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FRÉCHET AL-SPACES HAVE THE DUNFORD-PETTIS PROPERTY

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A Fréchet lattice E is an AL-space if its topology can be defined by a family of lattice seminorms that are additive in the positive cone of E. Grothendieck proved that AL-Banach spaces have the Dunford-Pettis property. This result was recently extended by Fernández and Naranjo to AL-Fréchet spaces with a continuous norm and weak order unit. In this note we show how to remove both hypotheses.

Dunford and Pettis proved in [**4**] that every weakly compact operator from $L_1(\mu)$ into itself carries weakly convergent sequences onto norm convergent sequences.

In general it is said that a Hausdorff locally convex space E has the Dunford-Pettis property (following [**7**, Definition 1]) if for any Banach space F and every linear continuous mapping $T : E \to F$ that carries bounded sets of E to weakly relatively compact sets of F, then $T(C)$ is relatively compact in F for any absoutely convex, weakly compact subset C in E.

This property has been intensively studied and characterised in different contexts. In particular, Grothendieck [**7**] established that both classes of Banach AL and AM spaces possess the Dunford-Pettis property. (See also [**1**, Theorem 19.6].)

This result was proved by using the duality between these classes, together with the representation of AL- and AM-spaces by means of spaces of integrable functions and continuous functions, respectively. First of all Grothendieck proved that AM-spaces have the Dunford-Pettis property. For AL-spaces it then follows by duality.

The concept of generalised AL-spaces was introduced by Wong in [**9**] (see also [**10**]) in the setting of locally convex lattices.

We recall that a Fréchet lattice E is said to be a Fréchet AL-space if its topology can be defined by a family of lattice seminorms $|\cdot|$ that are additive in the positive cone E^+ of E, that is,

$$|x+y| = |x| + |y|, \quad x, y \in E^+.$$

In [**5**] the second and third named authors proved, by using a representation result for certain Fréchet lattices due to Dodds, de Pagter and Ricker, [**3**], that every Fréchet

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AL-space with a continuous norm and weak order unit has the Dunford-Pettis property. Here we establish that every Fréchet AL-space has the Dunford-Pettis property. Let us observe at this point that there is no well established duality relation between locally convex AL-spaces and AM-spaces.

The proof will require some preliminary results.

LEMMA 1. *Every separable closed vector subspace of a Fréchet lattice is contained in a closed separable Fréchet sublattice.*

PROOF: See [**1**, Exercises 8 and 9 (p.197)]. ∎

LEMMA 2. *If all closed and separable sublattices of a Fréchet lattice E have the Dunford-Pettis property, then E also has this property.*

PROOF: Let F be a Banach space and let $T : E \to F$ be a linear continuous mapping for which $T(B)$ is weakly relatively compact in F for any bounded set B in E. Taking into account [**8**, Theorem 1] it suffices to prove that $\left(T(x_n)\right)_n$ is a Cauchy sequence in F, for any weakly Cauchy sequence $(x_n)_n$ in E.

Consider the closed and separable sublattice H which is generated by the sequence $(x_n)_n$. The restriction mapping $T : H \to F$ is linear, continuous and carries bounded sets of H into weakly relatively compact sets of F. Since H has the Dunford-Pettis property and $(x_n)_n$ is weakly Cauchy in H, it follows that $\left(T(x_n)_n\right)$ is a Cauchy sequence in F. ∎

PROPOSITION 3. *Let E be a Fréchet AL-space with a continuous norm. Then E has the Dunford-Pettis property.*

PROOF: Let H be any closed and separable sublattice of E. Then H is a Fréchet AL-space with a continuous norm and a weak order unit. From [**5**, Corollary 4.5] H has the Dunford-Pettis property. Now apply Lemma 2 to obtain the result. ∎

PROPOSITION 4. *Let $(E_n)_n$ be a sequence of Fréchet spaces, each one with the Dunford-Pettis property. Then, the product Fréchet space $E := \prod_{n=1}^{\infty} E^n$ also has the Dunford-Pettis property.*

PROOF: Fix, in every E_k, a countable increasing system $\mathcal{P}_k$ of continuous seminorms giving its topology. Let $x = (x_n)_n \in E$, with $x_n \in E_n$. Then the topology of E is determined by the family of seminorms $|\cdot|_k$ given by

$$\left|(x_n)_n\right|_k := \sum_{j=1}^{k} |x_j|_k\,, \quad k = 1,\, 2,\, \ldots,$$

where $|x_j|_k$ is the value of the k-th seminorm of $\mathcal{P}_j$ acting on the element $x_j \in E_j$.

Let F be a Banach space with norm $\|\cdot\|$ and let $T : E \to F$ be a linear and continuous mapping such that $T(B)$ is weakly relatively compact in F for every bounded B in E. Since $T : E \to F$ is continuous, there are constants $M > 0$ and $k \geqslant 1$ (that we fix) such that

$$\|Tx\| \leqslant M\,|x|_k\,, \quad x \in E. \tag{0.1}$$

If $G := \{(x_n)_n \in E : x_n = 0, \ \text{for all } n > k\}$ then we have $|x|_k = 0$ for all $x \in G$. From (0.1) we deduce that $T(x) = 0$ for every $x \in G$. Therefore, T admits a natural factorisation through the space $\prod_{j=1}^{k} E_j$. Now the result follows because the Dunford-Pettis property is inherited by finite products. ∎

We are now ready to establish the main result.

Theorem 5. *Every Fréchet AL-space has the Dunford-Pettis property.*

Proof: Let E be a Fréchet AL-space. Then E has the Lebesgue property [**6**, Theorem 2]. Now, from [**2**, Remark 2 of Theorem 1] we know that either E has a continuous norm or E is isomorphic to the product of a sequence of Fréchet spaces each one having a continuous norm. In the first case, E has the Dunford-Pettis property by applying Proposition 3. In the second case, we obtain the result by applying Proposition 3 and Proposition 4. ∎

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