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A NOTE ON DOUBLES OF 4-MANIFOLDS

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If M is a simply-connected 4-manifold with boundary, let $D(M)$ denote its double $M \cup_{\partial M} (-M)$. If M is closed, let $D(M)$ denote $M \# -M$. In either case, $D(M)$ is a simply-connected 4-manifold of index zero, and so by a theorem of Wall [8], $M \# k(S^2 \times S^2)$ must be standard for k sufficiently large, where by standard we mean diffeomorphic to the connected sum of copies of $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$, the non-trivial S^2 bundle over S^2 (which is itself diffeomorphic to $\mathbb{C}P^2 \# -\mathbb{C}P^2$ [7]). In this paper we give a bound on k, in the case where M has no 3-handles.

THEOREM 1. *Let M be a simply-connected 4-manifold with one 0-handle, m 1-handles, $m+n$ 2-handles, and no 3-handles. Then $D(M) \# (m-1)(S^2 \times S^2)$ is diffeomorphic to $\#(n+m-1)(S^2 \times S^2)$ if M has even intersection form, and $\#(n+m-1)(S^2 \tilde{\times} S^2)$ if M has odd intersection form.*

COROLLARY 2. *If M as above has no 1-handles, $D(M)$ is standard.*

This corollary applies in particular to the Kummer surface ([2]), or in fact to any simply-connected algebraic surface ([6]).

Note that we have shown in [7] that if the 2-handles are attached so as to correspond to a plumbing diagram, M itself is standard.

COROLLARY 3. *If M is a homotopy 4-cell as above, $D(M) \# (m-1)(S^2 \times S^2)$ is diffeomorphic to $(m-1)(S^2 \times S^2)$.*

Proof. $D(M)$ is the boundary of $M \times I$, and $M \times I$ has the same handle structure as M. Consider M as built up first by attaching the 1-handles to obtain $M_1 = D^4 \natural m(D^3 \times S^1)$, where $\natural$ denotes boundary connected sum, and then attaching the 2-handles to M_1. Since M is simply-connected, this gives a presentation of the trivial group

$$\langle x_1, \ldots, x_m; r_1 = r_2 = \cdots = r_{m+n} = 1 \rangle$$

Then $D(M_1)$ is the boundary of $M_1 \times I$, $D(M_1) = \# m(S^3 \times S^1)$, and $M \times I$ is constructed from $M_1 \times I$ by adding 2-handles "crossed with I" (i.e. if the attaching map of a 2-handle of M is $\varphi : D^2 \times S^1 \to M$, the attaching map of the corresponding 2-handle of $M \times I$ is $\varphi \times id : (D^2 \times S^1) \times I \to M \times I$).

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Now to M add $(m-1)$ additional trivial 2-handles (i.e. handles whose core circles are unknotted, unlinked circles lying in a disk on ∂M), and cross these handles with I. We now have a manifold $M' \times I$, and $\partial(M' \times I) = D(M) \# (m-1)(S^2 \times S^2)$.

Consider $M' \times I$. It is constructed by adding 2-handles to $M \times I$, so the core circles of these handles lie in $D(M)$. Since $D(M)$ is a 4-manifold, any two homotopic curves in $D(M)$ are isotopic. Thus we may isotope our $(m-1)$ trivial circles to represent the generators $x_1, \ldots, x_{m-1}$. On the one hand, this does not change $M' \times I$ (up to diffeomorphism), but on the other hand, these $(m-1)$ 2-handles now geometrically cancel the first $(m-1)$ 1-handles, leaving a handle structure on $M' \times I$ consisting of one 1-handle and $m+n$ 2-handles, giving a presentation of the trivial group

$$\langle x_m ; r_1' = r_2' = \cdots = r_{m+n}' = 1 \rangle$$

At this stage all of the r_i' are words in x_m, i.e. powers of x_m, $r_i' = x_m^{n_i}$, and so by sliding handles over each other (which has the effect of multiplying the words) we may arrange that $r_1' = x_m$, in which case we may cancel the 1-handle with this 2-handle, leaving $M' \times I$ with a handle structure of only 2-handles. But then the core circles of these 2-handles, being circles in D^4, may be isotoped to standardly embedded circles, and so $M' \times I = \natural E_i$, where each E_i is either the trivial or non-trivial D^2-bundle over S^2, depending on the framing, so $D(M') = \partial(M' \times I) = \partial(\natural E_i) = \#(S^2 \times S^2) \# (S^2 \tilde{\times} S^2)$, (and it is well-known that $(S^2 \times S^2) \# (S^2 \tilde{\times} S^2) = (S^2 \tilde{\times} S^2) \# (S^2 \tilde{\times} S^2)$), completing the proof.

Our argument is of course quite similar to the argument in [1], proving that the Andrews-Curtis conjecture implies that a homotopy 4-sphere without 3-handles is homeomorphic to S^4. In fact, our argument shows that a weak form of the Andrews-Curtis conjecture implies a corresponding, but weaker result.

To be precise, let $AC_{k,m}$ be the following conjecture:

$AC_{k,m}$. Every presentation of the trivial group with no more than m generators may be changed to the trivial group by Andrews-Curtis moves together with the adjunction of no more than k consequences of the relations.

Then we have

THEOREM 4. *Let M be as in Theorem 1. Then $AC_{k,m} \Rightarrow D(M) \# k(S^2 \times S^2)$ is standard.*

Proof. We may trivialize the presentation either by sliding handles, or by adding at most k new 2-handles, adding these handles as in the fourth paragraph of the proof of Theorem 1 so as to represent the additional necessary consequences of the relations.

Trivially, $AC_{k,m} \Rightarrow AC_{\ell,n}$ for $k \leq \ell$ and $n \leq m$, $AC_{k,m} \Rightarrow AC_{k+\ell,m+\ell}$ by using

ℓ relations to kill generators, and as observed above, $AC_{m-1,m}$ is true (by adjoining the relations $x_1 = \cdots = x_{m-1} = 1$). Also, the Andrews-Curtis conjecture AC is equivalent to $\bigcup_m AC_{0,m}$.

Consider, however, the presentation of the trivial group

$$\{a, b, c : [a, b]b = [b, c]c = [c, a]a = 1\}.$$

This is considered to be a possible counterexample to AC (see [4, Problems, 5.1 and 5.2]), but clearly does satisfy $AC_{1,3}$-adding $a = 1$, for example, trivializes the presentation.

We conclude by observing that Wall's Theorem, mentioned above, can be combined with other known results to give a purely low-dimensional proof (no homotopy theory!) of the following well-known theorem:

THEOREM. *Every orientable 3-manifold embeds in $\mathbb{R}^5$ and immerses in $\mathbb{R}^4$.*

Proof. If M is an orientable 3-manifold, then M bounds a simply-connected ([5]) and even parallelizable ([3]) 4-manifold N. Then $M \subset D(N)$, and hence $M \subset D(N) \# k(S^2 \times S^2) = \ell(S^2 \times S^2)$ for k sufficiently large. But $\ell(S^2 \times S^2)$ embeds in $\mathbb{R}^5$, and $\ell(S^2 \times S^2) - D^4$ immerses in $\mathbb{R}^4$, so M does too.

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