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# On Benson's Definition of Area in Minkowski Space

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*Abstract.* Let $(X, \|\,.\,\|)$ be a Minkowski space (finite dimensional Banach space) with unit ball $B$. Various definitions of surface area are possible in $X$. Here we explore the one given by Benson [1], [2]. In particular, we show that this definition is convex and give details about the nature of the solution to the isoperimetric problem.

## 1 Introduction

In [14, Chapter 5] several possible definitions of $n-1$-dimensional content in an $n$-dimensional space are given. "Possible" here means definitions that satisfy a minimal set of axioms. Questions that arise are: how large is the set of possible definitions? can some sort of structure be imposed on them to make sense out of the variety? and, what extra axioms might sensibly be imposed to reduce the variety?

A first step towards answers to these questions, especially the last one, is to undertake a detailed study of each of the reasonable definitions. In [14, Chapters 6, 7] this was done for two of them. The purpose of this note is to do the same for a third, due to R. V. Benson [1], that received only a brief mention (pp. 140 and 170) in [14]. We give a complete proof (different from Benson's) of the convexity of the definition (see below for the meaning of undefined terms). We also give a description of the isoperimetrix, an aspect that is not treated in Benson's work [1], [2]. In particular, if $\mathbf{I}$ denotes the map that assigns to each unit ball the corresponding solution to the isoperimetric problem, the *isoperimetrix*, then we show that $\mathbf{I}$ maps polytopes to polytopes, and that $\mathbf{I}$ is not injective. Moreover, there are balls in $\mathrm{R}^3$, other than ellipsoids, that are fixed points of $\mathbf{I}$ and yet others that, while not fixed by $\mathbf{I}$, are invariant under $\mathbf{I}^2$.

The fact that for Benson's definition the mapping $\mathbf{I}$ is not one-to-one is shared by several similar definitions. If this is thought to be a disadvantage, then such definitions can be ruled out by imposing the extra axiom that $\mathbf{I}$ be injective.

The background, terminology and notation will all be consistent with [14] except that we shall use $n$ (rather than $d$) to denote the dimension of the ambient space $X$. For the background in convexity theory, see Schneider [12]. However, to make the discussion self-contained we begin with a brief introduction of the main ideas.

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## 2 Background

The situation is an $n$-dimensional Minkowski space (*i.e.*, a real normed space $(X, \|\,.\,\|)$) with unit ball $B$. We can think of $X$ as $\mathbb{R}^n$ and $B$ as a centrally symmetric convex body. Since translations are isometries, in order that measures (like length, area, volume) be geometrically meaningful, we require that they be translation invariant which implies that they are all Haar measures. Thus, $n-1$-dimensional content (which we shall describe as *area*) should be a Haar measure on each hyperplane in $X$ and the same Haar measure on parallel hyperplanes. Therefore, given a direction on $X$ (*i.e.*, a linear functional $f$ in the dual space $X^*$) we need some way of normalizing Haar measure in the hyperplanes $H(f,\alpha) := \{x : f(x) = \alpha\}$. There are two equivalent ways of doing this. The first is geometrical and basis free; the second requires the introduction of a coordinate system and thereby an inner product and Lebesgue measure. The first method is to specify a number $\mu(B \cap f^{\perp})$ to be assigned as the area of $B \cap f^{\perp}$ where $f^{\perp} := H(f, 0)$. Then, for all Borel subsets $A$ of all hyperplanes $H(f,\alpha)$, we have

$$\mu(A) := \mu(B \cap f^{\perp}) \frac{\lambda(A)}{\lambda(B \cap f^{\perp})}, \tag{1}$$

where $\lambda$ is an arbitrary (but fixed) Haar measure on $f^{\perp}$. Since all such measures differ only by a multiplicative constant the ratio in (1) is independent of $\lambda$. The second method, given Lebesgue measure $\lambda$ on $f^{\perp}$, requires that we specify a number $\sigma(f)$ to be the ratio of the desired measure $\mu$ to $\lambda$ in all the hyperplanes $H(f,\alpha)$. Then we have

$$\mu(A) := \sigma(f)\lambda(A). \tag{2}$$

Evidently, (1) and (2) are related by using the same measure $\lambda$ in both and setting

$$\sigma(f) = \mu(B \cap f^{\perp})/\lambda(B \cap f^{\perp}).$$

The first method is preferable because it is coordinate free and independent of any Euclidean construction. The second is more practical for computational purposes. The various definitions of area alluded to in Section 1 are different ways of assigning the number $\mu(B \cap f^{\perp})$ or, equivalently, of defining $\sigma$.

The hyperplane $f^{\perp}$ is unaffected by a scalar change to the direction $f$. However, in (2), it is useful to first insist that $f$ be a Euclidean unit vector and that $\sigma$ be defined initially on the Euclidean unit ball in $X^*$ and satisfy $\sigma(-f) = \sigma(f)$. Then $\sigma$ may be extended by non-negative homogeneity to the whole of $X^*$. If $\sigma$ defined in this way is a convex function on $X^*$ then it is the support function of a convex body $\mathbf{I}(B)$ which (up to homothety) is the solution to the isoperimetric problem (see Busemann [3], [4], or [14]). If $\sigma$ is convex then "flat surfaces minimize area"; *i.e.*, if $M$ is a Borel set in $f^{\perp}$ whose boundary is an $n-2$-dimensional manifold then for all rectifiable $n-1$-dimensional manifolds $M'$ in $X$ that have the same boundary as $M$ we have $\mu(M) \le \mu(M')$. Thus the question of whether or not $\sigma$ is convex is of some importance. Benson's definition has this property (which is what was meant in Section 1 by saying that it is convex).

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## 3 The Function $\sigma$ Using Benson's Definition

Let $(X, \|\,.\,\|)$ be an $n$-dimensional Minkowski space with unit ball $B$. If $Y$ is an $n-1$-dimensional subspace of $X$ then $B \cap Y$ is a centrally symmetric convex body in $Y$. Among all the $n-1$-dimensional parallelotopes circumscribed to $B \cap Y$ there will be at least one minimal one $P_Y$ (minimal with respect to all Haar measures on $Y$). This statement is a consequence of the Blaschke selection theorem and the fact that parallelotopes form a closed subset of the space of convex bodies endowed with the Hausdorff metric.

***Definition 1*** The *Benson area* $\mu$ in $Y$ is the unique Haar measure for which

$$\mu(P_Y) = 2^{n-1}.$$

***Remark*** When $n = 2$ then $B \cap Y$ is an interval. Hence, $B \cap Y = P_Y$ and the definition coincides with the usual one.

From now on we shall assume that $X$ is equipped with an auxiliary Euclidean norm that induces Lebesgue measure (of the appropriate dimension) on each subspace. All these Lebesgue measures will be denoted by $\lambda$ (with subscripts if area and volume occur in the same equation). Therefore, the homogeneous function $\sigma$ is given by:

$$\sigma(f) := \frac{2^{n-1}|f|}{\lambda(P_Y)} \tag{3}$$

where $f \in X^*$ is a linear functional such that $Y = f^{\perp}$ and $|f|$ is the Euclidean norm of $f$.

The following simple lemma from linear algebra will be used repeatedly in what follows.

***Lemma 1*** *If $P$ is a parallelotope centred at the origin and if $\{u_1, u_2, \ldots, u_n\}$ are the normals to the facets of $P$ whose lengths are such that the facets with normal $u_i$ are contained in $H(u_i, \pm 1/2)$ then $\lambda(P) = (\det U)^{-1}$ where $U$ is a matrix whose rows are the vectors $u_1, u_2, \ldots, u_n$ in some coordinate system.*

**Proof** If we translate $P$ so that one vertex is at 0 then the volume is unchanged and the facets lie in the hyperplanes $H(u_i, 0)$ and $H(u_i, 1)$. If $\{b_1, b_2, \ldots, b_n\}$ are the edges of $P$ that emanate from 0 then these vectors form a basis for $X$ and $\lambda(P) = \det B$ where $B$ is the matrix that has the vectors $b_i$ as columns. The vectors $u_i$ are a basis for $X^*$ that is dual to the basis $b_i$ and hence, for the matrices $B$ and $U$, we have $UB = I$. Therefore $\lambda(P) = \det B = (\det U)^{-1}$ as required. ∎

***Remark*** Since volumes are to be positive, we can either suppose that the vectors are ordered so that all the determinants we consider are positive, or, wherever a determinant appears we can insert an absolute value.

***Corollary 2*** *A parallelotope $C := \{x : |u_i(x)| \le 1, i = 1, 2, \ldots, n\}$ is a minimal parallelotope circumscribed about $B$ if and only if* $\operatorname{co}\{0, u_1, u_2, \ldots, u_n\}$ *(where* co *denotes convex hull) is a maximal simplex inscribed to $B^\circ$ and with one vertex at* 0.

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**Proof** Let $S := \operatorname{co}\{0, u_1, u_2, \dots, u_n\}$, then, from the Lemma, we have

$$\lambda(C)^{-1} = 2^{-n}\det[u_1, u_2, \dots, u_n] = n!\,2^{-n}\lambda(S).$$

Moreover, $u_i \in B^\circ$ if and only if the hyperplane $H(u_i, 1)$ does not intersect the interior of $B$, *i.e.*, $S$ is inscribed to $B^\circ$ if and only if $C$ is circumscribed to $B$. The result is now clear. ■

***Definition 2* (Carathéodory)** A vector $x$ in $X$ is said to be *normal* to $y$ in $X$, written $x \dashv y$, if $\|x + \alpha y\| \geq \|x\|$ for all $\alpha$ in R.

Geometrically, this means that the line $\{x + \alpha y : \alpha \in \mathrm{R}\}$ is tangent to the ball $\{x' : \|x'\| = \|x\|\}$ at $x$.

***Theorem 3* (Benson)** *If $(X, \|\,.\,\|)$ is an $n$-dimensional normed space with unit ball $B$ and if $P_0$ is a minimal parallelotope circumscribed to $B$ and if $\{\pm x_1, \pm x_2, \dots, \pm x_n\}$ are the centres of the facets of $P_0$ then $\|x_i\| = 1$ and $x_i \dashv x_j$ for all $i$, $j$, $i \neq j$.*

**Proof** Since Blaschke's theorem assures us that a minimal parallelotope exists, we can prove the theorem by showing that a parallelotope circumscribed to $B$ and having a facet that does not touch $B$ at its centre is not minimal. Suppose, then, that $P$ is circumscribed to $B$, that its facets are $\pm F_i$, $i = 1, 2, \dots, n$, with corresponding normals $u_i$ and contained in hyperplanes $\pm H_i := H(u_i, \pm 1/2)$. Suppose, further, that $F_1$ does not touch $B$ at its centre $x_1$. We have $u_1(x_1) = 1/2$; the key to the proof is to observe that for $i \geq 2$, $u_i(x_1) = 0$ and therefore $x_1 = \alpha u_2 \times u_3 \times \cdots \times u_n$, the generalized cross-product of these normals suitably scaled by $\alpha$. Since $F_1$ does not touch $B$ at $x_1$ we have $\|x_1\| > 1$. Let $x_1' := x_1/\|x_1\|$ and let $H_1'$ be a hyperplane that touches $B$ at $x_1'$. Let $u_1'$ be the normal to $H_1'$ such that $x_1' \in H_1' = H(u_1, 1/2)$. Now, from Lemma 1, we have:

$$\begin{aligned}
\lambda(P)^{-1} = \det[u_1, u_2, \dots, u_n] &= u_1(u_2 \times u_3 \times \cdots \times u_n)\\
&= u_1(\alpha^{-1}x_1) = 1/(2\alpha)\\
&= u_1'(\alpha^{-1}x_1')\\
&< u_1'(\alpha^{-1}\|x_1\|\,x_1')\\
&= u_1'(u_2 \times u_3 \times \cdots \times u_n)\\
&= \det[u_1', u_2, u_3, \dots, u_n] = \lambda(P')^{-1}
\end{aligned}$$

where $P'$ is the parallelotope enclosed by $\pm H_1', \pm H_2, \pm H_3, \dots, \pm H_n$. Now $P'$ circumscribes $B$ and has volume strictly less than that of $P$, hence $P$ is not minimal which establishes the theorem. ■

Thus the centres of the facets of a minimal circumscribing parallelotope form a mutually normal system of unit vectors. For this reason we shall call such a parallelotope a *hypercube* in $X$, (a hypercube will always have edges of length 2). In [13], Taylor showed that among all $n$-simplices with one vertex at the origin and the others in $B$, a maximal one is also

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characterized by the non-zero vertices forming an "orthonormal" system. Corollary 2 and Theorem 3 give an alternative proof of Taylor's result. It is this connection and the theorem of Busemann, Ewald and Shephard [6, Corollary 10] on the convexity of the volume of maximal simplices inscribed in projections of a convex set, that Benson [1] used to prove the convexity of his area function. Note that in [2] he refers to Busemann and Straus [5] for the proof, a reference that seems to be misleading.

***Proposition 4*** *If $B$ is a polytope then there is a hypercube whose facets contain facets of $B$.*

**Proof** The proof is the same as the central part of the previous one. Let $C$ be a hypercube circumscribed to $B$ and let the facets $\pm F_i$ of $C$ be contained in hyperplanes $\pm H_i := H(u_i, \pm 1/2)$ $i = 1, 2, \ldots, n$, and let $x_i$ be the centre of $F_i$ as before. If $x_1$ is in the (relative) interior of a facet $F$ of $B$ then $F_1 \supseteq F$. On the other hand, if $x_1$ is on the boundary of a facet $F$ of $B$ with normal $u$, one can suppose that $F$ is contained in the hyperplane $H := H(u, 1/2)$. Then, as previously,

$$\begin{aligned}\lambda(C)^{-1} = \det[u_1, u_2, \ldots, u_n] &= u_1(\alpha^{-1}x_1)\\ &= 1/(2\alpha) = u(\alpha^{-1}x_1)\\ &= \det[u, u_2, u_3, \ldots, u_n] = \lambda(C')^{-1}\end{aligned}$$

where $C'$ is the parallelotope enclosed by $\pm H, \pm H_2, \ldots, \pm H_n$. Hence, $C'$ is also a hypercube and the facets of $C'$ that contain $\pm x_1$ also contain the facets $\pm F$ of $B$. We may continue this process with $x_2, x_3, \ldots x_n$ to obtain the required hypercube. ■

***Theorem 5*** *If $B$ is a polytope then the function $\sigma$ defined by Equation (3) is convex.*

**Proof** Let $f$ be a Euclidean unit linear functional on $X$ and let $P_f$ be an $(n-1)$-hypercube circumscribed to $B \cap f^{\perp}$. Then, by Proposition 4, we may assume that the facets of $P_f$ contain facets of $B \cap f^{\perp}$ which we will denote by $\pm F_1, \pm F_2, \ldots, \pm F_{n-1}$. For each $i$, there is a facet $\hat{F}_i$ of $B$ such that $F_i = \hat{F}_i \cap f^{\perp}$; (it is possible that $F_i$ is the intersection of two adjacent facets of $B$ in which case choose one of them as $\hat{F}_i$). As usual, choose $u_i$ so that the hyperplane that contains $\hat{F}_i$ is $H_i = H(u_i, 1/2)$. These $2n-2$ hyperplanes bound a cylinder

$$C_f := \{x : -1/2 \le u_i(x) \le +1/2, i = 1, 2, \ldots, n-1\}$$

such that $B \subseteq C_f$ and $P_f = C_f \cap f^{\perp}$.

In order to calculate the area, $\lambda_{(n-1)}(P_f)$, of $P_f$ consider a parallelotope $P'_f$ of (Euclidean) height 1 in the direction $f$ and bounded by the cylinder $C_f$. The "top" and "bottom" of $P'_f$ are at $\pm f/2$ ($f$ is of Euclidean length 1) and the hyperplanes that determine these two surfaces are $H(f, \pm 1/2)$. Therefore, by using Lemma 1 again,

$$\lambda_{(n-1)}(P_f) = \lambda_{(n)}(P'_f) = \det[f, u_1, u_2, \ldots, u_{n-1}]^{-1}. \tag{4}$$

Conversely, for every choice of $n-1$ pairs of opposite facets of $B$ $\pm G_1, \pm G_2, \ldots, \pm G_n$ with normals $w_j$ chosen so that $G_j \subset H(w_j, 1/2)$ there is a corresponding cylinder $C :=$

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$\{x : -1/2 \le w_j(x) \le 1/2, j = 1, 2, \dots n-1\}$ that circumscribes $B$. Hence, $C \cap f^{\perp}$ is an $n-1$-dimensional parallelotope $P$ (possibly unbounded) that circumscribes $B \cap f^{\perp}$. Moreover, the same calculation as the one just done, shows that

$$\lambda_{(n-1)}(P) = \det[f, w_1, w_2, \dots w_{n-1}]^{-1}.$$

Thus, among all such parallelotopes, the minimal one, $P_f$, is characterized by choosing $u_1, u_2, \dots, u_{n-1}$ so that

$$\det[f, u_1, u_2, \dots, u_{n-1}] = \max\{\det[f, w_1, w_2, \dots, w_{n-1}]\}.$$

Here the maximum is taken over all possible choices of $n-1$ normals from the facets of $B$. Finally,

$$\begin{aligned}\sigma(f) = \frac{2^{n-1}|f|}{\lambda(P_f)} &= 2^{n-1}\max\{\det[f, w_1, w_2, \dots, w_{n-1}]\}\\ &= \max\{\det[f, v_1, v_2, \dots, v_{n-1}]\}\\ &= \max\{f(v_1 \times v_2 \times \cdots \times v_{n-1})\}\end{aligned} \tag{5}$$

where $v_j := 2w_j$ is precisely the vertex of $B^{\circ}$ corresponding to the facet of $B$ with normal $w_j$. Hence the function $\sigma$ is the supremum of finitely many linear functionals and is convex. ■

***Corollary 6* (Benson)** *If $(X, \|\,.\,\|)$ is an arbitrary normed space then $\sigma$ defined by Equation (3) is convex.*

**Proof** An examination of the previous proof shows that with only slight modifications it can be used to show that in the general case $\sigma(f)$ is given by Equation (5) except that now the $v_j$'s are to be interpreted as arbitrary points of $B^{\circ}$. (It is clear that the maximum will occur when they are chosen on the boundary, and when they are chosen to be linearly independent and oriented so the determinant is positive.) The compactness of $B^{\circ}$ shows that it is an attained maximum in (5). Alternatively, one may use an argument that approximates $B$ by polytopes and then use the continuity of $\sigma$. ■

***Remarks*** 1. Equation (5) for $\sigma$ is very simple. It is especially gratifying that all the extra factors of 2 disappear.

2. In the discussion $f$ was assumed to be of Euclidean length 1 but that is no longer necessary in the final Equation (5).

3. Likewise, we used Euclidean constructions and arguments freely throughout the proof but in (5) the vectors $v_j$ are in $X^*$ and so $v_1 \times v_2 \times \cdots \times v_{n-1}$ is in $X$ and $f(v_1 \times v_2 \times \cdots \times v_{n-1})$ makes no use of an inner product.

4. Equation (5) may be viewed as follows. The right hand side of (5) is the volume of a parallelotope with one vertex at 0, and spanned by the vectors $f$ and $v_1, v_2, \dots, v_{n-1}$. Up to a factor of $n!$, this is also the volume of an $n$-simplex with vertices at 0, $f, v_j$. Thus, $\sigma(f)/n!$ is the volume of a maximal simplex with one vertex at 0, another at $f$ and the rest in $B^{\circ}$.

5. Corollaries 6 and 2 give an alternative proof of the result of Busemann, Ewald and Shephard [5] mentioned above.

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## 4 The Isoperimetrix Using Benson's Definition

As we indicated in Section 2, the isoperimetrix **I** is the convex body with support function $\sigma$. In this section we will use Equation (5) to obtain information about the isoperimetrix.

***Theorem 7*** *If $B$ is the unit ball in a Minkowski space $(X, \|\,.\,\|)$ then*

$$\mathbf{I}(B) = \mathrm{co}\{g_1 \times g_2 \times \cdots \times g_{n-1} : g_i \in B^\circ\}.$$

**Proof** For brevity let $K := \{g_1 \times g_2 \times \cdots \times g_{n-1} : g_i \in B^\circ\}$. The proof of Corollary 6, Equation (5) and the definition of support function show that $\sigma(f)$ is the support function of co $K$. The set $K$ need not be convex as can be seen by letting $B^\circ$ be an octahedron in $\mathrm{R}^3$, then the basis vectors are in $K$ but $(1/3, 1/3, 1/3)$ is not.

***Remarks*** 1. The extreme points of $\mathbf{I}(B)$ are obtained by choosing the $g_i$'s from among the extreme points of $B^\circ$. We state the polytope case separately.

2. If $n = 2$ then the cross-product is a unary operation. In Euclidean terms its action rotates a vector through 90°. In this case, therefore, $\mathbf{I}(B)$ is $B^\circ$ rotated through 90° as it is with all valid definitions of $\sigma$.

3. To discuss the effect of a linear transformation on $\mathbf{I}(B)$ we shall use the language of coordinates and matrices. Using columns to represent vectors in $X$ and rows to represent dual vectors in $X^*$, it is natural to write the matrices of linear transformations on the left in $X$ and on the right in $X^*$. With this convention, if $B$ in $\mathrm{R}^n$ is transformed by $T$, then $B^\circ$ is transformed by $T^{-1}$. Moreover, if $f_1, f_2, \ldots, f_{n-1}$ are vectors in $X^*$ and if $S$ is an $n \times n$ invertible matrix then

$$\begin{aligned} f_1 S \times f_2 S \times \cdots \times f_{n-1} S &= (f_1 \times f_2 \times \cdots \times f_{n-1})(\mathrm{adj}\, S)^t \\ &= (f_1 \times f_2 \times \cdots \times f_{n-1})(\det S) S^{-1}. \end{aligned}$$

Combining these we see that $\mathbf{I}\big(T(B)\big) = (\det T)^{-1} T\big(\mathbf{I}(B)\big)$. To avoid the extraneous factor of $\det T$ one should use a different normalization of $\mathbf{I}(B)$. A discussion of this can be found in [14, Chapter 5].

4. It follows from Remark 3 that all ellipsoids are (up to a scalar multiple) fixed points of the mapping **I**.

***Theorem 8*** *If $B$ is a polytope then $\mathbf{I}(B)$ is a polytope.*

**Proof** This time let $V := \{v_1 \times v_2 \times \cdots \times v_{n-1} : v_i$ vertices of $B^\circ\}$. Then Equation (5) shows that $\mathbf{I}(B) = \mathrm{co}\, V$ and, because $V$ is finite, this set is a polytope. ■

***Corollary 9*** *If $B$ is a polytope then each vertex of $\mathbf{I}(B)$ is a multiple of a vertex of $\Pi(B)^\circ$ where $\Pi(B)$ is the projection body of $B$ and, to avoid excessive brackets, $\Pi(B)^\circ = \big(\Pi(B)\big)^\circ$.*

**Proof** Each vertex of $\mathbf{I}(B)$ is of the form $v_1 \times v_2 \times \cdots \times v_{n-1}$ where the $v_i$'s are vertices of $B^\circ$. On the other hand, $\Pi(B) = \sum[-\alpha_i \hat{v}_i/2, +\alpha_i \hat{v}_i/2]$ where each $\hat{v}_i := v_i/|v_i|$ is a vertex of $B^\circ$ normalized to be a Euclidean unit vector, $\alpha_i$ is the (Euclidean) area of the corresponding

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facet of $B$, $[x, y]$ indicates the line segment from $x$ to $y$ and the summation (taken over all vertices of $B^\circ$) is Minkowski addition of convex sets. For details about projection bodies see, for example, Gardner [9], Goodey and Weil [10] and the references therein. It follows that each vertex of $\Pi(B)$ is of the form $\sum \pm\alpha_i\hat{v}_i/2$ for some suitable choice of signs and each edge of $\Pi(B)$ is in the direction of some $\hat{v}_i$. Therefore, the normal to a facet of $\Pi(B)$ and, hence, each vertex of $\Pi(B)^\circ$ is a multiple of a vector of the form $v_1 \times v_2 \times \cdots \times v_{n-1}$. ■

***Remarks*** 1. If $v_1, v_2, \ldots, v_{n-1}$ are linearly independent vertices of $B^\circ$ then there is a facet of $\Pi(B)$ with edges in these directions and $v_1 \times v_2 \times \cdots \times v_{n-1}$ as normal. Hence, there is a vertex of $\Pi(B)^\circ$ in the direction $v_1 \times v_2 \times \cdots \times v_{n-1}$. This may not be the case for $\mathbf{I}(B)$, *i.e.*, $\mathbf{I}(B)$ may have fewer vertices than $\Pi(B)^\circ$.

2. Remark 1 is a special case of the fact that the multiples involved in Corollary 9 are usually not all the same. In some cases the multiple may be so small that the "vertex" of $\mathbf{I}(B)$ is inside the convex hull of the other vertices and so does not appear. In other words, not all choices of $n-1$ linearly independent vertices of $B^\circ$ give rise to maximal simplices in Equation (5). Thus, $\mathbf{I}(B)$ is not, in general, a multiple of $\Pi(B)^\circ$. This clarifies some of the remarks in [14, p. 170]. We give some examples of the various possibilities below.

3. Note that both $\Pi(\,.\,)$ and $(\,.\,)^\circ$ are mappings from convex sets in $X$ to convex sets in $X^*$. The mappings $\mathbf{I}(\,.\,)$ and $\Pi(\,.\,)^\circ$ map the set of centrally symmetric convex bodies in $X$ into itself and can be iterated.

## 5 Examples

In each of the examples we take $X = \mathrm{R}^3$, use $x, y, z$ as coordinates and (to save space) write vectors as rows.

***Example 1*** Let $B_1$ be the cube with vertices at $(\pm1, \pm1, \pm1)$.

In this case, $B_1^\circ$ is the octahedron with vertices at $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$ and the cross-product of any two of these is another of them. Hence $\mathbf{I}(B_1) = B_1^\circ$. Note also, that in this case, $\Pi(B_1) = 2B_1$ and $\mathbf{I}(B_1) = 2\Pi(B)^\circ$.

***Example 2*** Let $B_2$ be the same cube but with the vertices $\pm(1, 1, 1)$ cut off to create two equilateral triangular facets.

The dual ball $B_2^\circ$ has the same vertices as $B_1^\circ$ together with $\pm(\xi, \xi, \xi)$ with $\xi > 1/3$. The cross-product $(1, 0, 0) \times (\xi, \xi, \xi) = (0, -\xi, \xi)$ is in the octahedron $B_1^\circ$ if $\xi \leq 1/2$ and outside otherwise. Thus, if $\xi \leq 1/2$ then $\mathbf{I}(B_2) = B_1^\circ = \mathbf{I}(B_1)$. If $\xi > 1/2$ then $\mathbf{I}(B_2) = \operatorname{co}\bigl(B_1^\circ \cup \{\pm(0, -\xi, \xi), \pm(\xi, 0, -\xi), \pm(-\xi, \xi, 0)\}\bigr)$. However, in all cases $\Pi(B_2)^\circ$ has 12 vertices in the same directions as those of $\mathbf{I}(B_2)$ in the case when $\xi > 1/2$ but the proportions are different.

This example can be extended.

***Example 3*** Let $B_3$ be the cubo-octahedron with vertices at

$$(\pm1, \pm1, 0), \quad (\pm1, 0, \pm1), \quad (0, \pm1, \pm1).$$

The dual ball is the rhombic dodecahedron with vertices at $\pm(1,0,0)$, $\pm(0,1,0)$ $\pm(0,0,1)$ and at $(\pm 1/2, \pm 1/2, \pm 1/2)$. The cross-product of vertices of the first type with those of the second and of two vertices of the second type are all on the boundary of the octahedron $\mathbf{I}(B_1) = B_1^\circ$. Therefore, $\mathbf{I}(B_3) = \mathbf{I}(B_1)$.

***Example 4*** Let $B_4$ be the octahedron $B_1^\circ$.

The dual ball is the cube $B_1$. Cross-products of the vertices of $B_1$ yield the vertices of the cubo-octahedron $2B_3$.

***Example 5*** Let $B_5$ be the rhombic dodecahedron $B_3^\circ$.

The dual ball is the cubo-octahedron $B_3$. Cross-products of vertices of $B_3$ of the type $(1,1,0) \times (1,-1,0)$ yield vertices of the form $(0,0,-2)$ and cross-products of vertices of $B_3$ of the type $(1,1,0) \times (0,1,1)$ yield vertices of the form $(1,-1,1)$. Permutation of the entries and the signs show that $\mathbf{I}(B_5) = 2B_5$.

***Example 6*** Let $B_6$ be the cylinder $\{(x,y,z) : x^2 + y^2 \le 1; |z| \le 1\}$.

In this case $B_6^\circ$ is the double cone $\{(x,y,z) : \sqrt{x^2+y^2} + |z| \le 1\}$. The cross-product of two extreme points of $B_6^\circ$ is either of the form

$$(0,0,1) \times (\cos\theta, \sin\theta, 0) = (-\cos\theta, \sin\theta, 0)$$

or

$$(\cos\theta, \sin\theta, 0) \times (\cos\phi, \sin\phi, 0) = \big(0, 0, \sin(\phi - \theta)\big).$$

The latter is extreme when $\phi = \theta \pm \pi/2$. Thus $\mathbf{I}(B_6) = B_6^\circ$.

Note that $\Pi(B_6)$ is the cylinder $\{(x,y,z) : x^2 + y^2 \le 4, |z| \le \pi\}$ which is not similar to $B_6^\circ$. Its polar $\Pi(B_6)^\circ$ is the double cone $\{(x,y,z) : 4(x^2 + y^2) - \pi|z| \le 1\}$ which is not a multiple of $\mathbf{I}(B_6)$.

***Example 7*** Let $B_7$ be the double cone $B_6^\circ$.

Here $B_7^\circ = B_6$ and (up to a sign) the cross-product of two extreme points of $B_7^\circ$ is of the form:

$$\begin{aligned}(\cos\theta, \sin\theta, 1) \times (\cos\phi, \sin\phi, 1) &= \big(\sin\theta - \sin\phi, -\cos\theta + \cos\phi, \sin(\phi - \theta)\big)\\ &= (2\cos\alpha \sin\beta, 2\sin\alpha\sin\beta, \sin 2\beta),\end{aligned}$$

where $\alpha := (\theta + \phi)/2$ and $\beta := (\theta - \phi)/2$. This represents a circle of radius $r = 2\sin\beta$ in the plane $z = \sin 2\beta$. Note that $r$ and $z$ satisfy the equation

$$4z^2 = r^2(4 - r^2).$$

As $\beta$ increases from 0 to $\pi/2$, $z$ increases from 0 to 1 and then decreases. At the same time $r$ increases from 0 to 2. Thus, extreme points are given when $\pi/4 \le \beta \le \pi/2$ and $\mathbf{I}(B_7)$ is

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the solid of revolution formed when that part of the lemniscate $4z^2 = x^2(4 - x^2)$ for which $-1 \leq z \leq 1$ and $\sqrt{2} \leq x \leq 2$ is rotated about the $z$-axis.

On the other hand, $\Pi(B_7)$ is a multiple of the cosine curve $x = \cos z$ revolved about the $z$-axis (see, for example, Gardner [9, p. 133]. In this case $\mathbf{I}(B_7)$ and $\Pi(B_7)$ appear unrelated. This fact was stated in [14, p. 170] based on some calculations much more complicated than those given here.

A very similar analysis may be made when $B$ is a double cone over a regular $2k$-gon.

Examples 1, 2 and 3 form the proof of the following theorem.

***Theorem 10*** *The mapping* $\mathbf{I}$ *is not one-to-one.*

***Remarks*** 1. When $\xi \leq 1/2$ the cross-sections of $B_2$ and the corresponding ones of $B_1$ are close enough so that the minimal parallelograms circumscribed to each are identical. Therefore, this construction for $\sigma$ is insufficient to distinguish between the two balls. If $\xi > 1/2$ then there are cross-sections of $B_2$ that are so much smaller than those of $B_1$ that the minimal parallelogram is also smaller and hence $\sigma$ is larger in that direction.

2. It was pointed out in [14, p. 140] that Benson's idea may be modified in one of several ways. One can inscribe either a maximal cross-polytope or a maximal ellipsoid in $B \cap f^\perp$ or one can circumscribe a minimal ellipsoid about $B \cap f^\perp$. For information about such ellipsoids, see Gruber [11] or Danzer, Laugwitz and Lenz [8]. For all of these possibilities, the corresponding mapping $\mathbf{I}$ is not injective. Inscribed and circumscribed ellipsoids will yield the same isoperimetrices for $B_1$ and $B_2$ if $\xi$ is sufficiently small. While inscribed parallelograms will distinguish these two balls, other examples can easily be given.

Examples 3, 4 and 5 form the proofs of the following theorems.

***Theorem 11*** *In* $\mathrm{R}^3$ *there are balls B (other than ellipsoids) for which* $\mathbf{I}(B)$ *and B are similar.*

***Theorem 12*** *In* $\mathrm{R}^3$ *there are balls B (other than ellipsoids and those covered by Theorem 11) for which* $\mathbf{I}^2(B)$ *and B are similar.*

***Remarks*** 1. It is not clear (to me) whether there are other balls with the same property as the rhombic dodecahedron. A reasonable candidate might be the triacontahedron (see, for example, Coxeter [7, Section 2.7]) but calculations show that it fails. It is interesting to compare $\mathbf{I}(B_5)$ with $\Pi(B_5)^\circ$. The latter (up to a multiple) has vertices at $(\pm 1, \pm 1, \pm 1)$ and at $(\pm 3/2, 0, 0)$, $(0, \pm 3/2, 0)$, $(0, 0, \pm 3/2)$. The adjustment of these latter vertices from $(3/2, 0, 0)$ to $(2, 0, 0)$ *etc.*, converts the dual of a zonotope into a zonotope. Example 5 also shows that the range of $\mathbf{I}$ is not contained in the set of duals of zonoids and raises the question as to the nature of its range.

2. Likewise, it is not clear (to me) whether there are other balls with the same property as the octahedron. The double cone over a regular hexagon does not. The calculations for a double cone over an octagon got out of hand. Nor have I done the calculations for higher dimensions.

3. In [14, Chapters 6 and 7] two questions are raised pertaining to the definitions of area discussed there: For $n \geq 3$, are there balls other than ellipsoids for which $\mathbf{I}(B)$ and

$B$ are similar? and, for arbitrary $B$, is it true that $\mathbf{I}^n(B)$ (suitably scaled) converges to an ellipsoid? Theorems 11 and 12 show that for Benson's definition the general answer to the first question is 'Yes' and to the second is 'No'. However, these very specific examples shed little light on the structure of the questions. Are the examples very isolated or part of a more general phenomenon?

4. For all definitions of area, it appears that $\mathbf{I}(B)$ is shaped somewhat like $B^\circ$ and that $\mathbf{I}^2(B)$ resembles $B$ itself. In R. D. Holmes' words "$\mathbf{I}(B)$ serves as a surrogate of $B^\circ$ in the original space." Most of these examples and Theorem 12 reinforce this heuristic but, in this sense, Theorem 11 is counter-intuitive.

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