_i - \frac{1}{n}\rho^2\nabla_iR - \frac{1}{2n}\rho\nabla_iL_vR\right)\\ = R_{ij}\rho^i\rho^j - \frac{1}{n}R\rho_i\rho^i - \frac{1}{n}\rho L_{d\rho}R - \frac{1}{2n}\rho^i\nabla_iL_vR + \rho T^{ij}\nabla_i\rho_j\\ + \frac{\rho}{2n}[(n-4)L_{d\rho}R + 2\rho\Delta R + \Delta L_vR].\end{aligned} \tag{4.6}$$

On the other hand, integrating (4.6) over M^n, applying Green's formula (3.12) and substituting (4.5), (3.14) in the resulting equation we have

$$\begin{aligned}\left(R_{ij}\rho^i\rho^j - \frac{1}{n}R\rho_i\rho^i - \frac{1}{n}\rho L_{d\rho}R - \frac{1}{2n}\rho^i\nabla_iL_vR, 1\right)\\ = \frac{1}{2c}(L_v||W||, \rho) + \frac{2}{c}(||W||, \rho^2) - \frac{1}{2n}(nL_{d\rho}R + \Delta L_vR, \rho).\end{aligned} \tag{4.7}$$

Since $(||W||, \rho^2)$ is nonnegative, from (4.7), (2.12), (1.7) and the second inequality of (1.6) we obtain (3.7). Hence by Lemma 3.3, M^n is isometric to an n-sphere.

A proof of Theorem 1 based on the first condition of (1.6) can be obtained by following the proof of Theorem I in [**6**]. In fact, substituting (3.20) for $\rho\rho^i\nabla_iR$ and (4.4) for $\rho T^{ij}\nabla_i\rho_j$ in [**6**, (4.6)] and using (3.12), (2.5) and the first condition of (1.6) we can easily reach (3.8).

Proof of Theorem 2. Without loss of generality we may assume our manifold M^n to be oriented, as otherwise we need only to take an orientable two-fold covering space of M^n. On M^n consider the covariant tensor field T of order $2(2a+b)$:

$$\begin{aligned}T_{h_1i_1j_1k_1\ldots h_ai_aj_ak_au_1v_1\ldots u_bv_b}\\ = \prod_{r=1}^{a} R_{h_ri_rj_rk_r}\prod_{s=1}^{b} R_{u_sv_s}\\ - \frac{R^{a+b}}{n^{a+b}(n-1)^a}\prod_{r=1}^{a}(g_{h_rk_r}g_{i_rj_r} - g_{h_rj_r}g_{i_rk_r})\prod_{s=1}^{b} g_{u_sv_s}.\end{aligned} \tag{4.8}$$

From (4.8) it is easily seen that the length of T is

$$(4.9)\qquad [2(2a+b)]!\,\langle T, T\rangle = A^aB^b - \frac{2^aR^{2(a+b)}}{n^{a+b}(n-1)^a},$$

which, together with (1.10), implies

$$(4.10)\qquad [2(2a+b)]!\,L_v\langle T, T\rangle = \frac{-2^a}{n^{a+b}(n-1)^a}L_vR^{2(a+b)}.$$

Thus by the extension of formula (4.3) to the tensor T we immediately obtain

$$(4.11)\qquad L_v\langle T, T\rangle = 2\langle L_vT, T\rangle - 4(2a+b)\rho\langle T, T\rangle,$$

from which and (4.10) it follows that

$$(4.12)\qquad (\langle L_vT, T\rangle, \rho) = \left(2(2a+b)\rho\langle T, T\rangle - \frac{2^{a-1}L_vR^{2(a+b)}}{n^{a+b}(n-1)^2[2(2a+b)]!}, \rho\right).$$

On the other hand from (2.3) and (2.4) we have

$$(4.13)\qquad \begin{aligned} &L_vT_{h_1i_1j_1k_1\ldots h_ai_aj_ak_au_1v_1\ldots u_bv_b}\\ &= 2a\rho\prod_{r=1}^{a}R_{h_ri_rj_rk_r}\prod_{s=1}^{b}R_{u_sv_s}\\ &\quad - \sum_{r=1}^{a}[R_{h_1i_1j_1k_1}\ldots R_{h_{r-1}i_{r-1}j_{r-1}k_{r-1}}(g_{h_rk_r}\nabla_{i_r}\nabla_{j_r}\rho\\ &\qquad - g_{h_rj_r}\nabla_{i_r}\nabla_{k_r}\rho + g_{i_rj_r}\nabla_{k_r}\nabla_{h_r}\rho - g_{i_rk_r}\nabla_{j_r}\nabla_{k_r}\rho)\\ &\qquad \cdot R_{h_{r+1}i_{r+1}j_{r+1}k_{r+1}}\ldots R_{h_ai_aj_ak_a}]\prod_{s=1}^{b}R_{u_sv_s}\\ &\quad + \prod_{r=1}^{a}R_{h_ri_rj_rk_r}\sum_{s=1}^{b}\{R_{u_1v_1}\ldots R_{u_{s-1}v_{s-1}}\\ &\qquad \cdot[g_{u_sv_s}\Delta\rho - (n-2)\nabla_{u_s}\nabla_{v_s}\rho]R_{u_{s+1}v_{s+1}}\ldots R_{u_bv_b}\}\\ &\quad - \frac{2(2a+b)R^{a+b}\rho + L_vR^{a+b}}{n^{a+b}(n-1)^a}\prod_{r=1}^{a}(g_{i_rj_r}g_{h_rk_r}\\ &\qquad\qquad - g_{i_rk_r}g_{h_rj_r})\prod_{s=1}^{b}g_{u_sv_s}.\end{aligned}$$

By means of (4.8), (4.9), (4.13) and (1.8) an elementary calculation yields

$$(4.14)\qquad \begin{aligned} &[2(2a+b)]!\,\langle L_vT, T\rangle\\ &= 2a\rho[2(2a+b)]!\,\langle T, T\rangle - A^aB^b\left[\frac{4a}{A} + \frac{(n-2)b}{B}\right]R^{ij}\nabla_j\nabla_k\rho\\ &\qquad - (a+b)\frac{2^{a+1}R^{2a+2b-1}}{n^{a+b}(n-1)^{a-1}}\Delta\rho + bA^aB^{b-1}R\Delta\rho,\end{aligned}$$

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from which and (1.11) it follows readily that

$$(4.15)\quad \begin{aligned}(\langle L_v T, T\rangle, \rho) &= 2a(\langle T, T\rangle, \rho^2)\\ &- \frac{A^a B^b}{[2(2a+b)]!}\left(\frac{4a}{A} + \frac{(n-2)b}{B}\right)\left(R^{jk}\nabla_j\nabla_k\rho + \frac{R}{n}\Delta\rho, \rho\right).\end{aligned}$$

Substituting (4.12) in (4.15) and using (1.11), (2.5), (1.12) we can easily show that

$$(4.16)\quad (\langle T, T\rangle, \rho^2) \leqq 0.$$

This means that $\langle T, T\rangle = 0$ which implies

$$(4.17)\quad T_{h_1 i_1 j_1 k_1 \ldots h_a i_a j_a k_a u_1 v_1 \ldots u_b v_b} = 0.$$

Multiplying (4.17) by

$$g^{h_1 k_1} \prod_{r=2}^{a} g^{h_r k_r} g^{i_r k_r} \prod_{s=1}^{b} g^{u_s v_s},$$

and using (4.8) we obtain $R_{i_1 j_1} = R g_{i_1 j_1}/n$ which implies that M^n is an Einstein space. Hence by Theorem A, M^n is isometric to an n-sphere.

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