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DETERMINATION OF BRAUER CHARACTERS

B. M. PUTTASWAMAIAH

The purpose of this note is to show that the values of an irreducible (Brauer) character are the characteristic values of a matrix with non-negative rational integers. The construction of these integral matrices is done by a description of a representation of the Grothendieck ring of the category of modules over the group algebra. In particular a result of Solomon on characters and a result of Burnside on vanishing of a non-linear character on some conjugate class are generalized.

Let G be a finite group of order g, A a splitting field of characteristic p (which may be 0) for G, $R = AG$ the group algebra of G over A, n the number of distinct p-regular classes of G and $\mathscr{C}$ the category of all finite dimensional (right) R-modules. Then the isomorphism of R-modules is an equivalence relation in $\mathscr{C}$. The equivalence class determined by an R-module M in $\mathscr{C}$ will be denoted by $\bar{M}$. If $T = \{\bar{M} | M \in \mathscr{C}\}$, then the Grothendieck group $K(R)$ is defined to be the additive abelian group generated by T subject to the defining relations $\bar{M} = \bar{N} + \bar{P}$ whenever

$$0 \to N \to M \to P \to 0$$

is an exact sequence of R-modules. The Grothendieck group $K(R)$ is a free abelian group of rank n [**6**]. In fact if $\{M_1, M_2, \ldots, M_n\}$ is a full set of pair-wise non-isomorphic irreducible R-modules, then $\{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\}$ is a basis of $K(R)$ over the ring Z of integers.

If M and N are any modules in $\mathscr{C}$, then we denote their inner tensor product over A by $M \otimes N$. Obviously $M \otimes N$ is a member of $\mathscr{C}$. If we define $\bar{M}\bar{N} = \overline{M \otimes N}$, then $K(R)$ becomes a commutative associative ring with unity $\bar{I}$, where I is the R-module which affords the principal irreducible representation; that is the one-dimensional representation of G in which every element of G is mapped on the identity. The ring $K(R)$ is called the Grothendieck ring of the category $\mathscr{C}$ of R-modules. The integer n will be called the rank of the ring $K(R)$.

THEOREM 1. *Let $K(R)$ be the Grothendieck ring of the category of R-modules, n the rank of $K(R)$ and Z_n the ring of n by n matrices over the ring Z of integers. Then there is a monomorphism of $K(R)$ into Z_n.*

Proof. Let $\{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\}$ be a fixed basis of $K(R)$ over Z. Then every element of $K(R)$ can be written uniquely in the form $\sum_{i=1}^{n} a_i\bar{M}_i$ where a_i

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belongs to Z. Each element $\bar{M}$ of $K(R)$ determines a unique matrix $(a_{ij}{}^M)$ of Z_n by

$$\bar{M}_i\bar{M} = \sum_{j=1}^{n} a_{ij}{}^M \bar{M}_j$$

for $i = 1, 2, \ldots, n$. Then we show that the mapping f from $K(R)$ to Z_n defined by $\bar{M}f = (a_{ij}{}^M)$ is a ring monomorphism.

Suppose that $\bar{M}$ and $\bar{N}$ are arbitrary elements of $K(R)$. Then $\bar{M}_i\bar{M} = \sum_{j=1}^{n} a_{ij}{}^M\bar{M}_j$ and $\bar{M}_i\bar{N} = \sum_{j=1}^{n} a_{ij}{}^N\bar{M}_j$ for all i. Therefore we have

$$\begin{aligned}\bar{M}_i(\bar{M} + \bar{N}) &= \bar{M}_i\overline{M \oplus N} = \overline{M_i \otimes (M \oplus N)} = \overline{M_i \otimes M \oplus M_i \otimes N} \\ &= \overline{M_i \otimes M} + \overline{M_i \otimes N} = \sum_{j=1}^{n} a_{ij}{}^M M_j + \sum_{j=1}^{n} a_{ij}{}^N \bar{M}_j \\ &= \sum_{j=1}^{n} (a_{ij}{}^M + a_{ij}{}^N)\bar{M}_j\end{aligned}$$

so that $(\bar{M} + \bar{N})f = (a_{ij}{}^M + a_{ij}{}^N) = (a_{ij}{}^M) + (a_{ij}{}^N) = \bar{M}f + \bar{N}f$. Hence f is a group homomorphism. Also we have

$$\begin{aligned}\bar{M}_i(\bar{M}\bar{N}) &= \bar{M}_i(\overline{M \otimes N}) = \overline{M_i \otimes (M \otimes N)} = (\overline{M_i \otimes M) \otimes N} \\ &= \overline{M_i \otimes M}\,\bar{N} = \sum_{k=1}^{n} a_{ik}{}^M \bar{M}_k\bar{N} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}{}^M a_{kj}{}^N\right)\bar{M}_j\end{aligned}$$

from which it follows that $(\bar{M}\bar{N})f = \bar{M}f\bar{N}f$. Thus we have proved that f is a ring homomorphism.

It remains to show that the homomorphism f is injective. If $\bar{M}$ is any element in the kernel of f, then $\bar{M}f = 0$. There are unique integers a_i in Z such that

$$\bar{M} = \sum_{j=1}^{n} a_i\bar{M}_i.$$

Let $\bar{I}$ be the identity of $K(R)$. Since $\bar{M}_i\bar{M} = 0$ for all i, it follows that $\bar{I}\bar{M} = 0$; that is

$$\sum_{i=1}^{n} a_i\bar{I}\bar{M} = \sum_{i=1}^{n} a_i\bar{M}_i = 0.$$

Since $\{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\}$ is a basis of $K(R)$, it follows that $\bar{M} = \bar{0}$. Hence f is injective. This completes the proof.

This proposition is a generalization for representations over an arbitrary field of a similar result proved earlier by Robinson [**4**; **5**] for ordinary representations.

With each element $\bar{M}$ of $K(R)$, we associate an n-tuple

$$\chi^M = (\chi_1{}^M, \chi_2{}^M, \ldots, \chi_n{}^M)$$

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of complex numbers, where $\chi_1^M, \chi_2^M, \ldots, \chi_n^M$ are the characteristic values of the matrix (a_{ij}^M). Clearly χ^M is independent of a basis of $K(R)$ over Z and it is uniquely determined by $\bar{M}$.

The irreducible Brauer characters of G will be denoted by $\varphi^1, \varphi^2, \ldots, \varphi^n$ (in some fixed order) and the p-regular classes of G will be denoted by $C_1 = \{1\}$, $C_2, \ldots, C_n$ (in some fixed order). If $p = 0$ or prime to the order g of G, then every class of G is p-regular. The Brauer character φ^i can also be considered as an n-tuple $\varphi^i = (\varphi_1^i, \varphi_2^i, \ldots, \varphi_n^i)$ where $\varphi_j^i = \varphi^i(x)$ with x in C_j. The $n \times n$ matrix $\Phi = (\varphi_j^i)$ whose rows are φ^i, $i = 1, 2, \ldots, n$ is called the Brauer character table. Since the rows of Φ are linearly independent [**2**], the matrix is a non-singular (complex) matrix. Without loss of generality, we may assume that φ^1 is the principal character of G.

THEOREM 2. *Let M be an R-module, n the rank of $K(R)$ and χ^M the n-tuple associated with $\bar{M}$. Then there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $(\chi_{1\sigma}^M, \chi_{2\sigma}^M, \ldots, \chi_{n\sigma}^M)$ is a Brauer character of G.*

Proof. Let $\{M_1, M_2, \ldots, M_n\}$ be a set of pairwise non-isomorphic irreducible R-modules such that $\{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\}$ is a basis of $K(R)$. Then

$$\bar{M} = \textstyle\sum_{j=1}^n d_j \bar{M}_j$$

for some non-negative integers d_j. Let

$$\varphi^M = \sum_{j=1}^n d_j \varphi^j.$$

Then we shall show that the components of φ^M will coincide with those of χ^M (except possibly for a permutation).

Let $M = N_1 \supset \ldots \supset N_t \supset N_{t+1} = 0$ be a composition series of M. Then

$$\bar{M} = \overline{\bigoplus_1^t N_i/N_{i+1}} = \sum_{i=1}^t \overline{N_i/N_{i+1}} = \sum_{j=1}^n d_j \bar{M}_j.$$

Therefore φ^M depends only on $\bar{M}$ and φ^M is the Brauer character associated with the R-module M. By [**1**, p. 577], it follows that $\varphi^i \varphi^M$ depends only on $\overline{M_i \otimes M} = \bar{M}_i \bar{M}$ and $\varphi^i \varphi^M$ is the Brauer character associated with $M_i \otimes M$. Therefore the equations

$$\bar{M}_i \bar{M} = \sum_{j=1}^n a_{ij}^M \bar{M}_j, \qquad i = 1, 2, \ldots, n$$

are equivalent to the equations

$$\varphi_t^i \varphi_t^M = \sum_{j=1}^n a_{ij}^M \varphi_t^j$$

for $i, t = 1, 2, \ldots, n$. If D^M is the diagonal matrix whose diagonal entries are $\varphi_1^M, \varphi_2^M, \ldots, \varphi_n^M$, then the above equations can be written in the matrix form $\Phi D^M = (a_{ij}{}^M)\Phi$ where Φ is the Brauer character table. Hence $\Phi^{-1}(a_{ij}{}^M)\Phi = D^M$, so that there is a permutation σ of $\{1, 2, \ldots, n\}$ such that

$$\varphi^M = (\chi_{1\sigma}{}^M \chi_{2\sigma}{}^M, \ldots, \chi_{n\sigma}{}^M)$$

where $i\sigma$ is the image of i under σ. This completes the proof.

COROLLARY. *For any Brauer character φ^i, the quantity $\sum_{j=1}^n |\varphi_j{}^i|^2$ is a positive integer.*

PROPOSITION 3. *The t-th column c_t of the Brauer character table Φ is a characteristic vector of $(a_{ij}{}^k)$ corresponding to the characteristic value $\varphi_t{}^k$ for $k = 1, 2, \ldots, n$ where $(a_{ij}{}^k)$ is the matrix corresponding to the element $\bar{M}_k$ of $K(R)$ determined by an irreducible R-module M_k.*

Proof. This follows by rewriting the equations

$$\varphi_t{}^i \varphi_t{}^k = \sum_{k=1}^n a_{ij}{}^k \varphi_t{}^j$$

for $i, k = 1, 2, \ldots, n$ in the matrix form $(a_{ij}{}^k)c_t = \varphi_t{}^k c_t$ for $t = 1, 2, \ldots, n$. (Cf. [**5**].)

LEMMA 4. *Let M_k be an irreducible R-module of dimension f, $(a_{ij}{}^k)$ the matrix corresponding to $\bar{M}_k$ of $K(R)$ and $\omega_1, \omega_2, \ldots, \omega_r$ the non-zero characteristic values of $(a_{ij}{}^k)$. Then $\omega_1 + \omega_2 + \ldots + \omega_r$ is a non-negative rational integer and $\omega_1\omega_2 \ldots \omega_r/f$ is a rational integer.*

Proof. Let $\det(a_{ij}{}^k - \lambda\delta_{ij})$ be the characteristic polynomial of the matrix $(a_{ij}{}^k)$. Since $a_{ij}{}^k$ are integers, the coefficients of $\det(a_{ij}{}^k - \lambda\delta_{ij})$ are rational integers. Then $-(\omega_1 + \omega_2 + \ldots + \omega_r)$ is the coefficient of λ^{n-1} and $\pm\, \omega_1\omega_2, \ldots, \omega_r$ is the coefficient of λ^{r-1}. Hence they are rational integers. But $\omega_1 + \omega_2 + \ldots + \omega_r$ is the sum of the diagonal entries of the matrix $(a_{ij}{}^k)$ and hence it is a non-negative integer.

COROLLARY 1. *Let φ^M be any Brauer character of G and $\varphi_1{}^M, \varphi_2{}^M, \ldots, \varphi_r{}^M$ the non-zero values of φ^M. Then $\sum_{i=1}^r \varphi_i{}^M$ is a non-negative integer and $\varphi_2{}^M \ldots \varphi_r{}^M$ is a non-zero integer.*

COROLLARY (Solomon [7]) 2. *If A is the complex field, then $\sum_{i=1}^n \varphi_i{}^M$ is a rational integer.*

THEOREM 5. *Let the characteristic p of A be either 0 or prime to $g = |G|$, M_k an irreducible R-module and $(a_{ij}{}^k)$ the corresponding matrix of $\bar{M}_k$ of $K(R)$. Then $(a_{ij}{}^k)$ is non-singular if and only if the dimension of M_k is 1.*

Proof. If the dimension of M_k is 1, then $\det(a_{ij}{}^k) = \omega_1\omega_2 \ldots \omega_n = 1 \neq 0$, where $\omega_1, \omega_2, \ldots, \omega_n$ are the characteristic values of $(a_{ij}{}^k)$. Hence $(a_{ij}{}^k)$ is non-singular.

Conversely assume that $(a_{ij}{}^k)$ is non-singular. Let ζ^k be the Brauer character of G corresponding to M_k. Then by Theorem 2, we may assume that

$\zeta^k = (\zeta_1^k, \zeta_2^k, \ldots, \zeta_n^k)$ where $\zeta_1^k = \dim M_k$, $\zeta_2^k, \ldots, \zeta_n^k$ are the characteristic values of the matrix (a_{ij}^k). Since the arithmetic mean of n positive real numbers is greater than or equal to the geometric mean, we obtain from the orthoganality relations [**2**] for characters, that

$$g = \sum_{i=1}^{n} g_i|\zeta_i^k|^2 = (\zeta_1^k)^2 + \sum_{i=2}^{n} g_i|\zeta_i^k|^2$$
$$\geqslant (\zeta_1^k)^2 + (g-1)(|\zeta_2^k|^{2g_2}|\zeta_3^k|^{2g_3}\ldots|\zeta_n^k|^{2g_n})^{1/g-1}$$

where g_i is the number of elements in the ith conjugate class of G. By Lemma 4, the above inequality implies $g \geqq (\zeta_1^k)^2 + g - 1$ so that $\zeta_1^k = 1$. Hence the dimension of M_k is 1.

COROLLARY (Burnside) 1. *If $p = 0$ and ζ^k is an irreducible character of degree > 1, then $\zeta_i^k = 0$ for some conjugate class C_i.*

COROLLARY 2. *Let $p = 0$ or $(p, g) = 1$. If $M_1, M_2, \ldots, M_n$ are pairwise non-isomorphic irreducible R-modules, then the number of units in the basis $\{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\}$ of $K(R)$ is equal to the order of G/G' where G' is the commutator subgroup of G.*

COROLLARY 3. *Let $p = 0$ or $(p, g) = 1$, M_k an irreducible R-module and (a_{ij}^k) the corresponding matrix of $\bar{M}_k$ of $K(R)$. Then (a_{ij}^k) is invertible if and only if (a_{ij}^k) is a permutation matrix.*

The Corollary (Burnside) 1 is not true if p is a factor of $g = |G|$. By this we mean that the value of an irreducible Brauer character φ^k may not be 0 on any p-regular conjugate class. As an illustration, we consider an example.

Example. Let $G = L_2(5)$ and $p = 5$. The group G has three 5-regular conjugate classes and hence $K(R)$ has rank 3. If I, J and K are irreducible R-modules of dimensions 1, 3 and 5 over A, then $\{\bar{I}, \bar{J}, \bar{K}\}$ is a basis of $K(R)$ over Z. Then we have

$$\begin{array}{lll} \bar{I}\bar{I} = \bar{I} & \bar{I}\bar{J} = \bar{J} & \bar{I}\bar{K} = \bar{K} \\ \bar{J}\bar{I} = \bar{J} & \bar{J}\bar{J} = \bar{I} + \bar{J} + \bar{K} \quad \text{and} & \bar{J}\bar{K} = \bar{I} + 3\bar{J} + \bar{K} \\ \bar{K}\bar{I} = \bar{K} & \bar{K}\bar{J} = \bar{I} + 3\bar{J} + \bar{K} & \bar{K}\bar{K} = 3\bar{I} + 4\bar{J} + 2\bar{K}. \end{array}$$

Therefore the corresponding matrices (a_{ij}^k) are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

These matrices determine the irreducible Brauer characters $\varphi^1 = (1, 1, 1)$ $\varphi^2 = (3, -1, 0)$ and $\varphi^3 = (5, 1, -1)$ respectively. Contrary to the situation in Corollary 1, the irreducible Brauer character φ^3 does not vanish on any 5-regular conjugate class of G.

If the field A is a splitting algebraic number field for G, then the irreducible Brauer characters of G are called the irreducible ordinary characters of G. Let $\text{Int}(A)$ be the ring of all algebraic integers of A, $\mathfrak{p}$ a fixed prime ideal of $\text{Int}(A)$ containing a given prime p and $D = \{\alpha/\beta | \alpha, \beta \in \text{Int}(A) \text{ and } \beta \notin \mathfrak{p}\}$. Then D is an integral domain with a unique maximal ideal P such that $\mathfrak{p} \subseteq P$ and A is the quotient field of D. The field $\bar{A} = D/P$ has a finite number of elements and it is a splitting field for G [**2**]. Let $R = AG$ and $\bar{R} = \bar{A}G$ be the group rings over A and $\bar{A}$ respectively. The Brauer characters depend on the prime p. The Brauer characters determined by $K(\bar{R})$ will be denoted by $\varphi^1, \varphi^2, \ldots, \varphi^n$ and the ordinary irreducible characters of G will be denoted by $\zeta^1, \zeta^2, \ldots, \zeta^m$. Again assume that $C_1, C_2, \ldots, C_n$ are the p-regular classes and $C_{n+1}, C_{n+2}, \ldots, C_m$ are the remaining conjugate classes of G. The value of ζ^i at x of C_j will be denoted by $\zeta_j{}^i$ and the number of elements in C_j will be denoted by g_j.

THEOREM 6. *Let ζ^k be an irreducible ordinary character of G and ∂^k the $n \times n$ diagonal matrix whose diagonal entries are $\zeta_1{}^k, \zeta_2{}^k, \ldots, \zeta_n{}^k$. If Φ is the Brauer character table of G corresponding to the prime p, then the coefficients of $\Phi\partial^k\Phi^{-1}$ are non-negative rational integers.*

Proof. There are non-negative rational integers d_{kj} such that

$$\zeta_t{}^k = \sum_{j=1}^{n} d_{kj}\varphi_t{}^j \text{ for } t = 1, 2, \ldots, n \text{ [\textbf{2}]}.$$

Hence we have $\partial^k = \sum_{j=1}^{n} d_{kj}D^j$ where D^j is the diagonal matrix whose entries are $\varphi_1{}^j, \varphi_2{}^j, \ldots, \varphi_n{}^j$. If $(a_{uv}{}^j)$ are the n matrices determined by a basis of $K(\bar{R})$, then by Theorem 2, we have $\Phi^{-1}(a^j{}_{uv})\Phi = D^j$. Therefore

$$\partial^k = \sum_{j=1}^{n} d_{kj}\Phi^{-1}(a_{uv}{}^j)\Phi$$

so that

$$\Phi\partial^k\Phi^{-1} = \sum_{j=1}^{n} d_{kj}(a_{uv}{}^j).$$

Since the coefficients of the matrix on the right hand side of this equation are non-negative integers, it follows that the coefficients of $\Phi\partial^k\Phi^{-1}$ are non-negative rational integers.

COROLLARY. *Let ζ^k be any irreducible ordinary character of G, φ^i an irreducible Brauer character of G and η^l a projective indecomposable Brauer character of G. Then*

$$\frac{1}{g}\sum_{j=1}^{n} g_j\varphi_j{}^i\zeta_j{}^k\overline{\eta_j{}^l}$$

is a non-negative integer.

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Proof. For the definition of a projective indecomposable Brauer character, see [**2**]. According to [**2**], the irreducible Brauer character φ^i and the indecomposable Brauer character η^l of G are connected by the relations

$$\sum_{t=1}^{n} \varphi_t^{\,i} \frac{g_t \overline{\eta_t^{\,l}}}{g} = \delta_{il}$$

where $\delta_{il} = 0$ if $i \neq l$ and 1 if $i = l$. Therefore it follows that

$$\Phi^{-1} = \left(\frac{g_t \overline{\eta_t^{\,l}}}{g} \right)$$

where t ranges over the rows and l ranges over the columns. The (i, l)th coefficient of the matrix $\Phi \partial^k \Phi^{-1}$ is

$$\frac{1}{g} \sum_{t=1}^{n} g_t \varphi_t^{\,i} \zeta_t^{\,k} \overline{\eta_t^{\,l}}$$

which is a non-negative rational integer.

There are groups for which the matrices $(a_{ij}^{\,k})$ can be calculated without an elaborate use of representation theory. For example, there is a method for computing the matrices $(a_{ij}^{\,k})$ in the case of a symmetric group [**3**]. It appears that the result of this paper relating to ordinary representation theory may have some appeal to physicists [**4**; **5**], although the same may not be true of those relating to modular representation theory.

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Carleton University,
Ottawa, Ontario