CERTAIN TOPOLOGICAL SEMIRINGS IN R_1

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If we consider any particular topological semigroup S, it may seem reasonable to ask for a characterization of all additions on S which make it a topological semiring. We are interested here in this problem when

(i) S is an (I)-semigroup;

(ii) S is $[0, \infty)$ and the multiplication on S is such that 0 and 1 are zero and identity respectively.

We are able to give an almost complete solution in the first case while our solution for the second case is complete except in so far as it depends on the first case. The results given for (I)-semigroups contain Theorems 11 and 12 of [5] where Selden has given much information about the commutative additions.

Recall that by a *topological semiring* we mean a system $\{S, +, \cdot\}$ where S is a Hausdorff space, $\{S, +\}$ and $\{S, \cdot\}$ are topological semigroups, and the distributive laws

$$z \cdot (x+y) = (z \cdot x)+(z \cdot y), \tag{1}$$

$$(x+y) \cdot z = (x \cdot z)+(y \cdot z), \tag{2}$$

hold for all x, y, z in S.

The multiplications (i) and (ii) above have been completely classified by Mostert and Shields in [1], Theorem B and [2], Theorem A respectively. In particular, they are commutative so that (1) and (2) are equivalent.

Finally, in § 4, we record a brief observation concerning the multiplicative kernel of any compact connected semiring in R_1.

1. (I)-semigroups

An (I)-*semigroup* is defined to be any semigroup topologically isomorphic to a semigroup on $[0, 1]$ for which 0 and 1 are zero and identity respectively. Two important examples of (I)-semigroups are

(i) J_1 which is $[0, 1]$ with ordinary multiplication;

(ii) J_2 which is $[\frac{1}{2}, 1]$ with multiplication $*$ given by

$$x * y = \max\left(\tfrac{1}{2}, xy\right),$$

where xy represents ordinary multiplication.

In [1], Theorem B, Mostert and Shields have identified all multiplications of (I)-semigroups. We quote their result.

THEOREM 1. *Let S be an (I)-semigroup on $[0, 1]$ with 0 as its zero and 1 as its identity. The set E of idempotents is closed and if $x, y \in E$, then $xy = \min(x, y)$. The complement of E is the union of disjoint intervals. Let P be the closure of one of these. Then P is topologically isomorphic to either J_1 or J_2. Finally, if $x \in P$, $y \notin P$, then $xy = \min(x, y) = yx$.*

In [4], Theorem 5, the author has found all additions of topological semirings on J_1. Of these, the only additions $*$ which have $1 * 1 < 1$ are given by

$$x * y = \begin{cases} (x^c+y^c)^{1/c} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0, \end{cases}$$

where $c < 0$ and $+$ represents ordinary addition. The following lemma is an immediate consequence.

LEMMA 1. *If $*$ is an addition of a semiring on J_1 and $1 * 1 < 1$, then, for all x, $x * 0 = 0 * x = 0$, $1 * x < 1$ and $x * 1 < 1$.*

We shall need the following example.

EXAMPLE 1. Let $[0, 1]$ be an (I)-semigroup for which 0 and 1 are zero and identity respectively. Choose a, b, c in E so that $a \leqq b \leqq c$. If $c = 1$ put $d = 1$, while if $c < 1$ put

$$d = \inf \{f \mid f \in E \text{ and } f > c\}.$$

If $c = d$ we define a binary operation $+$ on $[c, d]$ by putting $c+c = c$. If $c < d$ then $[c, d]$ is isomorphic to J_1 or J_2 and we let $+$ be any addition of a semiring on $[c, d]$ for which $d+d < d$. (All additions of semirings on J_1 are given in [4], Theorem 5, but we are not able, as yet, to list all additions of semirings on J_2. In § 2, however, some results dealing with semirings on J_2 are given.) Let $\phi : [0, c] \to [a, 1]$ be any continuous function satisfying

(i) $\phi(x) = x$ if $a \leqq x \leqq c$;

(ii) ϕ is decreasing on $[0, a]$ and $\phi([0, a]) \subset E$.

Let $\psi : [0, c] \to [b, 1]$ be any continuous function satisfying

(i) $\min(c, \psi(x)) = \max(b, \min(c, \phi(x)))$ if $0 \leqq x \leqq c$;

(ii) ψ is decreasing on $[0, b]$ and $\psi([0, b]) \subset E$.

(Notice that if $b < c$ and $[a, c] \not\subset E$, then, given ϕ, ψ is completely determined by (i) above. If $b = c$ or $[a, c] \subset E$, then, in general, ψ is not uniquely determined by ϕ.) We can extend $+$ to $[0, 1]$ by

$$(3)\qquad x+y = \begin{cases} \min(x, d)+\min(y, d) & \text{if } x \geqq c \text{ and } y \geqq c, \\ \min(x, \phi(y)) & \text{if } x \geqq y \text{ and } y < c, \\ \min(y, \psi(x)) & \text{if } x < y \text{ and } x < c. \end{cases}$$

THEOREM 2. *Consider an (I)-semigroup on $[0, 1]$ for which 0 and 1 are zero and identity respectively.*

(i) *If + is given by* (3) *of Example* 1 *then + is an addition of a topological semiring on $[0, 1]$ and*

$$(4)\qquad \inf\{1+x|x \in [0, 1]\} \leqq \inf\{x+1|x \in [0, 1]\}.$$

(ii) *Conversely, let + be an addition of a topological semiring on $[0, 1]$ with*

$$(4')\qquad a = \inf\{1+x|x \in [0, 1]\} \leqq b = \inf\{x+1|x \in [0, 1]\}.$$

Put

$$c = \sup\{f|f \in E \text{ and } f+f = f\},$$

define d as in Example 1, *and let $\phi(x) = 1+x$ for $x \leqq c$, $\psi(x) = x+1$ for $x \leqq c$. Then $[c, d]$ is a subsemiring, $d+d < d$ if $c < d$, $[0, c]$ is an additively idempotent subsemiring, $a \leqq b \leqq c$ and $a, b, c \in E$, ϕ and ψ satisfy the conditions in Example* 1 *and + satisfies* (3).

REMARK. The conditions (4) and (4′) are not significant restrictions because we can, if necessary, change to the sum dual (i.e., consider $*$ defined by $x * y = y+x$).

PROOF. To see that + given by (3) of Example 1 is an addition of a topological semiring on $[0, 1]$, it is necessary to check very many routine cases. This is omitted. We remark, however, that it is essential to realize when showing + is continuous that, if $c < d$, then $c+x = x+c = c$ for all x in $[c, d]$. (If $[c, d]$ is isomorphic to J_1 this follows from Lemma 1, while if $[c, d]$ is isomorphic to J_2 it follows from Lemma 3 of § 2.)

Now suppose that + is an addition of a topological semiring on $[0, 1]$ and let c, d be defined as in the theorem. (Notice that c exists because $0+0 = 0(1+1) = 0$.) As $\{f|f \in E \text{ and } f+f = f\}$ is closed, $c \in E$ and $c+c = c$. If $x, y \in [c, d]$ then

$$\min(c, x+y) = c(x+y) = cx+cy = \min(c, x)+\min(c, y) = c+c = c,$$
$$\min(d, x+y) = d(x+y) = dx+dy = \min(d, x)+\min(d, y) = x+y.$$

Thus $[c, d]$ is a subsemiring. Also, because $d+d \in [c, d]$, we see from the definition of c that $d+d < d$ if $c < d$. If $x, y \in [0, c]$ then

$$x+x = xc+xc = x(c+c) = xc = x,$$
$$(x+y)c = xc+yc = x+y.$$

It follows that $[0, c]$ is an additively idempotent subsemiring.

Next we show that

$$x+y = \min(x, d)+\min(y, d)$$

if $x \geqq c$ and $y \geqq c$. As this is trivial if $d = 1$, we can assume that $d < 1$. Suppose firstly that $c < d$. If $x \geqq d$ and $y \geqq d$,

$$\min(x+y, d) = (x+y)d = xd+yd = d+d < d$$

and so $x+y = d+d$. If $c \leqq x < d \leqq y$, then

$$\min(x+y, d) = (x+y)d = xd+yd = x+d.$$

But we know from Lemmas 1 and 3 that $x+d < d$; hence $x+y = x+d$. Similarly, $x+y = d+y$ if $c \leqq y < d \leqq x$. Suppose secondly that $c = d$, and let $x > c$, $y > c$. It is clear from the definition of d that there exists a sequence $\{c_n\}$ such that $c_n \to c$ as $n \to \infty$ and, for all n, $c_n > c$, $c_n \in E$ and $c_n+c_n \neq c_n$. Thus there is an n_0 such that $c_n < x$ and $c_n < y$ if $n \geqq n_0$. Hence, if $n \geqq n_0$,

$$\min(x+y, c_n) = (x+y)c_n = xc_n+yc_n = c_n+c_n \neq c_n,$$

and so $x+y = c_n+c_n$. Therefore

$$x+y = \lim_{n\to\infty}(c_n+c_n) = (\lim_{n\to\infty} c_n)+(\lim_{n\to\infty} c_n) = c+c.$$

It now follows from the continuity of $+$ that $x+y = c+c$ if $x \geqq c$ and $y \geqq c$.

Let $T = \{1+x | x \in [0, 1]\}$; then $a = \inf T$. Now

$$1+c = \min(1, d)+c = d+c.$$

If $c = d$ it follows that $1+c = c+c = c$, while if $c < d$, we know from Lemmas 1 and 3 that $d+c = c$, and again $1+c = c$. Thus $a \leqq c$. For any x, y,

$$(1+x)(1+y) = 1+(x+y+xy).$$

Hence T is a multiplicative semigroup which means that $a \in E$. Similarly $b \leqq c$ and $b \in E$.

We now show that $\phi(x) = x$ if $a \leqq x \leqq c$. Because we have shown that $\phi(c) = 1+c = c$, we can assume that $a \leqq x < c$ (and hence that $a < c$). Because we have seen that $1+y = d+\min(y, d) \geqq c$ if $y \geqq c$, there exists x_1 with $1+x_1 = a$ and $x_1 < c$. Thus

$$c+x_1 = c(1+x_1) = ca = a.$$

Because $c+c = c$ and $+$ is continuous, there exists x_2 in $[x_1, c)$ with $c+x_2 = x$. Hence

$$c+x = c+(c+x_2) = (c+c)+x_2 = c+x_2 = x,$$

and therefore $\phi(x) = x$ since

$$\min(c, \phi(x)) = c(1+x) = c+cx = c+x = x.$$

Let $x \leqq y \leqq a$. Then there exists z with $zy = x$ and so

$$y+x = y+zy = (1+z)y = y$$

since $y \leqq a \leqq 1+z$. Hence

$$1+y = 1+(y+x) = (1+y)+x = (1+y)+x(1+y) = (1+x)(1+y) \leqq 1+x.$$

Thus ϕ is decreasing on $[0, a]$. Also, if $x \leqq a$, then since $x^2 \leqq x$,

$$(1+x)(1+x) = 1+(x+x)+x^2 = 1+(x+x^2) = 1+x$$

which means that $\phi([0, a]) \subset E$.

It can be shown similarly that $\psi(x) = x$ if $b \leqq x \leqq c$, ψ is decreasing on $[0, b]$ and $\psi([0, b]) \subset E$.

If $a \leqq x \leqq c$, then

$$c+x = c(1+x) = cx = x.$$

Also, if $x \leqq c$, then

$$c+x = c(1+x) \geqq ca = a.$$

Similarly, $x+c = x$ if $b \leqq x \leqq c$ and $x+c \geqq b$ if $x \leqq c$. Consider $x \leqq b$. Then

$$c+x+c = c+(x+c) = x+c$$

since $x+c \geqq b \geqq a$. If $c+x \geqq b$, then

$$c+x+c = (c+x)+c = c+x$$

and so $x+c = c+x$. If, on the other hand, $c+x < b$, then since $x+c \geqq b$,

$$b+x+c = b+(x+c) = b(x+c)+(x+c) = (b+c)(x+c) = b(x+c) = b.$$

Also,

$$b+x+c = (b+x)+c = b(c+x)+c = (c+x)+c = c+x+c = x+c,$$

and so $x+c = b$. We conclude that, for $x \leqq b$,

$$x+c = \max(b, c+x).$$

But if $b \leqq x \leqq c$, then $x+c = c+x = x$ and so

$$x+c = \max(b, c+x)$$

for $x \leqq c$. However for $x \leqq c$,

$$x+c = c(x+1) = c\psi(x) = \min(c, \psi(x)),$$
$$c+x = c(1+x) = c\phi(x) = \min(c, \phi(x)),$$

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and it follows that

$$\min(c, \psi(x)) = \max(b, \min(c, \phi(x))).$$

The proof will be complete when we verify the latter two lines of (3). Suppose $x \geqq y$ and $y < c$. Then there exists z in $[y, c]$ so that $xz = y$, which means that

$$x+y = x+xz = x(1+z) = x\phi(z).$$

If $y > a$ then, since $\phi(z) = z$ and $\phi(y) = y$,

$$x+y = xz = y = \min(x, y) = \min(x, \phi(y)).$$

If $y \leqq a$ we can choose $z \leqq a$ and so $\phi(z) \in E$. Hence

$$x+y = \min(x, \phi(z)).$$

If also $x \leqq a$ then $\phi(z) \geqq x$ and $\phi(y) \geqq x$ which means that

$$x+y = \min(x, \phi(z)) = x = \min(x, \phi(y)),$$

while if $x > a$, we can choose $z = y$ and

$$x+y = \min(x, \phi(y)).$$

The final line of (3) follows similarly.

2. J_2

Because it is the only gap in our knowledge of semirings on (I)-semigroups, we consider here semirings on J_2. We shall write multiplication as xy. Note that if $x < 1$, then there is an integer n so that $x^m = \frac{1}{2}$ for all integers $m \geqq n$.

LEMMA 2. *The only additions + of semirings on J_2 which have $1+1 = 1$ are given by*

(i) $x+y = x$;
(ii) $x+y = y$;
(iii) $x+y = \min(x, y)$;
(iv) $x+y = \max(x, y)$.

PROOF. This follows easily from Theorem 2, for here

$$1 = \sup\{f \mid f \in E \text{ and } f+f = f\},$$

and the only multiplicative idempotents are $\frac{1}{2}$ and 1.

LEMMA 3. *If + is an addition of a semiring on J_2 and $1+1 < 1$, then, for all x, $x+\frac{1}{2} = \frac{1}{2}+x = \frac{1}{2}$, $1+x < 1$ and $x+1 < 1$.*

PROOF. We see that

$$(1+\tfrac{1}{2})^2 = 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2} = 1+\tfrac{1}{2}(1+1+1) = 1+\tfrac{1}{2}$$

from which it follows that $1+\frac{1}{2}$ is either $\frac{1}{2}$ or 1. Similarly $\frac{1}{2}+1$ is either $\frac{1}{2}$ or 1.

Now suppose, if possible, that $1+\frac{1}{2} = 1$, and let $\theta = 1+1$. We prove that if there exists $z \geqq \frac{1}{2}$ so that $1+x = 1$ for all $x \leqq z$, then $1+x = 1$ for all $x \leqq \min(1, z/\theta)$. For let $x \leqq \min(1, z/\theta)$. Then

$$(1+x)^2 = 1+x+x+x^2 = 1+x\theta+x^2.$$

But $x\theta \leqq z$ and so $1+x\theta = 1$. Thus $(1+x)^2 = 1+x^2$. Now $x^2 \leqq x \leqq \min(1, z/\theta)$ and so similarly,

$$(1+x)^4 = (1+x^2)^2 = 1+x^4.$$

Thus by induction we see that

$$(1+x)^{2^n} = 1+x^{2^n}$$

for all $n \geqq 1$. But there is an integer m so that $x^{2^m} = \frac{1}{2}$ and thus

$$(1+x)^{2^m} = 1+\tfrac{1}{2} = 1$$

which means that $1+x = 1$.

Because $1+\frac{1}{2} = 1$, we put z in turn equal to $\frac{1}{2}$, $1/2\theta$, $1/2\theta^2, \cdots$ and see that, for all $n \geqq 1$, $1+x = 1$ if $x \leqq \min(1, 1/2\theta^n)$. But eventually $1/2\theta^n \geqq 1$ and hence $1+1 = 1$ which is a contradiction.

Thus $1+\frac{1}{2} = \frac{1}{2}$ and so, for any x,

$$x+\tfrac{1}{2} = x(1+\tfrac{1}{2}) = x\tfrac{1}{2} = \tfrac{1}{2}.$$

Similarly $\frac{1}{2}+x = \frac{1}{2}$ for all x.

Now consider any x. If $x = 1$ then $1+x = 1+1 < 1$. If $x < 1$, there exists an integer n so that $x^n = \frac{1}{2}$. But $(1+x)^n$ is the semiring sum of 2^n terms, the last of which is x^n. Therefore $(1+x)^n = \frac{1}{2}$ because $y+\frac{1}{2} = \frac{1}{2}$ for all y, and it follows that $1+x < 1$. Similarly $x+1 < 1$ for all x.

The following lemma shows that there are some rather badly behaved additions on J_2.

LEMMA 4. *Suppose $\frac{1}{2} \leqq d \leqq 1/\sqrt{2}$ and let ϕ; $J_2 \to J_2$ and ψ: $J_2 \to J_2$ be two continuous functions satisfying*

(i) $\phi(x) = \psi(x) = \frac{1}{2}$ *if* $\frac{1}{2} \leqq x \leqq d$;

(ii) $\phi(x) \leqq \min(d, x/2d)$ *if* $d < x \leqq 1$;

(iii) $\psi(x) \leqq \min(d, x/2d)$ *if* $d < x \leqq 1$;

(iv) $\phi(1) = \psi(1)$;

(v) *for some z_0 in J_2, either*

$$\phi(z_0) = d \text{ or } \psi(z_0) = d.$$

If a binary operation $+$ on J_2 is defined by

$$(5) \qquad x+y = \begin{cases} x\phi(y/x) & \text{if } x \geqq y, \\ y\psi(x/y) & \text{if } x < y, \end{cases}$$

then $+$ is an addition of a topological semiring on J_2, $x+y+z = \frac{1}{2}$ for all x, y, z, and $\sup \{x+y|x, y \in J_2\} = d$.

Conversely, let $+$ be an addition of a topological semiring on J_2 for which $x+y+z = \frac{1}{2}$ for all x, y, z, and put $d = \sup \{x+y|x, y \in J_2\}$, $\phi(x) = 1+x$, $\psi(x) = x+1$. Then $d \leqq 1/\sqrt{2}$, ϕ and ψ satisfy the five conditions above and $+$ satisfies (5).

PROOF. The first part of the lemma can be easily checked.

Now let $+$ be an addition of a semiring on J_2 for which $x+y+z = \frac{1}{2}$ for all x, y, z, and let d, ϕ, ψ be defined as in the second part of the lemma. We can assume that $d \neq \frac{1}{2}$ or else the conditions are trivially satisfied. If $x \geqq y$, then

$$x+y = x+x(y/x) = x[1+(y/x)] = x\phi(y/x),$$

while similarly, $x+y = y\psi(x/y)$ if $x < y$. Also,

$$\phi(1) = 1+1 = \psi(1).$$

Because $d = \sup \{x+y|x, y \in J_2\}$, there exist x_1, y_1 with $d = x_1+y_1$. If $x_1 \geqq y_1$, then $d = x_1\phi(y_1/x_1)$. It follows that $x_1 = 1$ (or else $1+(y_1/x_1) > d$) and so $\phi(y_1) = d$. If, on the other hand, $x_1 < y_1$, then $y_1 = 1$ and $\psi(x_1) = d$. Also

$$d^2 = x_1^2+(x_1y_1+y_1x_1)+y_1^2 = \tfrac{1}{2}$$

and we conclude that $d \leqq 1/\sqrt{2}$.

Let $x \leqq d$. Because $x_1+y_1 = d$ and $\frac{1}{2}+\frac{1}{2} = \frac{1}{2}$, it follows from the continuity of $+$ that there exist x_2, y_2 with $x_2+y_2 = x$. Hence, for all y,

$$x+y = x_2+y_2+y = \tfrac{1}{2}$$

since the semiring sum of any three numbers is $\frac{1}{2}$. Similarly $y+x = \frac{1}{2}$ for all y. In particular, $\phi(x) = \psi(x) = \frac{1}{2}$.

Finally, suppose that $d < x \leqq 1$. If $\phi(x) > x/2d$, then

$$\tfrac{1}{2} < (d/x)\phi(x) = (d/x)(1+x) = (d/x)+d$$

which contradicts the paragraph above. Also $\phi(x) \leqq d$ from the definition of d. Hence $\phi(x) \leqq \min(d, x/2d)$ and similarly $\psi(x) \leqq \min(d, x/2d)$.

If $c < 0$, it is easily seen that $*$, given by

$$(6) \qquad x * y = \max(\tfrac{1}{2}, (x^c+y^c)^{1/c}), \qquad x, y \in J_2,$$

where, for once, $+$ represents ordinary addition, is also an addition of a semiring on J_2. Whether or not these and the additions in Lemmas 2 and 4 are the only additions on J_2 is the remaining query. (It should be noted that for $-\ln 3/\ln 2 \leqq c < 0$, the additions $*$ given by (6) are contained in those given in Lemma 4.)

3. $[0, \infty)$ with 0 as zero, 1 as identity

In Theorem A of [2], Mostert and Shields have characterized all multiplications of this type. We quote their result.

THEOREM 3. *Suppose* $[0, \infty)$ *is a topological semigroup with zero at* 0 *and identity at* 1. *Then*

(i) *if there are no other idempotents, multiplication is (isomorphic to) the ordinary multiplication of real numbers on* $[0, \infty)$;

(ii) *if S contains an idempotent different from* 0 *and* 1, *then it contains a largest (in the sense of the regular order of real numbers) such idempotent* e. *Moreover,* $e < 1$, $[e, \infty)$ *is a subsemigroup topologically isomorphic to* $[0, \infty)$ *under ordinary multiplication of real numbers, and* $[0, e]$ *is an* (I)*-semigroup.*

If the multiplication is as in (i) of Theorem 3, all additions of semirings are given in Theorem 2 of [4]. Accordingly we shall assume that (ii) holds. Notice that if $x \leqq e \leqq y$, then

$$xy = (xe)y = x(ey) = xe = x,$$

and similarly $yx = x$.

EXAMPLE 2. Suppose $[0, \infty)$ is a semigroup whose multiplication is given in (ii) of Theorem 3 and let $+$ be any addition of a semiring on the (I)-semigroup $[0, e]$ for which $e+e < e$. Then we can extend $+$ to $[0, \infty)$ by putting

$$x+y = \min(x, e)+\min(y, e) \tag{7}$$

for all x, y in $[0, \infty)$.

EXAMPLE 3. Suppose $[0, \infty)$ is a semigroup whose multiplication is given in (ii) of Theorem 3. We can define a binary operation $+$ on $[0, \infty)$ in the following way. Let $+$ restricted to $[0, e]$ be any addition of a semiring on the (I)-semigroup $[0, e]$ for which $e+e = e$, and let $+$ restricted to $[e, \infty)$ be any addition of a semiring on $[e, \infty)$ subject to requirements that

(i) if $z+e < e$ for some $z \leqq e$, then $e+1 = e$;

(ii) if $e+z < e$ for some $z \leqq e$, then $1+e = e$.

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It follows from Theorem 2 of [4] that each of $1+e$ and $e+1$ is either e or 1. We complete the definition of $+$ as follows:

(i) If $0 \leqq x < e < y$, put $x+y = x+e$ if $e+1 = e$ and put $x+y = y$ if $e+1 = 1$;

(ii) If $0 \leqq y < e < x$, put $x+y = e+y$ if $1+e = e$ and put $x+y = x$ if $1+e = 1$.

THEOREM 4. *Let* $[0, \infty)$ *be a semigroup whose multiplication is given in* (ii) *of Theorem* 3.

If $+$ *is defined as in Example* 2 *or Example* 3 *then* $+$ *is an addition of a topological semiring on* $[0, \infty)$.

Conversely, if $+$ *is an addition of a topological semiring on* $[0, \infty)$, *then* $[0, e]$ *is a subsemiring. Further, if* $e+e < e$, *then* $+$ *satisfies* (7) *of Example* 2, *while if* $e+e = e$, *then* $[e, \infty)$ *is a subsemiring and* $+$ *satisfies all the conditions in Example* 3.

PROOF. It can be easily checked that $+$ defined in Example 2 is an addition of a semiring on $[0, \infty)$. If $+$ is as defined in Example 3, then again the verification that $+$ is an addition of a semiring on $[0, \infty)$ is routine, but involves the examination of many cases. This will be omitted.

Suppose that $+$ is an addition of a topological semiring on $[0, \infty)$. If $x, y \leqq e$, then

$$(x+y)e = xe+ye = x+y.$$

Hence $[0, e]$ is a subsemiring.

As in Theorem 2 we let

$$c = \sup \{f | f \in E,\ f \leqq e \text{ and } f+f = f\}.$$

Put

$$A = \{x | x \leqq c \text{ and } x+1 \leqq e\},$$
$$B = \{x | x \leqq c \text{ and } x+1 > e\}.$$

If $x \in B$, then

$$(x+1)^2 = x(x+1)+(x+1) = x+x+1 = x(c+c)+1 = xc+1 = x+1$$

which means that $x+1$ is a multiplicative idempotent. Because 1 is the only multiplicative idempotent greater than e, we see that

$$B = \{x | x \leqq c \text{ and } x+1 = 1\}.$$

Clearly A and B are closed and $A \cup B = [0, c]$. As $[0, c]$ is connected, it follows that either $A = [0, c]$ or $B = [0, c]$. Notice also that if $x \in A$, then

$$x+1 = e(x+1) = ex+e = x+e.$$

Consider any $x \leqq e$ for which $x+e < e$. Then as $e(x+1) = x+e < e$, we see that $x+1 = x+e$.

Suppose firstly that $e+e < e$; then $c < e$. As in Theorem 2 we let

$$d = \inf \{f | f \in E \text{ and } f > c\}.$$

We show that $x+1 = x+e$ for $x \leqq e$. If $x \geqq c$ and $d < e$, it follows from Theorem 2 that

$$x+e = \min (x, d)+d \leqq d < e$$

and hence $x+1 = x+e$. If $x \geqq c$ and $d = e$, it follows from Theorem 2 that $[c, e]$ is a subsemiring which is multiplicatively isomorphic to J_1 or J_2 and so, because $x+e < e$ (see Lemmas 1 and 3), again $x+1 = x+e$. In particular, $c+1 = c+e \leqq e$ and $c \in A$. It is a consequence that $A = [0, c]$ and thus $x+1 = x+e$ if $x \leqq c$. We can now show that (7) holds. If $x \leqq e < y$ then

$$x+y = xy+y = (x+1)y = (x+e)y = x+e.$$

Similarly, if $y \leqq e < x$ then $x+y = e+y$. Finally, if $x \geqq e$ and $y \geqq e$ then

$$e(x+y) = ex+ey = e+e < e$$

and $x+y = e+e$.

Secondly let $e+e = e$; then $c = e$. If $x, y \geqq e$, then

$$e(x+y) = ex+ey = e+e = e$$

and so $x+y \geqq e$. Thus $[e, \infty)$ is a subsemiring. We have seen that if $z+e < e$ for some $z \leqq e$, then $z+1 = z+e < e$, $z \in A$ and so $A = [0, e]$. In particular $e \in A$ and thus $e+1 = e+e = e$. Similarly, if $e+z < e$ for some $z \leqq e$ then $1+e = e$. If $e+1 = e$, we see that because $e \in A$, then $A = [0, e]$. Hence if $x < e < y$, then, because $e \geqq x+1 = x+e$,

$$x+y = xy+y = (x+1)y = x+1 = x+e.$$

If, on the other hand, $e+1 = 1$, then, because $e \in B$, $B = [0, e]$. Thus if $x < e < y$,

$$x+y = xy+y = (x+1)y = 1y = y.$$

Similar results hold according as $1+e$ is either e or 1.

4. Compact connected semirings in R_1

Let $\{S, +, \cdot\}$ be any compact connected semiring in R_1; then S is a closed interval. If K is the kernel of $\{S, \cdot\}$, then K, being compact and connected (Theorem 1.2.9 and Lemma 2.4.1 of [3]), is a closed interval.

If K is just a single point $\{0\}$, then 0 is a multiplicative zero and so $0+0 = 0(0+0) = 0$.

If K is not a single point, then, because K has a cutpoint, it follows from the Corollary to Theorem 2.4.6 of [3] that either $xy = x$ for all x, y in K or $xy = y$ for all x, y in K. In the former case,

$$x+y = xx+yx = (x+y)x \in K$$

for all x, y in K, and so $\{K, +, \cdot\}$ is a subsemiring with left-trivial multiplication. Similarly in the latter case, $\{K, +, \cdot\}$ is a subsemiring with right-trivial multiplication. It is clear from the distributive laws, however, that, because K has trivial multiplication, $\{K, +, \cdot\}$ is a topological semiring if and only if $\{K, +\}$ is an idempotent topological semigroup. Paalman-de Miranda has listed all topological semigroups $\{T, +\}$ on a closed interval T of R_1 for which $T+T = T$ (see § 2.6 of [3]). Any idempotent semigroup $\{T, +\}$ has this latter property and all idempotent topological semigroups on a closed interval can be identified from his results. Hence the structure of $\{K, +, \cdot\}$ is completely determined.

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