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AN OSCILLATION ESTIMATE TO A VARIATIONAL INEQUALITY

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We prove that solutions for elliptic equations and variational inequalities are continuous pointwisely if the obstacle is continuous pointwisely. The continuity of weakly monotone functions in a high Sobolev space is crucial. Also a comparison principle is useful in estimating oscillations of solutions.

1. INTRODUCTION

In this note, we study a pointwise continuity criterion of solutions for degenerate elliptic equations and variational inequalities.

We suppose $\Omega \subset R^n$ is a bounded domain and the obstacle ψ is in $W^{1,p}(\Omega)$. Here we assume that $n-1 < p \leqslant n$. We let the boundary data $u_0 \geqslant \psi$ and define $K = \{v \in W_0^{1,p}(\Omega) + u_0\,;\ v \geqslant \psi\}$. We say $u \in K$ is a solution to the variational inequality

$$-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) \geqslant 0$$

with respect to K if

$$\int |\nabla u|^{p-2}\,\nabla u \cdot (\nabla v - \nabla u)\,dx \geqslant 0$$

for all $v \in K$. Obstacle problems like this arise in many area such as optimal control, elasticity, et cetera. In fact there have been many studies on various aspects of these problems. In the case of degenerate obstacle problems several authors have shown that the solution is regular under various assumptions on the operators and obstacles. (See [**1, 2, 3, 4, 6, 7**].) We note that a fine pointwise analysis at a contact point was done by Michael and Ziemer [**6**]. Indeed they assumed that the obstacle is upper semicontinuous and hence the solution is lower semicontinuous.

Here we estimate the oscillation of the solution u in terms of the oscillation of the obstacle and the L^p energy. First we state interior oscillation estimates. Suppose $-\Delta_p u \equiv -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) \geqslant 0$ with respect to K.

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Theorem 1. (Interior Continuity) *There exists a set S with p-capacity zero such that for all $x_0 \in \Omega \backslash S$ and $B_R(x_0) \subset \Omega$*

$$\operatorname*{osc}_{B_\rho(x_0)} u \leqslant C_{n,p}\Big[\frac{1}{\log(R/\rho)}\int_\rho^R r^{p-n}\int_{\partial B_r(x_0)} |\nabla u|^p \, d\sigma_x dr\Big]^{1/p} + C_{n,p}\Big[\frac{1}{\log(R/\rho)}\int_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)} \psi\Big)^p dr\Big]^{1/p},$$

for all $0 < \rho < R$ and for some $C_{n,p}$ depending only on n and p.

The following theorem estimates the oscillation of solutions of obstacle problems at the boundary.

Theorem 2. (Boundary Continuity) *There exists a set $S \subset \partial\Omega$ with p-capacity zero such that for all $x_0 \in \partial\Omega \backslash S$ and $\rho < R$*

$$\operatorname*{osc}_{B_\rho(x_0)\cap\Omega} u \leqslant C_{n,p} \operatorname*{osc}_{\partial\Omega\cap B_R(x_0)} u_0 + \Big[\frac{C_{n,p}}{\log(R/\rho)}\int_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)\cap\Omega} \psi\Big)^p dr\Big]^{1/p} + \Big[\frac{C_{n,p}}{\log(R/\rho)}\int_\rho^R r^{p-n}\int_{\partial B_r(x_0)\cap\Omega} |\nabla u|^p \, d\sigma_x \, dr\Big]^{1/p}.$$

To prove our continuity theorems we employ the idea of weakly monotone functions and estimate the oscillation of Sobolev functions in terms of their L^p energy. Indeed, this method was used by Manfredi [5] in proving that functions of bounded dilations are continuous except on a p-capacity zero set. Here the assumption $p > n - 1$ is important.

2. Estimate of Oscillation

We define weakly monotone functions.

Definition 1. *Let Ω be an open set in R^n and $f : \Omega \to R$ be a function in the space $W^{1,p}_{\text{loc}}(\Omega)$. We say that f is* weakly monotone *if for every relatively compact subdomain Ω' of Ω and for every pair of constants $m \leqslant M$ such that*

$$(m - f)^+ \in W_0^{1,p}(\Omega') \text{ and } (f - M)^+ \in W_0^{1,p}(\Omega'),$$

we have

$$m \leqslant f(x) \leqslant M \quad \text{for almost all } x \in \Omega'. \tag{1}$$

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Since functions in $W^{1,p}_{\mathrm{loc}}(\Omega)$ are p-quasi-continuous and two p-quasi-continuous functions that agree almost everywhere, also agree except in a set of p-capacity zero, we see that (1) holds for any weakly monotone function f except in a set S of p-capacity zero.

First we consider a Poisson type equation and prove a continuity theorem. We take $x_0 \in \Omega$ and $R > 0$ such that $\overline{B_R(x_0)} \subset \Omega$. Suppose that $v \in W_0^{1,p}\big(B_R(x_0)\big) + v_0$ is a solution to

$$\operatorname{div}\big(|\nabla u|^{p-2}\,\nabla u\big) = \operatorname{div}\big(|\nabla\psi|^{p-2}\,\nabla\psi\big),$$

where ψ is the given obstacle and v_0 is a boundary data on $\partial B_R(x_0)$.

Lemma 1. *For all $\rho < R$ we have*

$$\operatorname*{osc}_{B_\rho(x_0)} v \leqslant \operatorname*{osc}_{B_\rho(x_0)} \psi + \left[\frac{C_{n,p}}{\log(R/\rho)} \int_\rho^R r^{p-n} \int_{\partial B_r(x_0)} |\nabla v|^p + |\nabla\psi|^p \, d\sigma_x dr\right]^{1/p},$$

where $C_{n,p}$ is an absolute constant depending only on n and p.

Proof: A consequence of p-quasi-continuity for $p > n-1$ is that $v-\psi$ is continuous on $\partial B_r(x_0)$ for $r \in (0,R)\backslash E(x_0,R)$, where the measure of $E(x_0,R)$ is zero. Take $r \notin E(x_0,R)$. Let $M_r = \inf\Big\{M : (v-\psi-M)^+ \in W_0^{1,p}\big(B_r(x_0)\big)\Big\}$. We take $(v-\psi-M_r)^+ \in W_0^{1,p}\big(B_r(x_0)\big)$ as a test function in

$$\operatorname{div}\big(|\nabla v|^{p-2}\,\nabla v - |\nabla\psi|^{p-2}\,\nabla\psi\big) = 0$$

and find that from the monotonicity of the operator $-\Delta_p$

$$(v-\psi-M_r)^+ \equiv 0 \text{ in } B_r(x_0) \text{ and } v-\psi \leqslant M_r \text{ almost everywhere.}$$

Similarly we can prove that

$$v - \psi \geqslant m_r \text{ almost everywhere,}$$

where $m_r = \sup\Big\{m : \big[m-(v-\psi)\big]^+ \in W_0^{1,p}\big(B_r(x_0)\big)\Big\}$. Consequently $v-\psi$ is weakly monotone and hence

$$\operatorname*{osc}_{B_r(x_0)}(v-\psi) \leqslant \operatorname*{osc}_{\partial B_r(x_0)}(v-\psi).$$

In fact,

$$m_r \leqslant v(x) - \psi(x) \leqslant M_r$$

for $x \in B_R(x_0)\backslash F(x_0,R)$, where $F(x_0,R)$ has p-capacity zero. Therefore, the Hausdorff dimension of $F(x_0,R)$ is at most $n-p < 1$. It follows immediately that $\partial B_r(x_0)\backslash F(x_0,R)$ is dense in $\partial B_r(x_0)$ and thus m_r is nonincreasing and M_r is nondecreasing in $(0,R)\backslash E(x_0,R)$, where $E(x_0,R)$ is of measure zero. Therefore, $\operatorname{osc}_{\partial B_r(x_0)}(v-\psi)$ is nondecreasing in $(0,R)\backslash E(x_0,R)$.

As in Manfredi [5] we have Gehring's embedding theorem

$$\Big(\operatorname*{osc}_{\partial B_r(x_0)}(v-\psi)\Big)^p \leqslant C_{n,p} r^{p-n+1} \int\limits_{\partial B_r(x_0)} |\nabla v - \nabla \psi|^p \, d\sigma_x$$

for $n \geqslant p > n-1$, where $C_{n,p}$ is an absolute constant depending only on n and p. This estimate follows from the Sobolev embedding theorem on the sphere. Hence we have

$$\int\limits_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{\partial B_r(x_0)}(v-\psi)\Big)^p dr \leqslant C_{n,p} \int\limits_\rho^R r^{p-n} \int\limits_{\partial B_r(x_0)} |\nabla v|^p + |\nabla \psi|^p \, d\sigma_x \, dr.$$

Thus we have

$$\Big(\operatorname*{osc}_{\partial B_\rho}(v-\psi)\Big)^p \leqslant \frac{C_{n,p}}{\log(R/\rho)} \int\limits_\rho^R r^{p-n} \int\limits_{\partial B_r(x_0)} |\nabla u|^p + |\nabla \psi|^p \, d\sigma_x \, dr,$$

and from weak monotonicity of $(v-\psi)$

$$\operatorname*{osc}_{B_\rho} v \leqslant \operatorname*{osc}_{B_\rho} \psi + \Big[\frac{C_{n,p}}{\log(R/\rho)} \int\limits_\rho^R r^{p-n} \int\limits_{\partial B_R} |\nabla v|^p + |\nabla \psi|^p \, d\sigma_x \, dr\Big]^{1/p}.$$

□

Let S denote the set of those $x_0 \in \Omega$ for which

$$\int\limits_0^R r^{p-n} \int\limits_{\partial B_r(x_0)} |\nabla v|^p \, d\sigma_x \, dr = +\infty. \tag{2}$$

Note that S is empty for $p = n$ and the p-capacity of S is zero for $p > n-1$. Refer to Manfredi [5] for the proof.

As result of this lemma we prove that if ψ is continuous at x_0, then v is also continuous at p-capacity almost every x_0. Since there are many $W^{1,p}$ function for $n-1 < p \leqslant n$ which are not continuous, the continuity assumption on ψ is necessary.

Now we consider the obstacle problem. Here we employ a perturbation technique which is used in Choe [1].

LEMMA 2. *Suppose $-\Delta_p u \geqslant 0$ with respect to K. Then for all $B_R(x_0) \subset \Omega$, and $\rho < R$, we have*

$$\operatorname*{osc}_{B_\rho} u \leqslant C_{n,p}\Big[\frac{1}{\log(R/\rho)} \int\limits_\rho^R r^{p-n} \int\limits_{\partial B_r(x_0)} |\nabla u|^p \, d\sigma_x \, dr\Big]^{1/p} + C_{n,p}\Big[\frac{1}{\log(R/\rho)} \int\limits_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)} \psi\Big)^p dr\Big]^{1/p}.$$

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PROOF: Let v be the solution to $\Delta_p v = \Delta_p \psi$ and $v = u$ on $\partial B_r(x_0)$. Since u is a supersolution to $\Delta_p u = 0$, we have

$$\inf_{B_r(x_0)} u = \inf_{\partial B_r(x_0)} u \geqslant \inf_{B_r(x_0)} v.$$

Let $w = u - \Big[u - \sup_{B_r(x_0)} v\Big]^+$. Then $w = u$ on $\partial B_r(x_0)$ and $w(x) \geqslant \psi(x)$ for all $x \in B_r(x_0)$.

Hence w is a legitimate test function and

$$0 \leqslant \int |\nabla u|^{p-2} \nabla u \cdot (\nabla w - \nabla u) = - \int_{u \geqslant \sup_{B_r(x_0)} v} |\nabla u|^p \, d\sigma_x.$$

Thus we have

$$\sup_{B_r(x_0)} u \leqslant \sup_{B_r(x_0)} v$$

and

$$\operatorname*{osc}_{B_r(x_0)} u \leqslant \operatorname*{osc}_{B_r(x_0)} v.$$

Therefore, we have

$$\begin{aligned} \operatorname*{osc}_{B_r(x_0)} v &\leqslant \operatorname*{osc}_{\partial B_r(x_0)} (v - \psi) + \operatorname*{osc}_{B_r(x_0)} \psi \\ &\leqslant \operatorname*{osc}_{\partial B_r(x_0)} u + 2 \operatorname*{osc}_{B_r(x_0)} \psi \end{aligned}$$

and

$$\begin{aligned} \operatorname*{osc}_{B_r(x_0)} u &\leqslant \operatorname*{osc}_{\partial B_r(x_0)} (u - \psi) + \operatorname*{osc}_{B_r(x_0)} \psi \\ &\leqslant \operatorname*{osc}_{\partial B_r(x_0)} u + 2 \operatorname*{osc}_{B_r(x_0)} \psi. \end{aligned}$$

By Gehring's theorem, we have

$$\Big(\operatorname*{osc}_{B_r(x_0)} v\Big)^p \leqslant C_{n,p} r^{p-n+1} \int_{\partial B_r(x_0)} |\nabla u|^p \, d\sigma_x + C_{n,p} \Big(\operatorname*{osc}_{B_r(x_0)} \psi\Big)^p,$$

where $C_{n,p}$ depends only on n and p. So integrating with respect to r from ρ to R

$$\int_\rho^R \frac{1}{r} \Big(\operatorname*{osc}_{B_r(x_0)} u\Big)^p dr \leqslant C_{n,p} \int_\rho^R r^{p-n} \int_{\partial B_r(x_0)} |\nabla u|^p \, d\sigma_x \, dr + C_{n,p} \int_\rho^R \frac{1}{r} \Big(\operatorname*{osc}_{B_r(x_0)} \psi\Big)^p dr$$

and

$$\operatorname*{osc}_{B_\rho} u \leqslant \left[\frac{C_{n,p}}{\log(R/\rho)} \int_\rho^R r^{p-n} \int_{\partial B_r(x_0)} |\nabla u|^p \, d\sigma_x \, dr\right]^{1/p} + \left[\frac{C_{n,p}}{\log(R/\rho)} \int_\rho^R \frac{1}{r} \Big(\operatorname*{osc}_{B_r(x_0)} \psi\Big)^p dr\right]^{1/p}.$$

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$\square$

Since the set S of $x_0 \in \Omega$ satisfying (2) is of p-capacity zero, we have Theorem 1.

To study the boundary behaviour of solutions to variational inequalities and elliptic equations, we need the following lemma.

Lemma 3. *Suppose $\partial\Omega$ is Lipschitz and $x_0 \in \partial\Omega$. Then*

$$\Big(\operatorname*{osc}_{\partial B_r(x_0)\cap\Omega} u\Big)^p \leqslant C_{n,p} r^{p-n+1} \int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla u|^p \, d\sigma_x.$$

Once we have the oscillation lemma we need to show weak monotonicity of solutions to variational inequalities and elliptic equations.

Lemma 4. *Suppose v is a solution to $\Delta_p v = \Delta_p \psi$ in $B_R(x_0) \cap \Omega$, where $x_0 \in \partial\Omega$. Then, for all $\rho \leqslant R$,*

$$\begin{aligned}\operatorname*{osc}_{B_\rho\cap\Omega} v \leqslant \operatorname*{osc}_{B_\rho\cap\Omega} \psi + \Big[\frac{C_{n,p}}{\log(R/\rho)} \int_\rho^R r^{p-n} \int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla v - \nabla\psi|^p \, d\sigma_x\Big]^{1/p} \\ + C_{n,p} \operatorname*{osc}_{B_R\cap\partial\Omega} v + \Big[\frac{C_{n,p}}{\log(R/\rho)} \int_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_\rho\cap\partial\Omega} \psi\Big)^p dr\Big]^{1/p}.\end{aligned}$$

Proof: Let $M_r = \inf\Big\{M : (v-\psi-M)^+ \in W_0^{1,p}\big(B_r(x_0)\cap\Omega\big)\Big\}$, then as in the proof Lemma 1 we get

$$v - \psi \leqslant M_r.$$

We also have

$$v - \psi \geqslant m_r,$$

where $m_r = \sup\big\{m : [m - (v-\psi)]^+ \in W_0^{1,p}(B_r(x_0)\cap\Omega)\big\}$. So we have

$$\operatorname*{osc}_{B_r(x_0)\cap\Omega}(v-\psi) \leqslant \operatorname*{osc}_{\partial B_r(x_0)\cap\Omega}(v-\psi) + \operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}(v-\psi).$$

Again from Lemma 3

$$\Big(\operatorname*{osc}_{B_r(x_0)\cap\Omega}(v-\psi)\Big)^p \leqslant C_{n,p} r^{p-n+1} \int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla v - \nabla\psi|^p \, d\sigma_x + \Big(\operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}(v-\psi)\Big)^p.$$

Hence, dividing by r and integrating from ρ to R, we conclude that

$$\begin{aligned}\int\limits_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)\cap\Omega}(v-\psi)\Big)^p dr \leqslant C_{n,p} \int\limits_\rho^R r^{p-n} \int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla v - \nabla\psi|^p \, d\sigma_x \\ + C_{n,p} \int\limits_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}(v-\psi)\Big)^p dr,\end{aligned}$$

and since

$$\begin{aligned}\operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}(v-\psi) &\leqslant \operatorname*{osc}_{B_r(x_0)\cap\partial\Omega} v + \operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}\psi\\ &\leqslant \operatorname*{osc}_{B_R\cap\partial\Omega} v + \operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}\psi,\end{aligned}$$

we have

$$\begin{aligned}\operatorname*{osc}_{B_\rho\cap\Omega} v \leqslant \operatorname*{osc}_{B_\rho\cap\Omega}\psi &+ \left[\frac{C_{n,p}}{\log(R/\rho)}\int_\rho^R r^{p-n}\int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla v - \nabla\psi|^p\, d\sigma_x\right]^{1/p}\\ &+ C_{n,p}\operatorname*{osc}_{B_R\cap\partial\Omega} v + \left[\frac{C_{n,p}}{\log(R/\rho)}\int_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)\cap\partial\Omega}\psi\Big)^p dr\right]^{1/p}.\end{aligned}$$

□

Finally, we consider the obstacle problems on the boundary.

Lemma 5. *Suppose u is a solution to the obstacle problem $-\Delta_p u \geqslant 0$ with respect to $K = \left\{w : w \in W_0^{1,p}(\Omega) + u_0 \text{ and } w \geqslant \psi\right\}$. Let $x_0 \in \partial\Omega$, then*

$$\begin{aligned}\operatorname*{osc}_{B_\rho\cap\Omega} u \leqslant C_{n,p}\operatorname*{osc}_{\partial\Omega\cap B_R} u_0 &+ \left[\frac{C_{n,p}}{\log(R/\rho)}\int\limits_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)\cap\Omega}\psi\Big)^p dr\right]^{1/p}\\ &+ \left[\frac{C_{n,p}}{\log(R/\rho)}\int\limits_\rho^R r^{p-n}\int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla u|^p\, d\sigma_x\, dr\right]^{1/p}.\end{aligned}$$

Proof: We know that u is a super solution and hence

$$\inf_{B_r(x_0)\cap\Omega} u \geqslant \inf_{\partial(\Omega\cap B_r(x_0))} u.$$

We let v be the solution to $\Delta_p v = \Delta_p\psi$ in $B_r(x_0)\cap\Omega$ and $v = u$ on $\partial\big(B_r(x_0)\cap\Omega\big)$.

As in the case of Lemma 2 we find that

$$\sup_{B_r(x_0)\cap\Omega} u \leqslant \sup_{B_r(x_0)\cap\Omega} v.$$

So we have

$$\operatorname*{osc}_{B_r(x_0)\cap\Omega} u \leqslant \operatorname*{osc}_{B_r(x_0)\cap\Omega} v \leqslant \operatorname*{osc}_{\partial\Omega\cap B_r(x_0)} u_0 + \operatorname*{osc}_{\Omega\cap\partial B_r(x_0)} u + 2\operatorname*{osc}_{\Omega\cap B_r(x_0)}\psi.$$

Notice that

$$\begin{aligned}\Big(\operatorname*{osc}_{B_r(x_0)\cap\Omega} u\Big)^p &\leqslant C_{n,p}\Big(\operatorname*{osc}_{\partial\Omega\cap B_r(x_0)} u_0\Big)^p + C_{n,p}\Big(\operatorname*{osc}_{\Omega\cap\partial B_r(x_0)} u\Big)^p \\ &\quad + C_{n,p}\Big(\operatorname*{osc}_{\Omega\cap B_r(x_0)} \psi\Big)^p \\ &\leqslant C_{n,p}\Big(\operatorname*{osc}_{\partial\Omega\cap B_r(x_0)} u_0\Big)^p + C_{n,p} r^{p-n+1} \int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla u|^p \, d\sigma_x \\ &\quad + C_{n,p}\Big(\operatorname*{osc}_{\Omega\cap B_r(x_0)} \psi\Big)^p.\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\Big(\operatorname*{osc}_{B_\rho(x_0)\cap\Omega} u\Big)^p &\leqslant C_{n,p}\Big(\operatorname*{osc}_{\partial\Omega\cap B_R(x_0)} u_0\Big)^p + \frac{C_{n,p}}{\log(R/\rho)}\int_\rho^R \frac{1}{r}\Big(\operatorname*{osc}_{B_r(x_0)\cap\Omega} \psi\Big)^p dr \\ &\quad + \frac{C_{n,p}}{\log(R/\rho)}\int_\rho^R r^{p-n} \int\limits_{\partial B_r(x_0)\cap\Omega} |\nabla u|^p \, d\sigma_x \, dr.\end{aligned}$$

□

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