LATTICE OCTAHEDRA

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Let $A_1, A_2, \ldots, A_n$ be n linearly independent points in n-dimensional Euclidean space of a lattice Λ. The points $\pm A_1, \pm A_2, \ldots, \pm A_n$ define a closed n-dimensional octahedron (or "cross polytope") K with centre at the origin O. Our problem is to find a basis for the lattices Λ which have no points in K except $\pm A_1, \pm A_2, \ldots, \pm A_n$.

Let the position of a point P in space be defined vectorially by

$$P = p_1A_1 + p_2A_2 + \ldots + p_nA_n, \tag{1}$$

where the p are real numbers. We have the following results.

When $n = 2$, it is well known that a basis is

$$(A_1, A_2). \tag{2}$$

When $n = 3$, Minkowski **(1)** proved that there are two types of lattices, with respective bases

$$(A_1, A_2, A_3),\ (A_1, A_2, \tfrac{1}{2}(A_1 + A_2 + A_3)). \tag{3}$$

When $n = 4$, there are six essentially different bases typified by A_1, A_2, A_3 and one of

$$\begin{array}{ll} A_4, \tfrac{1}{2}(A_2 + A_3 + A_4), & \tfrac{1}{2}(A_1 + A_2 + A_3 + A_4), \\ \tfrac{1}{3}(\pm A_1 \pm A_2 \pm A_3 \pm A_4), & \tfrac{1}{4}(\pm 2A_1 \pm A_2 \pm A_3 \pm A_4), \\ \tfrac{1}{5}(\pm 2A_1 \pm 2A_2 \pm A_3 \pm A_4). & \end{array} \tag{4}$$

In all expressions of this kind, the signs are independent of each other and of any other signs. This result is a restatement of a result by Brunngraber **(2)** and a proof is given by Wolff **(3)**.

The proofs for $n = 3, 4$ depend upon Minkowski's method of adaption of lattices, and that for $n = 4$ is very complicated. I notice another method of considering the question which gives the result more directly, more simply, and with less troublesome numerical detail.

The simplest required lattice is that with basis $(A_1, A_2, \ldots, A_n)$. This will not be a basis of the other lattices Λ. Hence there will be points A of Λ given by

$$pA = a_1A_1 + a_2A_2 + \ldots + a_nA_n, \tag{5}$$

where $a_1, a_2, \ldots, a_n$ and $p > 1$ are integers, and

$$(a_1, a_2, \ldots, a_n, p) = 1. \tag{6}$$

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For brevity, we shall denote such a point A by

$$A = \{a_1, a_2, \ldots, a_n\}/p.$$

There is no loss of generality in supposing that

$$|a_1| \leqslant \tfrac{1}{2}p,\ |a_2| \leqslant \tfrac{1}{2}p, \ldots, |a_n| \leqslant \tfrac{1}{2}p. \tag{7}$$

We may also suppose that no $a \equiv 0 \pmod{p}$. For if $a_1 \equiv 0 \pmod{p}$, we have an $n - 1$ dimensional problem which may be considered as solved in dealing with the n-dimensional problem.

By the conditions of the problem, the point A is such that for any integer x prime to p, and all integers $x_1, x_2, \ldots, x_n$

$$xA - x_1A_1 - x_2A_2 - \ldots - x_nA_n$$

is not in K; and there is no loss of generality in supposing that $|x| < p$. We shall call such points A admissible. Then A will be admissible if and only if

$$\left|\frac{a_1x}{p} - x_1\right| + \ldots + \left|\frac{a_nx}{p} - x_n\right| > 1, \tag{8}$$

since the point P in (1) lies in K if

$$|p_1| + |p_2| + \ldots + |p_n| \leqslant 1. \tag{9}$$

Now by Minkowski's theorem on convex bodies, the convex $n + 1$ dimensional body

$$|X_1| + |X_2| + \ldots + |X_n| < 1,\ |X| < p$$

of volume $2^{n+1}p/n!$ contains at least two points of the lattice given by

$$X_1 = \frac{a_1x}{p} - x_1, \ldots, X_n = \frac{a_nx}{p} - x_n, X = x,$$

of determinant one when $p > n!$ We may suppose that $X \neq 0$ since then $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Hence, as is well known, admissible points A can arise only when $p \leqslant n!$

In this paper, we shall be concerned only with the cases $n = 2, 3, 4$. We shall see that admissible points A arise only when $n = 3$, $p = 2$, and $n = 4$, $p = 2, 3, 4, 5$.

Suppose first that $n = 2$. We need only consider $p = 2$, and then $|a_1| \leqslant 1$, $|a_2| \leqslant 1$. Clearly the point $A = \frac{1}{2}\{a_1, a_2\}$ lies in K and so cannot be a point of Λ. Hence $(A_1, A_2,)$ is a basis of Λ.

Suppose next that $n = 3$. We have now to consider $p = 2, 3, 4, 5, 6$.

If $p = 2$, $|a_1| \leqslant 1$, $|a_2| \leqslant 1$, $|a_3| \leqslant 1$, and then $A = \frac{1}{2}\{a_1, a_2, a_3\}$. This will be a point of K unless $|a_1| = |a_2| = |a_3| = 1$, and so $A = \frac{1}{2}\{\pm 1, \pm 1, \pm 1\}$. This point is admissible since $xA \equiv A \pmod{\Lambda}$ when $x = \pm 1$. Hence we clearly have a lattice Λ typified by the basis $(A = \frac{1}{2}\{1, 1, 1\}, A_1, A_2)$, since $A_3 = 2A - A_1 - A_2$.

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If $p = 3$, $|a_1| \leqslant 1$, $|a_2| \leqslant 1$, $|a_3| \leqslant 1$, then $A = \frac{1}{3}\{\pm 1, \pm 1, \pm 1\}$ and lies in K and is not admissible.

If $p = 4$, $|a_1| \leqslant 2$, $|a_2| \leqslant 2$, $|a_3| \leqslant 2$. We may suppose that one at least of the a's is not even, say $|a_1| = 1$. Since A does not lie in K, the only possibility for A is $A = \frac{1}{4}\{\pm 1, \pm 2, \pm 2\}$. Then $2A \equiv \frac{1}{2}A_1$ (mod Λ) and so A is not admissible.

If $p = 5$, $|a_1| \leqslant 2$, $|a_2| \leqslant 2$, $|a_3| \leqslant 2$ and so since A is not in K, we must have $A = \frac{1}{5}\{\pm 2, \pm 2, \pm 2\}$. Then $2A \equiv \frac{1}{5}\{\pm 1, \pm 1, \pm 1\}$ (mod Λ), and so A is not admissible since $\frac{1}{5}\{\pm 1, \pm 1, \pm 1\}$ lies in K.

If $p = 6$, $|a_1| \leqslant 3$, $|a_2| \leqslant 3$, $|a_3| \leqslant 3$. Since we require $|a_1| + |a_2| + |a_3| > 6$, we have only the three cases typified by

$$(a_1, a_2, a_3,) = (\pm 1, \pm 3, \pm 3), \quad (\pm 2, \pm 2, \pm 3), \quad (\pm 2, \pm 3, \pm 3), \quad (\pm 3, \pm 3, \pm 3).$$

In all these, $2A$ is congruent mod Λ to a point of K and so A is not admissible.

Suppose finally that $n = 4$ and so now $p \leqslant 24$. We shall show that there exist admissible points if and only if $p \leqslant 5$. We first give some results of a general character which will simplify the arithmetic. We note

(I) A is not admissible if p contains a factor f such that every A with denominator f is not admissible. This is obvious from

$$pA/f = \{a_1, a_2, a_3, a_4,\}/f.$$

We note next

(II) A is not admissible if for d, the greatest common divisor of p and of any of the a's, $d > 2$.

For suppose that $(a_1, p) = d$. Then $pA/d \equiv \{0, a_2, a_3, a_4\}/d$ (mod Λ), and from the case $n = 3$, this cannot be admissible unless $d = 2$ and a_2, a_3, a_4 are all odd. Hence, whenever A is admissible, we may suppose that one of the a, say a_1 is odd and prime to p. On considering xA where $xa_1 \equiv \pm 1$ (mod p), we may then take

(III) $a_1 \equiv \pm 1$ (mod p).

We shall presently consider the admissible points with $|a_2| = 1, 2, 3$, but first we consider the smaller values of p.

When $p = 2, 3$, it is clear that the only admissible points A are

$$A = \{\pm 1, \pm 1, \pm 1, \pm 1\}/p.$$

Note $Ax \equiv \{\pm 1, \pm 1, \pm 1, \pm 1\}/p$ (mod Λ) for $x = \pm 1$.

When $p = 4$, $|a_1| \leqslant 2$, $|a_2| \leqslant 2$, $|a_3| \leqslant 2$, $|a_4| \leqslant 2$. Since A is admissible, $\sum|a| \geqslant 5$, and since all the $|a|$ cannot be less than 2, we can take say $|a_1| = 2$. Then from (II), a_2, a_3, a_4 are odd giving the admissible point

$$A = \tfrac{1}{4}\{\pm 2, \pm 1, \pm 1, \pm 1\}.$$

We note $2A \equiv \frac{1}{2}\{0, 1, 1, 1\}$ (mod Λ).

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When $p = 5$, $|a_1| \leqslant 2$, etc. We can take $|a_1| = 1$, and since $\sum|a| \geqslant 6$, we may take, say, $|a_2| = 2$, and then, say, $|a_3| = 2$. We can reject $|a_4| = 2$ since for $A = \frac{1}{5}\{\pm 1, \pm 2, \pm 2, \pm 2\}$, $2A$ is not admissible. When $|a_4| = 1$, we have the admissible point A typified by

$$A = \tfrac{1}{5}\{\pm 2, \pm 2, \pm 1, \pm 1\}.$$

We note $2A \equiv \frac{1}{5}\{\pm 1, \pm 1, \pm 2, \pm 2\} \pmod{\Lambda}$.

When $p = 6$, by means of (II), we can exclude the cases when any a is divisible by 3, and also when any a is divisible by 2, since then the only possible forms for A are given by $A = \frac{1}{6}\{\pm 2, \pm 1, \pm 1, \pm 1\}$, and these are obviously not admissible. Hence also from (I),

$$p = 12, 18, 24 \text{ are not admissible.}$$

When $p = 7$, we have $|a_1| = 1$ and then, say, $|a_2| = 3$. Hence $|a_3| = 2$ or 3. We reject $|a_3| = 3$ since then $2A \equiv \frac{1}{7}\{\pm 2, \pm 1, \pm 1, 2a_4\} \pmod{\Lambda}$ and is inadmissible. Then $|a_4| = 2$ or 3 and we can reject $|a_4| = 3$ leaving $A = \frac{1}{7}\{\pm 1, \pm 3, \pm 2, \pm 2\}$; and $3A \equiv \frac{1}{7}\{\pm 3, \pm 2, \pm 1, \pm 1\} \pmod{\Lambda}$ and is not admissible. Hence also

$$p = 7, 14, 21 \text{ are not admissible.}$$

When $p = 8$, suppose first that all the a are odd. Since $|a| = 1$ or 3, at least two of the $|a|$ are equal, and on considering $3A$, if need be, we can take $|a_1| = 1$, $|a_2| = 1$. Then $A = \frac{1}{8}\{\pm 1, \pm 1, a_3, a_4\}$ is obviously inadmissible: Suppose next that some of the a are even. Then by (II), we need only consider the case when $|a_1| = 2$, and $|a_2|, |a_3|, |a_4|$, are odd. Since at least two of these are equal, we may on considering $3A$ if need be, take $|a_2| = 1$, $|a_3| = 1$ and then A is inadmissible: Hence also

$$p = 16, 24, \text{ are not admissible.}$$

When $p = 9$, on considering $3A$, we see that each a satisfies $a \equiv \pm 1 \pmod 3$, that is, $|a| = 1$, 2, or 4. Since at least two of the $|a|$ are equal, we can on considering $2A$ or $4A$, if need be, take $|a_1| = 1$, $|a_2| = 1$. Hence $A = \frac{1}{9}\{\pm 1, \pm 1, \pm 4, \pm 4\}$, and $2A$ is not admissible. Hence also

$$p = 18, \text{ is not admissible.}$$

When $p = 10$, we have $|a_1| = 1$, and since $|a_2| + |a_3| + |a_4| \geqslant 10$, we must have, say, $|a_4| = 4$ or 5. By (II), we can reject $|a_4| = 5$, and when $|a_4| = 4$, a_3 and a_4 must be odd and so $|a_3| \leqslant 3$, $|a_4| \leqslant 3$. The only possibility is $A = \frac{1}{10}\{\pm 1, \pm 4, \pm 3, \pm 3\}$, but then $3A$ is not admissible. Hence also

$$p = 20 \text{ is not admissible.}$$

We have now dealt with all the even values of $p \leqslant 24$, except $p = 22$ which will be dealt with when $p = 11$ is considered, and which is not admissible. We must now consider the remaining odd values of $p > 9$. We shall show that

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no admissible points A arise when $p > 5$ and $|a_1| = 1$, $|a_2| = 1, 2$, or 3. This will then hold also for any two a, say a_r, a_s if $(a_r, p) = 1$ and $a_s \equiv \pm a_r$, $\pm 2a_r$, $\pm 3a_r \pmod{p}$.

(IV) Suppose $|a_1| = 1$, $|a_2| = 1$. Since $|a_3| + |a_4| \geqslant p - 1$, we must have $|a_3| = \frac{1}{2}(p-1)$, $|a_4| = \frac{1}{2}(p-1)$. Then $2A \equiv \{\pm 2, \pm 2, \pm 1, \pm 1\}/p \pmod{\Lambda}$ and $2A$ is not admissible if $p \geqslant 7$.

(V). Suppose $|a_1| = 1$, $|a_2| = 2$. Then $|a_3| + |a_4| \geqslant p - 2$ and so, say, $|a_3| = \frac{1}{2}(p-1)$. Then $|a_4| = \frac{1}{2}(p-1)$ or $\frac{1}{2}(p-3)$. The first value can be rejected by (IV) since

$$\left(\frac{p-1}{2}, p\right) = 1.$$

For the second, $2A = \{\pm 2, \pm 4, \pm 1, \pm 3\}/p$ and is not admissible if $p \geqslant 11$. We have seen that no admissible points arise when $p = 7$ or 9.

(VI). Suppose finally $|a_1| = 1$, $|a_2| = 3$. Since $|a_3| + |a_4| \geqslant p - 3$, we have $|a_3| = \frac{1}{2}(p-1)$ or $\frac{1}{2}(p-3)$. Since $a_1 \equiv \pm 2a_3$, we need only consider $|a_3| = \frac{1}{2}(p-3)$ and then $|a_4| = \frac{1}{2}(p-3)$. This can be rejected by (IV) when $(p, 3) = 1$, and by (II) when $(p, 3) = 3$.

We now consider the odd values of $p \geqslant 11$. We know from (IV), (V), and (VI), that we need consider only the cases when $|a_1| = 1$, and the other a satisfy $|a| \geqslant 4$; and of course all a satisfy $|a| \leqslant \frac{1}{2}p$. We can reject all $a = \pm\frac{1}{2}(p-1)$ or $a = \pm\frac{1}{3}(p-1)$.

$p = 11$. Here $|a_2| = 4$ or 5, and both can be rejected. Hence A is not admissible.

$p = 13$. Here $|a_2| = 4, 5$, or 6 and $|a_2| = 4, 6$ can be rejected, and so $|a_2| = 5$. Since $|a_3| = 4, 5$, or 6, we can reject 4, 6 and then $|a_2| = |a_3|$. Hence A is not admissible.

$p = 15$. Here $|a_2| = 4, 5, 6, 7$ and we can reject 5, 6, and also 7 from (II). Hence $|a_2| = 4$ and this is also the only possibility for $|a_3|$. Hence A is not admissible.

$p = 17$. Here $|a_2| = 4, 5, 6, 7, 8$ and we can reject 6, 8. Since $|a_2|$, $|a_3|$, $|a_4|$ are distinct by (IV), they must be 4, 5, 7 in some order, and then $|a_1| + |a_2| + |a_3| + |a_4| = 17$, so that A is not admissible.

$p = 19$. Here $|a_2| = 4, 5, 6, 7, 8, 9$.

We can reject 6 and 9. Hence $|a_2|$, $|a_3|$, $|a_4|$ are three out of 4, 5, 7, 8 and since $|a_2| + |a_3| + |a_4| \geqslant 19$, we can suppose that $A = \{\pm 1 \pm 7, \pm 8, a_4\}/19$ where $a_4 = \pm 4$ or ± 5. But now $3A \equiv \{\pm 3, \pm 2, \pm 5, \pm 3a_4\}/19 \pmod{\Lambda}$ and is not admissible since $3a_4 \equiv \pm 7$ or ± 4 {mod 19}.*

$p = 23$. Here $|a_2| = 4, 5, 6, 7, 8, 9, 10, 11$.

We can reject 8, 11. The cases $|a_2| = 6, 9, 10$ are included under $|a_2| = 4, 5, 7$ respectively on considering $4A$, $5A$, $7A$, respectively.

*I am indebted to the referee for these proofs for $n = 17, 19$, which are rather shorter than those I had given.

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When $|a_2| = 4$, $|a_3| + |a_4| \geqslant 19$ and so $|a_3| = 10$. Then $A = \{\pm 1, \pm 4, \pm 10, a_4\}/23$, and $5A \equiv \{\pm 5, \pm 3, \pm 4, 5a_4\}/23 \pmod{\Lambda}$ is not admissible.

When $|a_2| = 5$, $|a_3| + |a_4| \geqslant 18$ or, say, $|a_3| = 9, 10$. We can reject 10 since $|a_3| = 2|a_2|$. Hence $A = \{\pm 1, \pm 5, \pm 9, a_4\}$ and now $3A \equiv \{\pm 3, \pm 8, \pm 4, 3a_4\}$ is not admissible.

When $|a_2| = 7$, $|a_3| + |a_4| \geqslant 16$ and so $|a_3| = 9, 10$ and so $A = \{\pm 1, \pm 7, \pm 9 \text{ or } \pm 10, a_4\}/23$. Now $7A \equiv \{\pm 7, 3, \pm 6, \text{ or } \pm 1, 7a_4\}/23 (\bmod \Lambda)$ and is clearly not admissible.

We can now find the possible bases for Λ. We may suppose that not all of the bases of the three-dimensional sublattices are of the type $(A_1, A_2, \frac{1}{2}(A_1 + A_2 + A_3))$. For if $(A_1, A_2, \frac{1}{2}(A_1 + A_2 + A_4))$ were also allowable, then $\frac{1}{2}(A_3 - A_4)$ would be a point of Λ. Hence we may suppose that three of the A's, say, A_1, A_2, A_3 form a basis for the three-dimensional sublattice. Then the fourth basis element A must be such that $A_4 = bA + b_1A_1 + b_2A_2 + b_3A_3$ where the b are integers. Clearly we can typify A by one of $A_4, \frac{1}{2}(A_2 - A_3 - A_4)$ and $\frac{1}{2}(1, 1, 1, 1\}, \frac{1}{3}(\pm 1, \pm 1, \pm 1, \pm 1\}, \frac{1}{4}(\pm 2, \pm 1, \pm 1, \pm 1\}, \frac{1}{5}\{\pm 2, \pm 2, \pm 1, \pm 1\}$.

This completes the proof for $n = 4$. We note that we have shown that when $n = 4$, integers $x, x_1, x_2, \ldots, x_n$ not all zero exist for which

$$\left|\frac{a_1x}{p} - x_1\right| + \ldots + \left|\frac{a_nx}{p} - x_n\right| < 1, \; |x| < p$$

not only when $p > 4!$ but also when $p > 5$. It is an interesting problem to find the exact result for $n > 4$. Approximate results for large n have been given by Blichfeldt **(4)**.

References

1. H. Minkowski, *Gesammelte Abhandlungen*, Bd. II.
2. E. Brunngraber, *Ueber Punktgitter* (Dissertation, Wien, 1944).
3. K. M. Wolff, Monatsh. Math., *58* (1954), 38–56.
4. H. F. Blichfeldt, Monatsh. Math., *42–43* (1935–6), 410–414.

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