The Geometrical Interpretation of the Complete System of two Double Binary (2, 1) Forms.

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$$\text{Let* } f = a_x^2 \alpha_\xi = b_x^2 \beta_\xi = \xi_1 (a_{01} x_1^2 + 2a_{11} x_1 x_2 + a_{21} x_2^2) + \xi_2 (a_{02} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2)$$

and $f' = a'^2_x \alpha'_\xi = b'^2_x \beta'_\xi$
denote two double binary (2-1) forms in (x, ξ). It is proposed to discuss the geometrical significance of their simultaneous covariant complete system, which is here quoted without proof. †

When two quadrics which possess a common generator intersect, the remaining curve of intersection is in general a twisted cubic. Such a twisted cubic intersects each generator of one system in two points and the generators of the opposite system in one point. Let $x (= x_1 : x_2)$ be the parameter of the one system, and let ξ be that of the opposite system. This leads to a chord ξ of the cubic joining two points P, Q of the curve and through each of the points P and Q will pass a generator x of the opposite system. Thus to each ξ correspond two x's and to one x corresponds one ξ: namely the generator through the point of intersection of the cubic curve with x. There is thus a $(2-1)$ correspondence between the x and ξ, and as our double binary $(2-1)$ form is of such a nature *we represent our double binary forms f and f' as twisted cubics of the same kind on a quadric surface.* We recall these facts: if θ_x^2 and ϕ_x^2 are binary quadratics then $(\theta\phi)\, \theta_x \phi_x$ is simultaneously harmonic to both quadratics: and if $(\theta\phi)^2$ vanishes, θ_x^2 and ϕ_x^2 are harmonic pairs.

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* H. W. TURNBULL. *Proceedings of Royal Society of Edinburgh*, Vol. XLIII., 1923, pp. 44.

KASNER. *Transactions of American Mathematical Society*, Vol. I., 1900.
PEANO. *Battaglini Giorn. di. Math.*, XX., 1882.

† SADDLER. *Proceedings of Royal Society of Edinburgh*, Vol. XLV., 1924.

The complete list of the irreducible covariants is given for reference.

$f,\ f'$	$a_x^2\alpha_\xi$.		
$D,\ D'$	$(ab)(\alpha\beta)a_x b_x$	$=(f,f)_{11}$	§ 2
$\Delta,\ \Delta'$	$(ab)^2\alpha_\xi\beta_\xi$	$=(f,f)_{20}$	,,
$P,\ P'$	$-(ab)^2(\alpha\gamma)c_x^2\beta_\xi$	$=(f,\ D)_{10}=(f\Delta)_{01}$	,,
$R,\ R'$	$(ab)^2(cd)^2(\alpha\gamma)(\beta\delta)$	$=(f,\ P)_{21}$	,,
E	$(aa')a_x a'_x\alpha_\xi\alpha'_\xi$	$=(f,f')_{10}$	,,
F	$(\alpha\alpha')a_x^2 a'^2_x$	$=(f,f')_{01}$	,,
D_{12}	$(aa')(\alpha\alpha')a_x a'_x$	$=(f,f')_{11}$	§ § 2 & 3
Δ_{12}	$(aa')^2\alpha_\xi\alpha'_\xi$	$=(f,f')_{20}$	§ 3
A	$(aa')^2(\alpha\alpha')$	$=(f,f')_{21}$	,,
$G,\ G'$	$(aD')\ a_x D'_x\alpha_\xi$	$=(f,\ D')_{10}$	,,
$\kappa,\ \kappa'$	$(aD')^2\alpha_\xi$	$=(f,\ D')_{20}$	§ 2
$H,\ H'$	$(a'b')^2(\alpha\alpha')a_x^2\beta'_\xi$	$=(f,\ \Delta')_{01}$	§ 3
U	$(aD')\ (\alpha\beta)b_x^2$	$=(D,\ D')_{10}$	,,
R_2	$(ab)(\alpha\beta)(aD')(bD')$	$=(D,\ D')_{20}$	,,
u	$(ab)^2(\alpha\Delta')\beta_\xi\Delta'_\xi$	$=(\Delta,\ \Delta')_{01}$	,,
R_3	$(ab)^2(\alpha\Delta')(\beta\Delta')$	$=(\Delta,\ \Delta')_{02}$	,,
$\tau,\ \tau'$	$(aa')^2(\alpha'\Delta')\alpha_\xi\Delta'_\xi$	$=(f,\ P')_{20}$	,,
$Q,\ Q'$	$a_x^2 a'^2_x(\alpha\Delta)(\alpha'\Delta)$	$=(f,\ P')_{01}$	,,
$M,\ M'$	$(aa')(a'D')(\alpha\alpha')a_x D'_x$	$=(f,\ P')_{11}$	,,
$B,\ B'$	$(aa')^2(\alpha\Delta')(\alpha'\Delta')=(D_{12},\ D')_{20}$	$=(f,\ P')_{21}$	,,
$\rho,\ \rho'$	$-(ab)^2(\alpha\gamma)(cD')^2\beta_\xi$	$=(P,\ D')_{10}$	,,
R_4	$(aa')^2(\alpha\Delta)(\alpha'\Delta')(\Delta\Delta')$	$=(P,\ P')_{21}$	,,
$X,\ Y$	$a_x^2 b_x^2(\alpha\Delta')(\beta\Delta')$	$=(f^2,\ \Delta')_{02}$ (only 1 necessary)	
$\sigma,\ \sigma'$	$(ab)^2(Da')^2(\alpha\alpha')\beta_\xi$	$=(D\Delta,f')_{21}$	,,

The symbolic equivalent and transvectant are given for the first member: thus $\rho=-(ab)^2(\alpha\gamma)\ cD'^2\beta_\xi=(P,\ D')_{10}$.

§ 2. When ξ varies, the points P, Q where the generator cuts the cubic curve are pairs of points in involution. P is uniquely determined by Q, given ξ, and when Q moves to P, then P moves to Q.

$\Delta=(ab)^2\alpha_\xi\beta_\xi$. The values of x given by $f=0$ coincide if $\Delta=0$. Thus the double points D_1, D_2 of the involution on the curve are given by Δ. *The values of ξ satisfying $\Delta_\xi^2=0$ answer therefore to the two ξ tangent generators of the curve*: (throughout we shall consider these points as D_1, D_2 and similar points for the second cubic as D_1', D_2'). Since $(f, D)_{20}$ vanishes identically,* *the values of x satisfying $D_x^2 = (ab)(\alpha\beta)a_x b_x$ answer to the two x generators through $D_1 D_2$.*

$P = p_x^2 \pi_\xi = (f, D)_{10}$ is an associated twisted cubic which is the locus of the harmonic conjugates of the pairs of points in which any ξ generator cuts $f=0$ and $D_x^2 = 0$.

(1) The x generators through the intersection of f and P being given by $(fP)_{01} = \frac{1}{2}D^2 = 0$, we infer that P has double contact with f at the points $D_1 D_2$ and no other intersection with f at all.

(2) Since $(fP)_{20} \equiv 0$ any ξ generator cuts f and P harmonically.

Thus f and P are corresponding curves: the covariant D of P is that of f multiplied by the Resultant R: similarly for the other covariants of P. Thus $(P, P)_{20} = \frac{1}{2}R\Delta$.

$R = (f, P)_{21} = (D, D)_{20} = (\Delta, \Delta)_{0,2}$ is the resultant of the two quadratics $a_x^2 \alpha_1$ and $a_x^2 \alpha_2$ and vanishes if f decomposes into an x generator and a (1, 1) form, or a conic.

$F_x^4 = (\alpha\alpha')\, a_x^2 a_x'^2 = 0$ gives the four x generators through the points of intersection of the two cubic curves. The four generators will be equianharmonic if $A^2 + 3(R_3 - R_2) = 0$. Reciprocally the corresponding ξ generators will be obtained by eliminating x from $a_x^2 \alpha_\xi = 0$ and $a_x'^2 \alpha'_\xi = 0$, and are given by

$$(ab)^2 \alpha_\xi \beta_\xi \,.\, (a'b')^2 \alpha'_\xi \beta'_\xi - \{(aa')^2 \alpha_\xi \alpha'_\xi\}^2 = 0 \text{ or}$$
$$\Delta \,.\, \Delta' - \Delta_{12}^2 = 0.$$

$E = e_x^2 \epsilon_\xi^2 = 0 = (aa')\, a_x a_x' \alpha_\xi \alpha'_\xi$. This is a two-two curve generated as the locus of the pairs of points harmonic to each of the two pairs in which any ξ generator cuts the two twisted cubics. It is the same for all pencils of twisted cubics which pass through the intersection of the two given ones, since

$$(\lambda f + \mu f', \lambda' f + \mu' f')_{10} = (\lambda\mu' + \lambda'\mu)(f, f')_{10}.$$

It passes through the four common points of the two cubics.

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* $(fD)_{20} = (a_x^2 a_\xi, (bc)(\beta\gamma) b_x c_x)_{20} = -(ab)(bc)(ca)(\beta\gamma)a_\xi$
$= -\frac{1}{3}(ab)(bc)(ca)\{(\beta\gamma)\alpha_\xi + (\gamma\alpha)\beta_\xi + (\alpha\beta)\gamma_\xi\} \equiv 0$

The x generators common to E and f are given by

$$(aa')\,a_x a'_x\,.\,(\alpha\beta)(\alpha'\gamma)b_x^2\,c_x^2=\tfrac{1}{2}(ab)(\alpha\beta)\,a_x b_x\,.\,(\alpha'\gamma)a_x'^2 c_x^2=\tfrac{1}{2}D\,.\,F=0.$$

The ξ generators common to E and f are obtained by elimination of x, and are thus $(ff)_{20}\,.\,(EE)_{20}-\{(fE)_{20}\}^2=0$ or $\Delta\,.\,(\Delta\Delta'-\Delta_{12}^2)=0$, since $(fE)_{20}\equiv 0$.

Thus E, whose branch quartic * is $\Delta\Delta'-\Delta_{12}^2=0$, touches the four ξ generators through the common points of intersection of the two cubics and passes through the points corresponding to the double points of the involution on each of the cubics:—the points corresponding to (D,Δ) and (D',Δ').

The intermediate covariant, Δ, of the pencil $a_x^2\alpha_\xi+\lambda a_x'^2\alpha'_\xi=0$

is $\qquad (ab)^2\alpha_\xi\,\beta_\xi+2\lambda(aa')^2\alpha_\xi\,\alpha'_\xi+\lambda^2(a'b')^2\alpha'_\xi\,\beta'_\xi=0$

or $\qquad \Delta+2\lambda\Delta_{12}+\lambda^2\Delta'=0,$

where $\Delta_{12}=0$ gives the two ξ generators which cut the two twisted cubics in a harmonic range.

Now when this quadratic in λ has equal roots $\Delta_{12}^2-\Delta\Delta'=0$ giving, as before, the branch quartic of the two-two curve E.

The second branch quartic of this two two curve is $(E,E)_{02}=0$ or $DD'-D_{12}^2=0$ where $D_{12}=(aa')(\alpha\alpha')a_x a'_x$ which may be obtained by considering the intermediate covariant D of $f+\lambda f'=0$, *i.e.* $D+2\lambda D_{12}+\lambda^2 D'=0$.

The Linear Covariant $\kappa_\xi=(fD')_{20}=-(D_{12},f')_{20}=0$. $\kappa_\xi=0$ can be regarded as the ξ generator which cuts f and D' harmonically, or better:—let the two x generators of the second cubic through $D'_1 D'_2$ meet the first cubic in $D_3 D_4$: the harmonic pair to $D_1 D_2$; $D_3 D_4$ will be the points in which κ_ξ meets the first cubic: similarly for $\kappa'_\xi=(f'D)_{20}$. Again $\kappa_\xi=-(D_{12},f')_{20}$ where $D_{12}=(aa')(\alpha\alpha')\,a_x\,a'_x$. Let κ_ξ meet the second cubic in $D''_1 D''_2$: and let P, Q be the pair of points where the two x generators corresponding to $D_{12}=0$ meet f'. Then the common pair of harmonic points to $P\,Q$; $D'_1 D'_2$ will be the points $D''_1 D''_2$.

§ 3. Polar Conic of a given x – Generator.

Fix a ξ generator and take the fourth harmonic to where an x generator (say y) meets this ξ with respect to the two points where the generator cuts the cubic: namely $a_x a_y \alpha_\xi=0$. Vary ξ – this

* Turnbull. *Proc. Royal Soc. Edinburgh*, Vol. XLIV. (1923-4) p. 36.

gives us a *polar conic* on the quadric corresponding to all points on y: a conic, since it is a $(1,1)$ correspondence on the quadric between x and ξ

Similarly take the polar conic of y with respect to the second cubic, so that $a'_x a'_y \alpha'_\xi = 0$.

These two polar conics have their planes conjugate with respect to the quadric, if y satisfies $(aa')(\alpha\alpha')a_y a'_y = 0$, *i.e.* if y belongs to the covariant $D_{12} = 0$.

It is to be noticed that two conics on the quadric $p_x \pi_\xi = 0$ and $q_x \rho_\xi = 0$ are conjugate to one another when the forms are apolar :—namely when $(p\,q)(\pi\,\rho) = 0$.

PROPERTIES OF THE TWO-TWO CURVE, $E_x^2 \epsilon_\xi^2 = 0$.

Its J covariant $(E,E)_{11}$ is another two-two form which may be shown to be $f.\kappa + f'\kappa' = 0$.

J passes through the four points of intersection of f and f' and also where κ cuts f' and where κ' cuts f. Again

$$(E_x^2 \epsilon_\xi^2,\ J)_{02} = UD_{12} + D'.M' - DM - ADD' \equiv 0,\ (Syzygy 34),$$

proving that any ξ generator cuts E and J harmonically : similarly for any x generator.*

The x generators through the intersection of E and J obtained by eliminating ξ between these two equations, are given by

$\lambda.F(DD' - D_{12}^2) = 0$ where λ is a constant factor, $AR_2 - R_4$.

But $F = 0$ gives the four x generators through the intersection of E and f and $DD' - D_{12}^2 = 0$ is the branch quartic corresponding to the four x rays which touch E.

The second branch quartic of E has been shown to be

$$\Delta\Delta' - \Delta_{12}^2 = 0$$

which determines the ξ rays touching E at the points of intersection of E and f and, as shown above, of E and J.

Thus the curves E and J intersect at *8* points

$$I_1 I_2 I_3 I_4\,;\ I_1^1 I_2^1 I_3^1 I_4^1.$$

The x generators through the four I points touch E at these points ; similarly for the ξ generators.*

The condition that J breaks up into linear factors is the vanishing of the third degree invariant of E: *i.e.*

$$(EJ)_{22} = \tfrac{3}{2}\,(AR_2 - 2R_4).*$$

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* TURNBULL, *loc. cit.*, pp. 36, 34.

Now $(\kappa\kappa')_{01} = AR_2 - 2R_4$. Thus *when the two ξ generators corresponding to κ and κ' coincide the J covariant of the two-two curve E breaks up into linear factors.*

APOLARITY OF f AND f'.

If $A = (aa')^2(\alpha\alpha')$, then A vanishes when f and f' are apolar. Two interpretations may be given as follows.

If the *polar conic* corresponding to y with respect to f is conjugate to the *polar conic* of z with respect to f', then

$$(aa')(\alpha\alpha')\,a_y\,a'_z = 0.$$

If the reciprocal relation also holds then

$$(aa')(\alpha\alpha')\,a_z\,a'_y = 0.$$

Thus $(aa')(\alpha\alpha')\,\{a_z\,a'_z - a_z\,a'_y\} = 0$

or $\qquad (aa')^2(\alpha\alpha')(yz) = 0 \qquad (yz) \neq 0$

thus $\qquad (aa')^2(\alpha\alpha') = A = 0.$

Again, the two ξ generators which are cut harmonically by the cubics f and P' are given by $\tau_\xi{}^2 = 0 = (aa')^2(\alpha'\Delta')\,\alpha_\xi\,\Delta'_\xi$.

These two generators will be harmonic to the pair given by $\Delta_{12} = 0$ if $(\tau_\xi{}^2, \Delta_{12\xi}{}^2)_{02}$ vanishes.

But $(r_\xi{}^2, \Delta_{12\xi}{}^2)_{02} = \frac{1}{2}(aa')^2(\alpha'\Delta')(bb')^2\,\{(\alpha\beta)(\Delta'\beta') + (\alpha\beta')(\Delta'\beta)\}$

$$= \tfrac{1}{2}(aa')^2(\alpha'\Delta')(bb')^2(\alpha\Delta')(\beta\beta') + (aa')^2(bb')^2(\alpha'\Delta')(\alpha\beta)(\Delta'\beta')$$

$$= -\tfrac{1}{2}A.(fp')_{21} + R_p$$

where R_p vanishes identically on interchanging a and b and also a' and b'. $= -\frac{1}{2}A(fp')_{21} = -\frac{1}{2}A.B.$

Thus, *if B does not vanish the two cubics will be apolar when the two generators corresponding to Δ_{12} are harmonic to the two given by $\tau_\xi{}^2$:*

Similarly the apolarity of

$$P \text{ and } G'(=(a'D)a'_xD'_x\alpha_\xi') = \tfrac{1}{2}AR = 0,$$
$$P' \text{ and } G(=(aD')a_xD_x\alpha_\xi) = \tfrac{1}{2}AR' = 0,$$
$$P \text{ and } H'\ (=(\alpha'\Delta)a'^2_x\Delta_\xi) = -\tfrac{1}{2}AR = 0.$$

$G_x^2\gamma_\xi = 0 = (f, D')_{10} = -(\Delta_{12}, f')_{01}.$

This is the two-one curve which is the locus of the common harmonic conjugates of the pair of points in which any ξ generator cuts the cubic f and the pair of generators $D'_x{}^2 = 0$.

Any ξ generator naturally cuts f and G harmonically, since $(f, G)_{20}$ identically vanishes.

The x generators through the point of intersection of f and G are $-(aa')^2(\alpha'\beta')(\alpha\beta)b_x^2 b'^2_x = 0 = D.D'$, and the ξ generators corresponding are got by eliminating x between

$a_x^2 \alpha_\xi = 0 = f$ and $(aD')a_x D'_x \alpha_\xi = G = 0$ and are $\Delta\{R'\Delta - \kappa_\xi^2\} = 0$.

Thus G passes through the common points of f and P and has two x generators $D'^2_x = 0$ common with f′ and P′: Similarly for G'. G will be apolar to f if $R_2 = (DD')^2$ vanishes, which is the condition that the x generators corresponding to D and D' may be harmonic.

$H_x^2 \eta_\xi = (\alpha\Delta')a_x^2 \Delta'_\xi = (f\Delta')_{01}$. Let a ξ generator cut the first cubic in $A_1 A_2$: the line harmonic to $A_1 A_2$ with respect to Δ'^2_ξ will be cut by the two x generators through $A_1 A_2$ in points which lie on a new cubic $H_x^2 \eta_\xi = 0$.

$H_x^2 \eta_\xi = 0$ passes through the points where the two lines given by Δ meet f: the x generators through the points of intersection of H and f_1' being given by $(\alpha\Delta')(\alpha'\Delta')a_x^2 a'^2_x = 0$, form the covariant $Q = 0$ and are thus the same lines as pass through the intersection of f and P'.

Similarly the x generators through f' and P are the same as those through f and H' and are given by $Q'^4_x = 0$.

$H_x^2 \eta_\xi = 0$ will be apolar to $f' = 0$ if $(a'^2_x \alpha'_\xi, (\alpha\Delta')a_x^2 \Delta'_\xi) = 0$,

$$\text{or} \quad (aa')^2(\alpha\Delta')(\alpha'\Delta') = 0 = (fP')_{21} = B.$$

But if B vanish $\tau_\xi^2 = 0$ will be harmonic to $\Delta_{12\xi}^2 = 0$

$$\text{or} \quad D'^2_x = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ to } D_{12x}^2 = 0.$$

Thus when f and P' are apolar so are H and f_1' and τ is harmonic to Δ_{12}.

Again, $\tau = (fP')_{20} = 0$ will be harmonic to $\Delta_\xi^2 = 0$ when $(\tau_\xi^2, \Delta_\xi^2)_{02} = 0 = R_4 = (P, P')_{31}$ *so when P is apolar to P′, τ will be harmonic to Δ and τ′ will be harmonic to Δ′.*

Just as we have $\kappa_\xi = (fD')_{20}$ so we have $\rho_\xi = (p_x^2\pi_\xi, D'^2_x)_{20}$ and our linear covariant corresponding to $\rho_\xi = 0$ can be identified with a ξ generator cutting across the curve $p_x^2 \pi_\xi = 0$ in a similar manner to that corresponding to $\kappa_\xi = 0$ and $f = 0$: now the x generators through κ and P are the same as those through ρ *and* f: both are

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given by $(\kappa_\xi, p_x^2 \pi_\xi)_{01} = RD' - R_2 D = 0$, while exactly the same pair are given by κ' and $G' = 0$. Thus the x generators through the intersection of

$$\begin{array}{lll} 1 & \rho_\xi = 0 \text{ and } & a_x^2 \alpha_\xi = 0 \\ 2 & \kappa_\xi = 0 \quad ,, & p_x^2 \pi_\xi^2 = 0 \\ 3 & \kappa'_\xi = 0 \quad ,, & G'^2_x \gamma'_\xi = 0 \end{array}$$

are all given by $RD' - R_2 D = 0$,

and of

$$\begin{array}{lll} 1 & \rho'_\xi = 0 \text{ and } & a'^2_x \alpha'_\xi = 0 \\ 2 & \kappa'_\xi = 0 \quad ,, & p'^2_x \pi'_\xi = 0 \\ 3 & \kappa_\xi = 0 \quad ,, & G_x^2 \gamma_\xi = 0 \end{array}$$

by $R'D - R_2 D' = 0$.

These being linear combinations of D and D', it follows that $U_x^2 = 0 = (DD') D_x D'_x$ will be harmonic to such generators.

e.g., $\rho_\xi = (p_x^2 \pi_\xi, D'^2_x)_{20}$

$$= (aD)(aD')(DD')\alpha_\xi :$$

$$\begin{aligned} (a_x^2 \alpha_\xi, \rho_\xi)_{01} &= (\alpha\beta)(bD)(bD')(DD')a_x^2 \\ &= (DD')(aD)(bD')(\alpha\beta) - (ab)(bD')(DD')(\alpha\beta)\ldots \\ &= \tfrac{1}{2}(\alpha\beta)(ab)\ldots(DD')^2 - (D_1D')(DD')D_xD_{1x} \\ &= \tfrac{1}{2}D.R_2 - \tfrac{1}{2}\{2(DD')^2D_{1x}^2 - (DD_1)^2D'^2_x\} \\ &= \tfrac{1}{2}RD' - \tfrac{1}{2}D.R_2 \end{aligned}$$

The two x generators corresponding to $(DD')D_xD_x' = U_x^2 = 0$ are the two generators through the intersection of κ_ξ and f or through κ'_ξ and f' since $(DD')_{10} = (\kappa_\xi, f)_{01} = (\kappa', f')_{01}$

$$\begin{aligned} (\kappa_\xi, f)_{01} = \{(aD')^2 \alpha_\xi, b_x^2 \beta_\xi\}_{01} &= (\alpha\beta)(aD')^2 b_x^2 \\ &= \tfrac{1}{2}(\alpha\beta)\{(aD')^2 b_x^2 - (bD')^2 a_x^2\} \\ &= (\alpha\beta)(bD')(ab)a_xD_x' \\ &= (DD')D_xD_x'. \end{aligned}$$

Suppose $\kappa_\xi = 0$ is produced to meet the second cubic $f' = 0$ in the points $A_3 A_4$: the x generators through these points are given by

$$(aD')^2(\alpha\alpha')a'^2_x = 0$$

$$\text{or} \quad (aa')(a'D')(\alpha\alpha')D'_x a_x + A.D'^2_x = 0$$

$$\text{or} \quad M_x^2 + A.D'^2_x = 0.$$

Producing these lines through A_3 and A_4 to meet f again in $A_5 A_6$ and taking the pair of points harmonic to $A_5 A_6$, $D_1 D_2$ we get $(aD')^2 (\alpha\alpha') (a' b)^2 \alpha'_\xi = 0$

$$\text{or} \quad A\kappa_\xi - \rho'_\xi = 0.$$

Similarly we can construct $A\kappa'_\xi - \rho_\xi = 0$.

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These lines can be taken to represent the linear covariants instead of $\rho'_\xi = 0$, $\rho_\xi = 0$; so for $M_x{}^2 = 0$ and $M'_x{}^2 = 0$. Again $M_x{}^2 = (fP')_{11} = 0$ can be regarded as the two x generators whose *polar conics* on the quadric with respect to f and P' are conjugate: or since $(fP')_{11} = (aa')(\alpha\Delta')(\alpha'\Delta')\, a_x a'_x = \{(aa')\, a_x a'_x \;\, \alpha_\xi \alpha'_\xi,\;\; \Delta'_\xi{}^2\}_{02}$ as the two generators whose corresponding ξ generators of the two-two curve $E_x{}^2 \epsilon_\xi{}^2 = 0$ are harmonic to the pair $\Delta'_\xi{}^2 = 0$.

Most of the covariants can be interpreted in a similar manner. Thus $X_x{}^4 = a_x{}^2 b_x{}^2 (\alpha\Delta')(\beta\Delta') = 0$ will give the four x generators through the points of intersection of $\Delta'_\xi{}^2 = 0$ with f: or the lines through the intersection of $H_x{}^2 \eta_\xi = 0$ and f since

$$\begin{aligned} a_x{}^2 b_x{}^2 (\alpha\Delta')(\beta\Delta') &= -\{a_x{}^2 (\alpha\Delta')\, \Delta'_\xi,\; b_x{}^2 \beta_\xi\}_{01} \\ &= +\{f,\; H\}_{01}. \end{aligned}$$

The (2,1) covariant $p_x{}^2 \pi_\xi$ of $a_x{}^2 \alpha_\xi + \lambda a'_x{}^2 \alpha'_\xi$ is

$$P + \lambda(Af + 2G' + H') + \lambda^2(H + 2G - Af') + \lambda^3 P' = 0 = P_{f+\lambda f'}$$

while the invariant $R\cdot$ of the form is

$$\begin{aligned} &\left(a_x{}^2 \alpha_\xi + \lambda a'_x{}^2 \alpha'_\xi,\; P_{f+\lambda f'}\right)_{21} \\ &= R + 4\lambda B' + 2\lambda^2(2R_2 + R_3 - A^2) + 4\lambda^3 B + \lambda^4 R' \\ &= R_{f+\lambda f'} \end{aligned}$$

There are thus four members of the pencil which degenerate into a linear form in ξ and a conic: the values of λ will satisfy the equation $R_{f+\lambda f'} = 0$.