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Potential Theory of the Farthest-Point Distance Function

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Abstract. We study the farthest-point distance function, which measures the distance from $z \in \mathbb{C}$ to the farthest point or points of a given compact set E in the plane.

The logarithm of this distance is subharmonic as a function of z, and equals the logarithmic potential of a unique probability measure with unbounded support. This measure σ_E has many interesting properties that reflect the topology and geometry of the compact set E. We prove $\sigma_E(E) \le \frac{1}{2}$ for polygons inscribed in a circle, with equality if and only if E is a regular n-gon for some odd n. Also we show $\sigma_E(E) = \frac{1}{2}$ for smooth convex sets of constant width. We conjecture $\sigma_E(E) \le \frac{1}{2}$ for all E.

1 Introduction

Throughout the paper, E is a compact set in the complex plane that contains at least two points. Write $D(z, R)$ for the closed disk centered at z with radius R.

The function that measures the distance from a point $z \in \mathbb{C}$ to the farthest point or points of E is

$$d_E(z) := \max_{t \in E} |z - t| > 0, \quad z \in \mathbb{C}.$$

This distance function is Lipschitz continuous with constant 1, because if $z_1, z_2 \in \mathbb{C}$ and $t \in E$ then

$$\begin{aligned} |z_1 - t| &\le |z_1 - z_2| + |z_2 - t| \\ &\le |z_1 - z_2| + d_E(z_2), \end{aligned}$$

and so $d_E(z_1) \le |z_1 - z_2| + d_E(z_2)$ by maximizing over t. Now interchange z_1 and z_2, also.

Next, $\log d_E(z) = \max_{t \in E} \log |z - t|$ is subharmonic since it is the maximum of the subharmonic functions $z \mapsto \log |z - t|$. In this paper we will study the interplay between the analytic properties of the distance function, particularly the potential theory of $\log d_E$, and the topological and geometric properties of the set E.

The Riesz decomposition theorem (*cf.* [7, p. 76]) applied to $\log d_E$ gives that

$$\log d_E(z) = \int_{\mathbb{C}} \log |z - t| \, d\sigma_r(t) + h_r(z), \quad |z| < r,$$

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where h_r is a harmonic function and σ_r is a Borel measure supported in $|z| < r$, for any $r > 0$. By uniqueness, if $r < s$ then $\sigma_r = \sigma_s$ in $|z| < r$. Letting $r \to \infty$, then, we obtain the Riesz measure σ_E on $\mathbb{C}$ corresponding to the subharmonic function $\log d_E$. Moreover, one can show h_r vanishes in the limit, so that the Riesz decomposition takes the following form.

Theorem 1.1 $\log d_E(z)$ *is a subharmonic function in* $\mathbb{C}$ *and*

$$\log d_E(z) = \int_{\mathbb{C}} \log|z-t|\, d\sigma_E(t), \quad z \in \mathbb{C}, \tag{1.1}$$

where the integral converges absolutely and σ_E *is a unique positive unit Borel measure in* $\mathbb{C}$ *with unbounded support, so that*

$$\sigma_E(\mathbb{C}) = 1 \quad \textit{and} \quad \infty \in \operatorname{supp}(\sigma_E).$$

Furthermore, if $d_E \in C^2(U)$ *for some domain* U*, then* σ_E *can be calculated in* U *by*

$$d\sigma_E(z) = \frac{1}{2\pi}\Delta\big(\log d_E(x+iy)\big)\, dx\, dy, \quad z = x+iy \in U. \tag{1.2}$$

And if $E = \overline{V}$ *for some bounded domain* V*, then* $\operatorname{supp}(\sigma_E) \cap V$ *is nonempty.*

Except for the uniqueness of σ_E (which we prove in Section 3), the theorem was previously obtained by the second author [6, Lemma 5.1], in connection with certain inequalities for norms of products of polynomials. We also give an alternative proof of the Riesz representation (1.1), in Section 3 of this paper. Note that Theorem 1.1 generalizes a result of D. W. Boyd [1, Lemma 2] in which E consists of a finite number of points. Incidentally, (1.2) follows of course from the Riesz decomposition for potentials (see [7, p. 76]).

We exclude the case of a singleton set in this paper; in fact several of our results and proofs break down when $E = \{a\}$ is a singleton, because $d_{\{a\}}$ vanishes at the point a. But in any case, the representation of $\log d_{\{a\}}$ is trivial:

$$\log d_{\{a\}}(z) = \log|z-a| = \int \log|z-t|\, d\delta_a$$

where δ_a is a unit point mass at a. That is, $\sigma_{\{a\}} = \delta_a$.

The most elementary examples for Theorem 1.1 are disks and segments.

Example 1.2 Consider the closed disk $E = D(0,R)$, where $R > 0$. It is easy to see that

$$d_{D(0,R)}(z) = |z| + R, \quad z \in \mathbb{C},$$

and so $d_{D(0,R)} \in C^2(\mathbb{C} \setminus \{0\})$. Therefore we immediately obtain that

$$d\sigma_{D(0,R)}(z) = \frac{1}{2\pi}\Delta(\log d_{D(0,R)})(z)\, dx\, dy = \frac{R}{2\pi|z|(R+|z|)^2}\, dx\, dy, \quad z = x+iy \in \mathbb{C}. \tag{1.3}$$

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(We need not worry what happens at the origin, since $\sigma_{D(0,R)}(\{0\}) = 1 - \sigma_{D(0,R)}(\mathbb{C} \setminus \{0\}) = 0$.)

Observe that σ_E is supported on the whole plane, in this example. We show in Proposition 2.1 (b) that this is always the case when ∂E is smooth. ■

Example 1.3 Consider the segment $E = [-a, a]$, where $a > 0$. Then

$$u(z) := \log d_{[-a,a]}(z) = \max(\log|z-a|, \log|z+a|), \quad z \in \mathbb{C}.$$

Note that $u(z)$ is harmonic in $\Re z > 0$ and $\Re z < 0$, so that $\sigma_{[-a,a]}$ is supported on the imaginary axis $\Re z = 0$. Using Theorem II.1.5 of [9, p. 92], we obtain that

$$d\sigma_{[-a,a]}(iy) = \frac{1}{2\pi}\left(\frac{\partial u}{\partial \mathbf{n}_-}(iy) + \frac{\partial u}{\partial \mathbf{n}_+}(iy)\right) dy = \frac{a\,dy}{\pi(a^2+y^2)}, \quad y \in \mathbb{R}, \tag{1.4}$$

where $\mathbf{n}_+$ and $\mathbf{n}_-$ are unit normals to the y-axis, in the positive and negative directions.

Observe that σ_E is supported precisely on the set where the "farthest point" changes from $-a$ to $+a$. We will generalize this observation in Proposition 2.1 (a). ■

In Section 2, we study the fundamental properties of the measure σ_E, especially its support and concentration properties, and its moments.

We also study the scale-invariant quantity $1 - \sigma_E(E)$, which appears to act (roughly) as a measure of the "width-variation" of E. For instance, we see from Examples 1.2 and 1.3 that $\sigma_E(E)$ equals $\frac{1}{2}$ for the disk and 0 for the line segment. We will show in Theorem 2.5 that $\sigma_E(E) = \frac{1}{2}$ for every regular n-gon with n odd, and further in Theorem 2.6 that $\sigma_E(E) = \frac{1}{2}$ for every smooth set of constant width. We conjecture $\sigma_E(E) \leq \frac{1}{2}$ for every set E, and prove this for polygons inscribed in a circle, in Theorem 2.5.

2 Properties of the Distance function and Its Representing Measure

The results in this section are all proved in Section 3.

The first proposition deals with the support of σ_E: when is an open set *not* in the support, and when does the support equal the whole plane?

Proposition 2.1

(a) *Let G be a domain in the plane. Then* $\log d_E$ *is harmonic in G, that is*

$$\operatorname{supp}(\sigma_E) \cap G = \varnothing, \tag{2.1}$$

if and only if there exists a point $\zeta \in \partial E \setminus G$ *such that*

$$d_E(z) = |z - \zeta| \quad \forall z \in G.$$

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(b) *If ∂E is C^1-smooth then* $\operatorname{supp}(\sigma_E) = \mathbb{C}$.

The next result shows how the mass of σ_E is distributed in the plane with respect to small and large disks. The diameter of E occurs in our estimates.

Theorem 2.2 *For each $z \in \mathbb{C}$,*

$$\sigma_E\big(D(z,r)\big) \le \frac{r}{d_E(z) - r} \quad \text{for all sufficiently small } r > 0, \tag{2.2}$$

$$1 - \frac{3\operatorname{diam}(E)}{r + \operatorname{diam}(E)} < \sigma_E\big(D(z,r)\big) \le 1 \quad \text{for all sufficiently large } r > 0. \tag{2.3}$$

Hence $\sigma_E\big(D(z,r)\big) \le O(r)$ as $r \to 0$, and $1 - O(\frac{1}{r}) \le \sigma_E\big(D(z,r)\big) \le 1$ as $r \to \infty$.

Remarks 1. One cannot hope for a lower bound on $\sigma_E\big(D(z,r)\big)$ as $r \to 0$, because σ might be identically zero on $D(z,r)$, as happens say in Example 1.3 where E is a line segment.

2. Perhaps one could prove an upper bound of the form $\sigma_E\big(D(z,r)\big) \le 1 - O(\frac{1}{r})$ as $r \to \infty$, but we have not done this.

3. The estimates in Theorem 2.2 are sharp in order of magnitude, as one finds easily by considering the disk as in Example 1.2.

Now that we know the distribution of mass, we show certain moments of σ_E are finite.

Corollary 2.3 *For all $z \in \mathbb{C}$ and $-1 < p < 1$,*

$$\int_{\mathbb{C}} |z - t|^p \, d\sigma_E(t) < \infty.$$

The range $p \in (-1,1)$ is sharp, as can be seen from the case $E = D(0,R)$ in Example 1.2.

We now ask how much of the σ_E-measure can be captured within E itself. Since $\sigma_E(\mathbb{C}) = 1$ and we know that σ_E concentrates, roughly speaking, at points where the "farthest point" changes, it seems plausible that the more the width of E varies, the less will $\sigma_E(E)$ be. Indeed for the closed disk $E = D(0,R)$ one finds $\sigma_E(E) = \frac{1}{2}$, using Example 1.2, whereas for the line segment $E = [-a,a]$ we have $\sigma_E(E) = 0$. Further, by integrating (1.2) over the interior of E and applying Green's theorem, one arrives formally at the expression

$$\sigma_E(E) = \frac{1}{2\pi} \int_{\partial E} \frac{1}{d_E(z)} \frac{\partial}{\partial n} d_E(z) \, |dz|,$$

which plainly measures in some fashion the width variation of E.

After investigation of numerous examples, one arrives at the following:

Conjecture 2.4 $\sigma_E(E) \le \frac{1}{2}$.

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We can prove the conjecture in the class of polygons inscribed in a circle. (We regard these polygons as consisting of their interior as well as the boundary.)

Theorem 2.5 *If E is a polygon that can be inscribed in a circle then $\sigma_E(E) \le \frac{1}{2}$, with equality if and only if E is a regular n-gon for some odd n.*

The disks and the regular n-gons with n odd are not the only sets with $\sigma_E(E) = \frac{1}{2}$:

Theorem 2.6 *If E is a C^2-smooth convex body of constant width, then $\sigma_E(E) = \frac{1}{2}$.*

We suspect that Theorem 2.6 could be extended to the non-smooth case by some approximation argument, though we have not yet succeeded in doing so.

Recall that there are many convex bodies of constant width (see [10, §7.6] and [2]). Perhaps the most famous one, other than the ball, is the Reuleaux triangle.

Even with the addition of the sets of constant width, we still have not identified all possible extremal sets for our conjecture. For if we continuously expand the equilateral triangle E_0 (with sidelength 1) into the corresponding Reuleaux triangle E_1, by deforming with arcs of circles whose radii vary from ∞ down to 1, then we can show (using tools from the proofs of Theorems 2.5 and 2.6) that $d_{E_\lambda} = d_{E_0}$ on E_1 and hence $\sigma_{E_\lambda}(E_\lambda) = \frac{1}{2}$, for all $\lambda \in [0, 1]$.

Perhaps the conjecture that $\sigma_E(E) \le \frac{1}{2}$ might be proved as follows. It is known that E can be "completed" to a compact convex set F, with F having constant width equal to the diameter of E (see [2, p. 61]). If one could show $\sigma_E(E) \le \sigma_F(F)$, then Theorem 2.6 would finish the proof (at least when F is smoothly bounded). Thus the question can be phrased: how does the completion procedure affect the σ-measure?

Our final proposition connects the distance function to convex geometry. Let $\operatorname{conv}(E)$ be the convex hull of E. Notice that $\operatorname{conv}(E)$ is compact because E is compact [10, p. 57]. And write $\operatorname{extr}(E)$ for the set of extreme points of $\operatorname{conv}(E)$, so that $\operatorname{extr}(E)$ is compact because $\operatorname{conv}(E)$ is compact, convex and two-dimensional [10, p. 90].

Proposition 2.7 *The following distance functions coincide:*

$$d_E = d_{\operatorname{conv}(E)} = d_{\operatorname{extr}(E)}.$$

Furthermore, $\operatorname{conv}(E)$ *is determined by d_E, by means of the following reconstruction formula:*

$$\operatorname{conv}(E) = \bigcap_{z \in \mathbb{C}} D\big(z, d_E(z)\big).$$

Thus two compact sets have the same distance function if and only if their convex hulls agree.

The proposition implies $\sigma_E = \sigma_{\operatorname{conv}(E)}$, since d_E determines σ_E uniquely. Thus one might as well assume E is convex, when trying to prove $\sigma_E(E) \le \frac{1}{2}$, because $E \subset \operatorname{conv}(E)$ and so

$$\sigma_E(E) \le \sigma_E\big(\operatorname{conv}(E)\big) = \sigma_{\operatorname{conv}(E)}\big(\operatorname{conv}(E)\big).$$

Higher Dimensions The analogue of $\log d_E(z)$ in dimension $n \geq 3$ is $-1/d_E(z)^{n-2}$. One can again develop the potential theory of this function, using Riesz potentials. In this setting, Conjecture 2.4 claims that:

$$\sigma_E(E) \leq 2^{1-n}$$

with equality holding if and only if E is a closed ball. Our results on the measure σ_E in higher dimensions will be published separately.

3 Proofs

Proof of Theorem 1.1 We will use Theorem 2.2, which does not depend on Theorem 1.1, in this alternative proof of (1.1).

We first show that the integral in (1.1) is absolutely convergent. The absolute convergence in any open disk $D^0(\zeta, 1)$ of radius 1 actually follows just by taking $z = \zeta$ in the Riesz decomposition formula for $\log d_E(z)$ in $D^0(\zeta, 1)$, and using that $\log|\zeta - t| < 0$ when $t \in D^0(\zeta, 1)$. The absolute convergence in $\mathbb{C} \setminus D^0(\zeta, 1)$ (near infinity) is less obvious, but can be deduced from the following estimate, which uses (2.3) and integration by parts:

$$\begin{aligned}
&\int_{|z-t|>1} \log|z - t|\, d\sigma_E(t) \\
&\qquad = -\int_1^\infty \log r \left(\frac{d}{dr}\sigma_E\big(\mathbb{C} \setminus D(z, r)\big)\right) dr \\
&\qquad = -\log r\sigma_E\big(\mathbb{C} \setminus D(z, r)\big)\Big|_1^\infty + \int_1^\infty r^{-1}\sigma_E\big(\mathbb{C} \setminus D(z, r)\big)\, dr \\
&\qquad = \int_1^\infty r^{-1}\sigma_E\big(\mathbb{C} \setminus D(z, r)\big)\, dr \\
&\qquad < \infty.
\end{aligned}$$

This estimate also shows that $\int_{\mathbb{C}} \log|z - t|\, d\sigma_E(t)$ defines a subharmonic function in $\mathbb{C}$, by part (b) of the Theorem in [3, §1.IV.9].

Observe that $\sigma_E(\mathbb{C}) = 1$ by (2.3). Hence

$$\begin{aligned}
&\liminf_{r\to\infty}\left(\sigma_E\big(D(0, r)\big)\log r - \frac{1}{2\pi}\int_0^{2\pi} \log d_E(re^{i\theta})\, d\theta\right) \\
&\qquad \leq \liminf_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \log\frac{r}{d_E(re^{i\theta})}\, d\theta \leq \liminf_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \log\frac{r}{r - d_E(0)}\, d\theta \leq 0.
\end{aligned}$$

It follows that the condition (9.3) of part (d) of the Theorem in [3, §1.IV.9] is satisfied, and we have the Riesz decomposition

$$\log d_E(z) = \int_{\mathbb{C}} \log|z - t|\, d\sigma_E(t) - h(z), \quad z \in \mathbb{C},$$

where $h(z)$ is the harmonic function in $\mathbb{C}$ given by

$$h(z) = \lim_{r\to\infty}\left(\sigma_E(\mathbb{C})\log r - \frac{1}{2\pi}\int_0^{2\pi}\log d_E(z+re^{i\theta})\,d\theta\right)$$
$$= \lim_{r\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{r}{d_E(z+re^{i\theta})}\,d\theta = \lim_{r\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{r}{r+O(1)}\,d\theta = 0.$$

Thus (1.1) is proved.

Suppose there exist two positive finite Borel measures σ_E and ν_E such that

$$\int_{\mathbb{C}}\log|z-t|\,d\sigma_E(t) = \log d_E(z) = \int_{\mathbb{C}}\log|z-t|\,d\nu_E(t), \quad z\in\mathbb{C},$$

with the integrals converging absolutely. Then we have for any open disk D that

$$\int_D\log|z-t|\,d\sigma_E(t) = \int_D\log|z-t|\,d\nu_E(t) + u(z), \quad z\in D,$$

where $u(z)$ is harmonic in D. Taking the Laplacian of both sides, or more precisely applying the unicity theorem for potentials [9, Theorem II.2.1], now implies that $\sigma_E = \nu_E$ and $u\equiv 0$ on D. Since D is arbitrary, we obtain that $\sigma_E = \nu_E$. ■

Proof of Proposition 2.1 (a) Fix $t\in G$ and choose $\zeta\in\partial E$ so that $|t-\zeta| = d_E(t)$. Observe that the function

$$u(z) := \log|z-\zeta| - \log d_E(z)$$

is subharmonic in G, and $u\le 0$ in G because $|z-\zeta|\le d_E(z)$ for all $z\in G$. But $u(t)=0$ and so $u\equiv 0$ in G by the strong maximum principle (*cf.* [7, p. 29]). That is, $d_E(z) = |z-\zeta|$ for all $z\in G$. Notice that $\zeta\notin G$, since $d_E(\zeta)>0$.

(b) Assume $\operatorname{supp}(\sigma_E)\ne\mathbb{C}$, so that $\operatorname{supp}(\sigma_E)\cap G=\varnothing$ for some domain G. Then by part (a) there exists $\zeta\in\partial E\setminus G$ such that $d_E(z)=|z-\zeta|$ for all $z\in G$. But the segment $[z,\zeta]$ must be orthogonal to ∂E at ζ, which is not possible for all $z\in G$. Contradiction. ■

Proof of Theorem 2.2 (a) Fix $z\in\mathbb{C}$ and denote the mean values of $\log d_E$ on circles around z by

$$M(r) := \frac{1}{2\pi}\int_0^{2\pi}\log d_E(z+re^{i\theta})\,d\theta, \quad r>0. \tag{3.1}$$

It is known (*cf.* [9, p. 85]) that the σ_E-measure of a disk can be calculated by

$$\sigma_E\big(D(z,r)\big) = r\lim_{h\to 0^+}\frac{M(r+h)-M(r)}{h} \tag{3.2}$$

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(formally this follows from integrating (1.2) and using Green's formula). Now,

$$M(r+h) - M(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{d_E\big(z+(r+h)e^{i\theta}\big)}{d_E(z+re^{i\theta})}\, d\theta, \tag{3.3}$$

and by using the Lipschitz continuity of the distance function, we obtain that

$$\left| \frac{d_E\big(z+(r+h)e^{i\theta}\big)}{d_E(z+re^{i\theta})} - 1 \right| \le \frac{h}{d_E(z+re^{i\theta})}.$$

Taylor's formula and the above estimate imply for small $h > 0$ that

$$\log \frac{d_E\big(z+(r+h)e^{i\theta}\big)}{d_E(z+re^{i\theta})} \le \frac{h}{d_E(z+re^{i\theta})} + O\left(\frac{h}{d_E(z+re^{i\theta})}\right)^2. \tag{3.4}$$

If $0 < r < d_E(z)$ then $d_E(z+re^{i\theta}) \ge d_E(z) - r > 0$, and therefore

$$\lim_{h\to 0^+} \frac{M(r+h)-M(r)}{h} \le \frac{1}{d_E(z)-r}$$

by (3.3) and (3.4). Equation (2.2) now follows from (3.2).

On the other hand, if $z \in E$ then $d_E(z+re^{i\theta}) \ge r$, and so

$$\lim_{h\to 0^+} \frac{M(r+h)-M(r)}{h} \le \frac{1}{r}$$

by the same argument. This gives $\sigma_E\big(D(z,r)\big) \le 1$ by (3.2), for all r, so that $\sigma_E(\mathbb{C}) \le 1$. The upper bound of (2.3) follows.

(b) Fix $z \in E$ and consider $r > \operatorname{diam}(E)$. Then there exists $0 < \alpha(r) < \pi/2$ such that for all $\theta \in [0, 2\pi]$, the set E is visible from $z + re^{i\theta}$ within a sector of aperture $2\alpha(r)$ and direction $-e^{i\theta}$, meaning that

$$|\arg(z+re^{i\theta}-\zeta) - \theta| \le \alpha(r) \quad \forall \zeta \in E. \tag{3.5}$$

We claim that $\alpha(r)$ can be chosen to satisfy

$$\sin\alpha(r) \le \frac{\operatorname{diam}(E)}{r}. \tag{3.6}$$

Indeed, consider a triangle formed by the points z, $z+re^{i\theta}$, and an arbitrary $\zeta \in E$. Writing $\alpha(r,\zeta) := |\arg(z+re^{i\theta}-\zeta)-\theta|$ for the angle opposite the side $[z,\zeta]$ and β for the angle opposite the side $[z+re^{i\theta}, z]$, we obtain from the Law of Sines that

$$\sin\alpha(r,\zeta) = \frac{|z-\zeta|}{|z+re^{i\theta}-z|}\sin\beta \le \frac{\operatorname{diam}(E)}{r}.$$

Furthermore, the triangle has sidelengths $A = |z - \zeta|$, $B = r$, $C = |z + re^{i\theta} - \zeta|$ and so the Law of Cosines gives

$$\begin{aligned} 2BC\cos\alpha(r,\zeta) &= B^2 + C^2 - A^2 = 2r\big(r + \Re[(z-\zeta)e^{-i\theta}]\big) \\ &\geq 2r\big(r - \operatorname{diam}(E)\big) > 0, \end{aligned}$$

so that $\alpha(r,\zeta) < \pi/2$. Therefore we can satisfy (3.6) simply by taking $\alpha(r) = \max_{\zeta\in E}\alpha(r,\zeta) < \pi/2$.

Given $\theta \in [0, 2\pi]$, assume $\zeta \in E$ is a point such that $d_E(z + re^{i\theta}) = |z + re^{i\theta} - \zeta|$. Then

$$\begin{aligned} d_E\big(z + (r+h)e^{i\theta}\big) - d_E(z + re^{i\theta}) &\geq |z + (r+h)e^{i\theta} - \zeta| - |z + re^{i\theta} - \zeta| \\ &= \frac{2h\Re[(z + re^{i\theta} - \zeta)e^{-i\theta}] + h^2}{|z + (r+h)e^{i\theta} - \zeta| + |z + re^{i\theta} - \zeta|} \\ &\geq h\cos\alpha(r) + O(h^2) \quad \text{as } h \to 0+ \end{aligned} \tag{3.7}$$

by (3.5), where the error term $O(h^2)$ is uniform in θ. Hence by Taylor's formula again,

$$\log\frac{d_E\big(z + (r+h)e^{i\theta}\big)}{d_E(z + re^{i\theta})} \geq \frac{h\cos\alpha(r)}{d_E(z + re^{i\theta})} + O(h^2) \geq \frac{h\cos\alpha(r)}{r + \operatorname{diam}(E)} + O(h^2).$$

Hence using (3.2) and (3.3), we have

$$\begin{aligned} \sigma_E\big(D(z,r)\big) &\geq \frac{r\cos\alpha(r)}{r + \operatorname{diam}(E)} \\ &> 1 - \frac{r\sin\alpha(r) + \operatorname{diam}(E)}{r + \operatorname{diam}(E)} \\ &\geq 1 - \frac{2\operatorname{diam}(E)}{r + \operatorname{diam}(E)}, \end{aligned}$$

where we also employed (3.6).

Now take $z_0 \in \mathbb{C}$. Then for all $r \geq 3|z - z_0|$ we have $D(z, 2r/3) \subset D(z_0, r)$, and so

$$\sigma_E\big(D(z_0,r)\big) \geq \sigma_E\big(D(z,2r/3)\big) \geq 1 - \frac{2\operatorname{diam}(E)}{2r/3 + \operatorname{diam}(E)} > 1 - \frac{3\operatorname{diam}(E)}{r + \operatorname{diam}(E)}$$

for all sufficiently large r, by above. This completes the proof. ∎

Proof of Corollary 2.3 When $p = 0$, the moment simply equals the total mass, $\sigma_E(\mathbb{C}) = 1$.

Suppose next that $-1 < p < 0$. Using integration by parts and (2.2), we obtain that

$$
\begin{aligned}
\int |z-t|^p \, d\sigma_E(t) &= \int_0^\infty r^p \left(\frac{d}{dr} \sigma_E\big(D(z,r)\big) \right) dr \\
&= r^p \sigma_E\big(D(z,r)\big) \Big|_0^\infty - p \int_0^\infty r^{p-1} \sigma_E\big(D(z,r)\big) \, dr \qquad (3.8)\\
&= |p| \int_0^\infty r^{p-1} \sigma_E\big(D(z,r)\big) \, dr \\
&< \infty.
\end{aligned}
$$

A similar idea works for $0 < p < 1$:

(3.9)
$$
\begin{aligned}
\int |z-t|^p \, d\sigma_E(t) &= -\int_0^\infty r^p \left(\frac{d}{dr} \sigma_E\big(\mathbb{C} \setminus D(z,r)\big) \right) dr \\
&= -r^p \sigma_E\big(\mathbb{C} \setminus D(z,r)\big) \Big|_0^\infty + p \int_0^\infty r^{p-1} \sigma_E\big(\mathbb{C} \setminus D(z,r)\big) \, dr \\
&= p \int_0^\infty r^{p-1} \sigma_E\big(\mathbb{C} \setminus D(z,r)\big) \, dr \\
&< \infty,
\end{aligned}
$$

where we have used that

$$
\sigma_E\big(\mathbb{C} \setminus D(z,r)\big) = 1 - \sigma_E\big(D(z,r)\big) = O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty,
$$

by (2.3).

Note Formulas (3.8) and (3.9) can be obtained without integrating by parts; they represent an L^p-norm in terms of a distribution function (*cf.* [8, Theorem 8.16]). ■

Proof of Theorem 2.5 Obviously $\sigma_E(E)$ is unchanged under translations and rotations of E, since these rigid motions leave distances invariant. But $\sigma_E(E)$ is also unchanged under dilations, for if $a > 0$ then Theorem 1.1 gives that for all $z \in \mathbb{C}$,

$$
\begin{aligned}
\int \log|az - v| \, d\sigma_{aE}(v) = \log d_{aE}(az) = \log\big(a d_E(z)\big) &= \log a + \log d_E(z) \\
&= \log a + \int \log|z-t| \, d\sigma_E(t) \\
&= \int \log|az - at| \, d\sigma_E(t),
\end{aligned}
$$

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so that $d\sigma_{aE}(at) = d\sigma_E(t)$, by uniqueness of σ_{aE}. In particular, $\sigma_{aE}(aE) = \sigma_E(E)$, which is the desired dilation invariance.

Therefore, it suffices to consider a polygon P inscribed into the unit circle $\{z : |z| = 1\}$. Let the vertices of P be $e^{i(\psi_k+\pi)}$ for $k = 1, \dots, n$, with $\psi_1 < \psi_2 < \cdots < \psi_n < \psi_1 + 2\pi$. For later use, we establish the convention that $\psi_{n+1} := \psi_1 + 2\pi$ and $\psi_0 := \psi_n - 2\pi$, and so on.

In Remark 2 of [1], Boyd found that σ_P is supported on the rays $L_k := \{re^{i\phi_k} : r \geq 0\}$ where

$$\phi_k = \frac{\psi_k + \psi_{k-1}}{2} \tag{3.10}$$

for $k = 1, \dots, n$. Note that

$$\psi_0 < \phi_1 < \psi_1 < \phi_2 < \psi_2 < \cdots < \phi_n < \psi_n = \psi_0 + 2\pi. \tag{3.11}$$

The density on the ray L_k is given by

$$\frac{d\sigma_P}{dr}(re^{i\phi_k}) = \frac{1}{\pi}\frac{\sin\theta_k}{r^2 + 2r\cos\theta_k + 1}, \quad r \geq 0, \tag{3.12}$$

where $\theta_k = (\psi_k - \psi_{k-1})/2$. (Note that there is a misprint in the above formula in [1]: the sign before $2r\cos\theta_k$ must be "+".) It is clear from (3.12) that the density of σ_P is symmetric with respect to the reflection $r \mapsto \frac{1}{r}$ in the unit circle, on each ray L_k. Thus

$$\sigma_P(\{re^{i\phi_k} : 0 \leq r \leq 1\}) = \sigma_P(\{re^{i\phi_k} : r \geq 1\}).$$

Since the total mass of σ_P is 1, we deduce that

$$\sigma_P(P) \leq \sigma_P\big(D(0,1)\big) = \frac{1}{2}, \tag{3.13}$$

as we wanted to prove.

For the equality case, suppose P is a regular n-gon with n odd. Then the intersection of L_k with P equals its intersection with the unit disk (because the perpendicular bisector of any side passes through the "opposite vertex" of the n-gon, as in Figure 1 on the next page).

Hence

$$\sigma_P(P) = \sigma_P\big(D(0,1)\big) = \frac{1}{2} \quad \text{when } n \text{ is odd.}$$

On the other hand, if n is even then the intersection of L_k with P is strictly smaller than its intersection with the unit disk (because the perpendicular bisector of any side passes through the "opposite side" of the n-gon). Hence

$$\sigma_P(P) < \sigma_P\big(D(0,1)\big) = \frac{1}{2} \quad \text{when } n \text{ is even.}$$

To complete the proof of the equality case, we suppose $\sigma_P(P) = \frac{1}{2}$ and prove P is a regular n-gon. Necessarily n is then odd, by above.

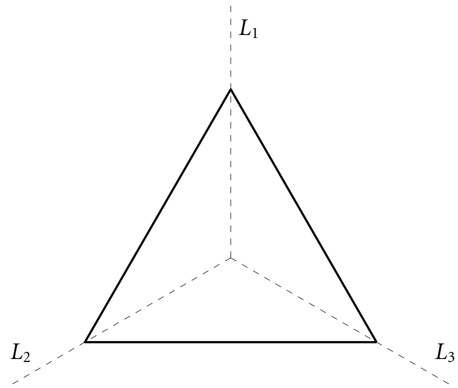

Figure 1: An equilateral triangle P, with σ_P supported on the rays $L_1,\ L_2,\ L_3$.

Observe that equality must hold at (3.13) above, since $\sigma_P(P) = \frac{1}{2}$, and so for each k the entire segment $[0, e^{i\phi_k}]$ lies in P. Thus the points $e^{i\phi_k}$ are precisely the n vertices of P. Therefore the role of the ψ_k can also be played by the numbers $\tilde{\psi}_k = \phi_k - \pi$, with the role of the ϕ_k being played by

$$\tilde{\phi}_k = \frac{\tilde{\psi}_k + \tilde{\psi}_{k-1}}{2} = \frac{\phi_k + \phi_{k-1}}{2} - \pi.$$

By the same reasoning as above, each point $e^{i\tilde{\phi}_k} = -e^{i(\phi_k+\phi_{k-1})/2}$ is a vertex of P, and that vertex must be $e^{i(\psi_{k-1}+\pi)}$, in view of (3.11). Thus

$$\psi_{k-1} = \frac{\phi_k + \phi_{k-1}}{2}$$

for $k = 1, \dots, n$. By this and (3.10), we have the averaging formulas

$$\phi_k = \frac{\psi_k + \psi_{k-1}}{2}, \quad \psi_k = \frac{\phi_{k+1} + \phi_k}{2}, \quad \phi_{k+1} = \frac{\psi_{k+1} + \psi_k}{2},$$

for $k = 1, \dots, n$. By solving the first and third equations for ψ_{k-1} and ψ_{k+1} respectively, and using the second equation for ψ_k as it stands, we can check that

$$\psi_{k+1} - \psi_k = \psi_k - \psi_{k-1}.$$

That is, the vertices are equally spaced and so P is a regular n-gon. ∎

Proof of Theorem 2.6 We first show that

$$d_E(t) = \operatorname{dist}(t, E) + \operatorname{diam}(E), \quad t \in E^c \cup \partial E, \tag{3.14}$$

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provided ∂E is C^1-smooth. Indeed, writing $\alpha \in E$ for the nearest point to t, and $\omega \in E$ for the farthest point, we have

$$d_E(t) = |t-\omega| \le |t-\alpha| + |\alpha-\omega| \le \operatorname{dist}(t,E) + \operatorname{diam}(E).$$

To get the reverse inequality, let $\zeta \in \partial E$ be a point such that the tangent lines to ∂E at ζ and α are parallel, with the distance between these tangent lines being the width of E in the normal direction. Then

$$\text{width }(E) \le |\alpha-\zeta| \le \operatorname{diam}(E).$$

But the width equals $\operatorname{diam}(E)$, since E has constant width, and so equality must hold throughout. Therefore the segment $[\alpha,\zeta]$ is actually normal to the tangent lines. Since the segment $[t,\alpha]$ is also normal, we deduce that t,α,ζ are collinear and so

$$d_E(t) \ge |t-\zeta| = |t-\alpha| + |\alpha-\zeta| = \operatorname{dist}(t,E) + \operatorname{diam}(E),$$

which proves (3.14).

Next we prove the theorem assuming ∂E is C^2-smooth. The smoothness of the boundary implies that $t \mapsto \operatorname{dist}(t,E)$ is C^2-smooth on $E^c \cup \partial E$ (see [4], [5, pp. 12 and 205] and the references therein). By (3.14), $d_E(t)$ inherits the same smoothness. Fix $z \in E$ and write by Green's theorem on E^c,

$$\begin{aligned}
\sigma_E(E^c) &= \lim_{R\to\infty} \sigma_E(E^c \cap \{|t-z|<R\}) \\
&= \lim_{R\to\infty} \int_{E^c\cap\{|t-z|<R\}} \frac{1}{2\pi}\Delta\big(\log d_E(t)\big)\, dxdy, \quad t = x+iy, \\
&= \lim_{R\to\infty} \frac{1}{2\pi}\int_{\{|t-z|=R\}} \frac{1}{d_E(t)}\frac{\partial}{\partial n} d_E(t)\,|dt| - \frac{1}{2\pi}\int_{\partial E} \frac{1}{d_E(t)}\frac{\partial}{\partial n} d_E(t)\,|dt| \\
&= \lim_{R\to\infty} \frac{1}{2\pi}\int_{\{|t-z|=R\}} \frac{1}{d_E(t)}\frac{\partial}{\partial n} d_E(t)\,|dt| - \frac{1}{2\pi}\frac{|\partial E|}{\operatorname{diam}(E)},
\end{aligned}$$

where we have used (3.14) and that the outward normal derivative of $\operatorname{dist}(t,E)$ equals 1, on ∂E. Now clearly,

$$\frac{1}{R+\operatorname{diam}(E)} \le \frac{1}{d_E(t)} \le \frac{1}{R} \quad \text{on } |t-z| = R. \tag{3.15}$$

And the Lipschitz continuity of d_E gives that

$$d_E\big(z+(R+h)e^{i\theta}\big) - d_E(z+Re^{i\theta}) \le h,$$

while from (3.6) and (3.7) we obtain that

$$d_E\big(z+(R+h)e^{i\theta}\big) - d_E(z+Re^{i\theta}) \ge h\left(1-\frac{\operatorname{diam}(E)}{R}\right) + O(h^2), \quad \text{as } h \to 0+.$$

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Hence

$$1 - \frac{\operatorname{diam}(E)}{R} \leq \frac{\partial}{\partial n} d_E(t) \leq 1 \quad \text{on } |t - z| = R,$$

and so

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{\{|t-z|=R\}} \frac{1}{d_E(t)} \frac{\partial}{\partial n} d_E(t)\, |dt| = 1,$$

using (3.15). It now follows that

$$\sigma_E(E^c) = 1 - \frac{1}{2\pi} \frac{|\partial E|}{\operatorname{diam}(E)}.$$

Finally $|\partial E| = \pi \operatorname{diam}(E)$ according to Barbier's theorem [10, Theorem 7.6.7], and so $\sigma_E(E^c) = \frac{1}{2}$, or $\sigma_E(E) = \frac{1}{2}$. ■

Proof of Proposition 2.7 Observe that $d_E \leq d_{\operatorname{conv}(E)}$ because $E \subset \operatorname{conv}(E)$. For the reverse inequality, let $\zeta_1, \dots, \zeta_m \in E$ and $t_1, \dots, t_m \in [0, 1]$ with $\sum_{j=1}^m t_j = 1$. Then

$$\Big| z - \sum_{j=1}^m t_j \zeta_j \Big| \leq \sum_{j=1}^m t_j |z - \zeta_j| \leq d_E(z), \quad \forall z \in \mathbb{C},$$

so that $d_{\operatorname{conv}(E)} \leq d_E$. Hence $d_E = d_{\operatorname{conv}(E)}$.

But $\operatorname{conv}(E)$ equals the convex hull of its extreme points by the Krein-Milman theorem [10, p. 86] (since $\operatorname{conv}(E)$ is compact and convex), and so $d_{\operatorname{conv}(E)} = d_{\operatorname{conv}(\operatorname{extr}(E))} = d_{\operatorname{extr}(E)}$ as desired.

We must still prove $F = \bigcap_{z\in\mathbb{C}} D\big(z, d_F(z)\big)$ where $F = \operatorname{conv}(E)$; notice $d_E = d_F$ by above. Now, clearly $F \subset D\big(z, d_F(z)\big)$ for all $z \in \mathbb{C}$. Suppose though that there exists $z_0 \in \bigcap_{z\in\mathbb{C}} D\big(z, d_F(z)\big)$ with $z_0 \notin F$. Since F is convex, we can find a line ℓ separating z_0 from F. Consider a line ℓ' through z_0 that is perpendicular to ℓ. One can immediately see that if $z \in \ell'$ is near infinity and is in the half-plane of $\mathbb{C} \setminus \ell$ containing F, then $F \subset D\big(z, \operatorname{dist}(z, \ell)\big)$. It follows that $d_F(z) < \operatorname{dist}(z, \ell) < |z - z_0|$. But $z_0 \in D\big(z, d_F(z)\big)$ and so $|z - z_0| \leq d_F(z)$, yielding a contradiction. ■

References

[1] D. W. Boyd, *Sharp inequalities for the product of polynomials.* Bull. London Math. Soc. **26**(1994), 449–454.

[2] G. D. Chakerian and H. Groemer, *Convex bodies of constant width.* In: "Convexity and its Applications" (eds. P. M. Gruber and J. M. Wills), Birkhäuser, Basel, 1983, 49–96.

[3] J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart.* Springer-Verlag, New York, 1984.

[4] S. G. Krantz and H. R. Parks, *Distance to C^k hypersurfaces.* J. Differential Equations **40**(1981), 116–120.

[5] ———, *The Geometry of Domains in Space.* Birkhäuser, Boston, 1999.

[6] I. E. Pritsker, *Products of polynomials in uniform norms.* Trans. Amer. Math. Soc. **353**(2001), 3971–3993.

[7] T. Ransford, *Potential Theory in the Complex Plane.* Cambridge Univ. Press, Cambridge, 1995.

[8] W. Rudin, *Real and Complex Analysis.* 3rd edition, McGraw-Hill, New York, 1987.

[9] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields.* Springer-Verlag, Heidelberg, 1997.
[10] R. Webster, *Convexity.* Oxford University Press, Oxford, 1994.

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