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MONOTONE CASE FOR AN EXTENDED PROCESS

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Abstract

We consider a nonnegative discrete time and bounded horizon process X for which 0 is an absorbing state and extend it by a random variable that is independent of X. We find a sufficient condition for the resulting process to satisfy, after a canonical time rescaling, the hypothesis of the monotone case theorem. If X describes a secretary type search on a poset with one maximal element or if we consider X with no extension then this condition assumes an especially simple log-concavity type form.

Keywords: Monotone case; secretary problem; best choice; optimal stopping time

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1. Introduction

In [10] the following problem was considered. An administrator has to choose the best candidate in a classical secretary search. However, if the administrator decides not to choose in this search then a new independent search starts. A general assumption is that there are n possible searches and the number of candidates in each search is a priori known. The administrator can choose only once, i.e. in only one of the independent n searches, and success is dependent on choosing the best candidate from the presently examined pool. The problem is solved recursively. Assume that we know where to choose optimally for the case of n searches, but now we have $n+1$ searches for disposition. Let $(X_i)_{i\leq m}$ be an appropriate process describing the first search (in this paper we call this process the maximum identification average; see Section 4), and let $(Y_i)_{i\leq n}$ be a process describing the next n searches. As we have assumed, we know an optimal stopping time τ for Y. Thus, to find the optimal stopping for the case of $n+1$ searches, it is enough to consider the process $(X_i)_{i\leq m}$ concatenated with just one random variable Y_τ which is independent of the whole process X. It was proved in [10] that, for the case of classical secretary searches, the concatenated process satisfied the so-called monotone case theorem that gave a method for calculating the optimal stopping time. In this paper we generalise this result of [10].

All random variables considered in this paper are assumed to be finitely integrable. We consider a discrete-time finite horizon stochastic process: X extended by a random variable Q; let us call the resulting process Z. Now we want to stop Z at some stopping time $\bar{\xi}$ in such a way that $\mathbb{E}Z_{\bar{\xi}}$ is maximal among $\mathbb{E}Z_\xi$ for all stopping times ξ. We assume that X is positive up to a certain moment and that if it is equal to 0 once then it remains equal to 0 till the end.

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(This feature of the process appears naturally when considering the secretary problem; see Section 4 and [10].) Thus, we can rescale time, restricting ourselves only to sensible times, namely those where $X > 0$, and add the last time that is the index of the random variable Q. We obtain an inequality which implies that the process consisting of the positive part of X concatenated with Q satisfies the monotone case theorem. If X is the process of conditional expectations of absolute maximum indicators taken at relative record times for the best-choice problem on a poset then this inequality simplifies, is of log-concave type, and depends only on X. We prove that this inequality is indeed satisfied by some well-known processes, e.g. the processes of conditional expectations of absolute maximum indicators taken at relative record times for the classical and the full-information secretary problems. Thus, in relation to the above, if $Q = Y_\tau$, we can apply the monotone case theorem to obtain the optimal stopping for concatenated independent secretary searches with only one choice for the selector whose success is dependent on choosing an element that is the best in its own pool.

The paper is organized as follows. In Section 2 we prove a general theorem (Theorem 2.2) stating an inequality which implies that the processes described above fall into the monotone case. In particular, if we do not consider a concatenation but just a single process, this inequality, which implies that the process satisfies the monotone case theorem, takes a simple log-concave form (Theorem 2.3).

In Section 3 we prove the general theorem that if a stochastic process $Y = (Y_i)_{i\le n}$ is adapted to a filtration $(\mathcal{L}_i)_{i\le n}$ and a σ-algebra $\mathcal{G}$ is independent of $\mathcal{L}_n$, then no stopping rule γ with respect to the enriched filtration $(\sigma(\mathcal{L}_i \cup \mathcal{G}))_{i\le n}$ can give a better result for the value of $\mathbb{E}Y_\gamma$ than an optimal stopping rule with respect to the original filtration $(\mathcal{L}_i)_{i\le n}$. In other words, no independent information increases the maximal value of $\mathbb{E}Y_\gamma$. This result is used later to find an optimal stopping time for maximizing the expected value of a stopped process C if the process C is a concatenation of two independent processes: a process X followed by an independent process Y. Namely, the knowledge of what happened in the X-part if no stopping occurred does not influence our optimal stopping in the second part. In Section 4 for the secretary problem on a poset we introduce the so-called maximum identification average (MIA) processes which are conditional expectations of the absolute record indicators taken taken *either* at only those times where the current element is the best (a relative record) or at the maximal time if the current element is not ever the best. We show that inequality (2.5) given in Theorem 2.2 assumes an especially simple form (2.6) (the same as for the single process case mentioned above) for MIA processes and, thus, it depends only on X, i.e. the random variable Q (extending X) is not involved. In Section 5 we prove that this inequality is indeed satisfied by MIA processes for both classical and full-information secretary searches. We also make use of the results of Sections 2, 3, and 4 to find optimal stopping times for concatenations of all possible independent pairs of classical and full-information secretary searches. In Section 6 we present two processes that are not MIAs for secretary-type searches on posets with one maximal element but are examples where our general results can still be applied. In Section 7 we give examples and counterexamples related to some natural questions about the main results of this paper.

2. General inequality implying the monotone case

Let us recall the monotone case theorem (see [1]).

Theorem 2.1. (Monotone case theorem.) *If $\{(X_i, \mathcal{F}_i)\colon i \le m\}$ is a stochastic process such that for almost every (a.e.) ω the inequality $X_i(\omega) \ge \mathbb{E}(X_{i+1} \mid \mathcal{F}_i)(\omega)$ implies the inequality*

$X_{i+1}(\omega) \geq \mathbb{E}(X_{i+2} \mid \mathcal{F}_{i+1})(\omega)$ for each $i \leq m-2$, then the stopping time

$$\bar{\tau}(\omega) = \min\{i : X_i(\omega) \geq \mathbb{E}(X_{i+1} \mid \mathcal{F}_i)(\omega)\}$$

(if the set under min is empty then we take $\bar{\tau} = m$) is optimal for maximizing $\mathbb{E}X_\tau$ over all stopping times τ.

Furthermore, we deal with a nonnegative discrete stochastic process $X = \{(X_i, \mathcal{F}_i) : i \leq m\}$ and a nonnegative random variable Q independent of X. Let us also assume that

$$X_1(\omega) > 0, \qquad X_2(\omega) > 0, \qquad X_{k-1}(\omega) > 0, \quad \text{and} \quad X_k(\omega) = \cdots = X_m(\omega) = 0 \tag{2.1}$$

for a.e. $\omega \in \Omega$, where k depends on ω. (For $i > m$ we formally set $X_i \equiv 0$.)

Such processes generalize, for instance, the processes consisting of conditional expected values of absolute record indicators that naturally appear for the secretary problem on posets (called MIA processes in Section 4).

A stopping time with respect to a filtration $(\mathcal{F}_t)$ will be called an $(\mathcal{F}_t)$-*stopping time*. Let

$$Z_i = X_i \quad \text{for } i \leq m \quad \text{and} \quad Z_{m+1} = Q,$$

and

$$\mathcal{E}_i = \mathcal{F}_i \quad \text{for } i \leq m \quad \text{and} \quad \mathcal{E}_{m+1} = \sigma(\mathcal{F}_m, \sigma(Q)).$$

We rescale time to define reasonable stopping times beyond which it does not make sense to stop if we want to maximize $\mathbb{E}Z_\tau$ over all $(\mathcal{E}_j)_{j \leq m+1}$-stopping times τ. Namely, let

$$\rho_i(\omega) = \begin{cases} i & \text{if } X_i(\omega) > 0, \\ m+1 & \text{otherwise.} \end{cases} \tag{2.2}$$

The main purpose of this paper is to establish conditions under which the process $(Z_{\rho_i}, \mathcal{E}_{\rho_i})_{i \leq m+1}$ satisfies the monotone case theorem.

Proposition 2.1. *For a.e. ω such that $\rho_t(\omega) = t < m$, we have*

$$\mathbb{E}(Z_{\rho_{t+1}} \mathbf{1}_{[\rho_{t+1} \leq m]} \mid \mathcal{E}_{\rho_t})(\omega) = \mathbb{E}(X_{t+1} \mathbf{1}_{[X_{t+1} > 0]} \mid \mathcal{F}_t)(\omega).$$

Proof. As $\rho_t \geq t$, and, consequently, $\mathcal{F}_t = \mathcal{E}_t \subseteq \mathcal{E}_{\rho_t}$, it is enough to show that, for each $A \in \mathcal{E}_{\rho_t}$,

$$\int_{[\rho_t = t] \cap A} Z_{\rho_{t+1}} \mathbf{1}_{[\rho_{t+1} \leq m]} \, \mathrm{d}P = \int_{[\rho_t = t] \cap A} \mathbb{E}(X_{t+1} \mathbf{1}_{[X_{t+1} > 0]} \mid \mathcal{F}_t) \, \mathrm{d}P.$$

We have

$$\begin{aligned} \int_{[\rho_t = t] \cap A} Z_{\rho_{t+1}} \mathbf{1}_{[\rho_{t+1} \leq m]} \, \mathrm{d}P &= \int_{[\rho_t = t] \cap A} X_{t+1} \mathbf{1}_{[X_{t+1} > 0]} \, \mathrm{d}P \\ &= \int_{[\rho_t = t] \cap A} \mathbb{E}(X_{t+1} \mathbf{1}_{[X_{t+1} > 0]} \mid \mathcal{F}_t) \, \mathrm{d}P, \end{aligned}$$

because $[\rho_t = t] \cap A \in \mathcal{F}_t$.

Proposition 2.2. *For a.e. ω such that $\rho_t(\omega) = t < m$, we have*

$$\mathbb{E}(Z_{\rho_{t+1}}\mathbf{1}_{[\rho_{t+1}>m]} \mid \mathcal{E}_{\rho_t})(\omega) = \mathbb{E}(Q\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega).$$

Proof. As $\rho_t \geq t$, and, consequently, $\mathcal{F}_t = \mathcal{E}_t \subseteq \mathcal{E}_{\rho_t}$, it is enough to show that, for each $A \in \mathcal{E}_{\rho_t}$,

$$\int_{[\rho_t=t]\cap A} Z_{\rho_{t+1}}\mathbf{1}_{[\rho_{t+1}>m]}\,\mathrm{d}P = \int_{[\rho_t=t]\cap A} \mathbb{E}(Q\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)\,\mathrm{d}P.$$

We have

$$\begin{aligned}\int_{[\rho_t=t]\cap A} Z_{\rho_{t+1}}\mathbf{1}_{[\rho_{t+1}>m]}\,\mathrm{d}P &= \int_{[\rho_t=t]\cap A} Q\mathbf{1}_{[X_{t+1}=0]}\,\mathrm{d}P\\ &= \int_{[\rho_t=t]\cap A} \mathbb{E}(Q\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)\,\mathrm{d}P,\end{aligned}$$

because $[\rho_t = t] \cap A \in \mathcal{F}_t$.

By Propositions 2.1 and 2.2, and the independence of the process X and the random variable Q, we have, for $\rho_t(\omega) = t < m$,

$$\begin{aligned}\mathbb{E}(Z_{\rho_{t+1}} \mid \mathcal{E}_{\rho_t})(\omega) &= \mathbb{E}(Z_{\rho_{t+1}}\mathbf{1}_{[\rho_{t+1}\leq m]} \mid \mathcal{E}_{\rho_t})(\omega) + \mathbb{E}(Z_{\rho_{t+1}}\mathbf{1}_{[\rho_{t+1}>m]} \mid \mathcal{E}_{\rho_t})(\omega)\\ &= \mathbb{E}(X_{t+1}\mathbf{1}_{[X_{t+1}>0]} \mid \mathcal{F}_t)(\omega) + \mathbb{E}(Q\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega)\\ &= \mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) + \mathbb{E}Q\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega).\end{aligned}$$

We need to prove the implication

$$Z_{\rho_t}(\omega) \geq \mathbb{E}(Z_{\rho_{t+1}} \mid \mathcal{E}_{\rho_t})(\omega) \quad\Longrightarrow\quad Z_{\rho_{t+1}}(\omega) \geq \mathbb{E}(Z_{\rho_{t+2}} \mid \mathcal{E}_{\rho_{t+1}})(\omega) \tag{2.3}$$

for a.e. ω.

Assume that $\rho_t(\omega) = t < m$. If $X_{t+1}(\omega) = 0$ then $\rho_{t+1}(\omega) = m+1$, $Z_{\rho_{t+1}}(\omega) = Q(\omega)$, and $Z_{\rho_{t+2}}(\omega) = Q(\omega)$. Hence,

$$\begin{aligned}\mathbb{E}(Z_{\rho_{t+2}} \mid \mathcal{E}_{\rho_{t+1}})(\omega) &= \mathbb{E}(Z_{\rho_{t+2}} \mid \mathcal{E}_{\rho_{t+1}})(\omega)\mathbf{1}_{[\rho_{t+1}=m+1]}(\omega)\\ &= \mathbb{E}(Z_{\rho_{t+2}}\mathbf{1}_{[\rho_{t+1}=m+1]} \mid \mathcal{E}_{\rho_{t+1}})(\omega)\\ &= \mathbb{E}(Q\mathbf{1}_{[\rho_{t+1}=m+1]} \mid \mathcal{E}_{\rho_{t+1}})(\omega)\\ &= Q(\omega).\end{aligned}$$

Thus, if $X_{t+1}(\omega) = 0$, the monotone case implication always holds and it is enough to consider the case $X_{t+1}(\omega) > 0$. Then (2.3) becomes

$$\begin{aligned}&X_t(\omega) \geq \mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) + \mathbb{E}Q\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega)\\ \Longrightarrow\quad &X_{t+1}(\omega) \geq \mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega) + \mathbb{E}Q\mathbb{E}(\mathbf{1}_{[X_{t+2}=0]} \mid \mathcal{F}_{t+1})(\omega).\end{aligned} \tag{2.4}$$

As $X_{t+1}(\omega) > 0$ implies that $X_t(\omega) > 0$, then the above inequalities give

$$1 - \frac{\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega)}{X_t(\omega)} - \mathbb{E}Q\frac{\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega)}{X_t(\omega)} \geq 0$$

and

$$1 - \frac{\mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega)}{X_{t+1}(\omega)} - \mathbb{E}Q\frac{\mathbb{E}(\mathbf{1}_{[X_{t+2}=0]} \mid \mathcal{F}_{t+1})(\omega)}{X_{t+1}(\omega)} \geq 0.$$

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Thus, obviously, the inequality

$$1 - \frac{\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega)}{X_t(\omega)} - \mathbb{E}Q\frac{\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega)}{X_t(\omega)} \\ \leq 1 - \frac{\mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega)}{X_{t+1}(\omega)} - \mathbb{E}Q\frac{\mathbb{E}(\mathbf{1}_{[X_{t+2}=0]} \mid \mathcal{F}_{t+1})(\omega)}{X_{t+1}(\omega)}$$

implies (2.4). This inequality is equivalent to

$$X_{t+1}(\omega)\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) + X_{t+1}(\omega)\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega)\mathbb{E}Q \\ \geq X_t(\omega)\mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega) + X_t(\omega)\mathbb{E}(\mathbf{1}_{[X_{t+2}=0]} \mid \mathcal{F}_{t+1})(\omega)\mathbb{E}Q. \tag{2.5}$$

Thus, we have proved the following theorem.

Theorem 2.2. *If a process $X = (X_t, \mathcal{F}_t)_{t\leq m}$ satisfies (2.1) and, for each ω, (2.5) holds whenever $X_{t+1}(\omega) > 0$, then the process $(Z_{\rho_i}, \mathcal{E}_{\rho_i})_{i\leq m+1}$ satisfies the monotone case theorem.*

The assumption that $Q \equiv 0$ for the process Z from Theorem 2.2 implies the following theorem concerning processes X without concatenation.

Theorem 2.3. *If a process $X = (X_i)_{i\leq m}$ satisfies (2.1), in particular, if X is strictly positive, and, for each ω, the inequality*

$$X_{t+1}(\omega)\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) \geq X_t(\omega)\mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega) \tag{2.6}$$

holds whenever $X_{t+1}(\omega) > 0$, then X satisfies the monotone case theorem.

3. Independent information does not change the optimal policy

The *concatenation* of two stochastic processes $X = \{(X_t, \mathcal{K}_t),\ 1 \leq t \leq m\}$ and $Y = \{(Y_t, \mathcal{L}_t),\ 1 \leq t \leq n\}$ is the process $C = \{(C_t, \mathcal{M}_t),\ 1 \leq t \leq m+n\}$ such that $C_t = X_t$ for $t \leq m$, $C_{m+t} = Y_t$ for $t \leq n$, $\mathcal{M}_t = \mathcal{K}_t$ for $t \leq m$, and $\mathcal{M}_{m+t} = \sigma(\mathcal{K}_m \cup \mathcal{L}_t)$.

Let X and Y be processes on the same probability space with independent σ-algebras $\mathcal{K}_m$ and $\mathcal{L}_n$ (thus, the processes X and Y are independent), and let $C = \{(C_t, \mathcal{M}_t),\ 1 \leq t \leq m+n\}$ be the concatenation of the processes X and Y. Let $\bar{\tau}$ be a stopping time maximizing $\mathbb{E}Y_\tau$ over all $(\mathcal{L}_t)$-stopping times τ, and let $Z = \{(Z_t, \mathcal{E}_t),\ 1 \leq t \leq m+1\}$ be the concatenation of the process X and a one element process $(Y_{\bar{\tau}}, \mathcal{L}_{\bar{\tau}})$. We will call Z the *merge* of X and Y.

Intuitively, to maximize $\mathbb{E}C_\tau$ over all $(\mathcal{M}_t)_{t\leq m+n}$-stopping times τ if no stopping occurs in the first part, namely for $t \leq m$, we should play in the second part optimally according to an optimal stopping time with respect to the filtration $(\mathcal{L}_t)_{t\leq n}$. In other words, we should maximize $\mathbb{E}Z_\tau$ over all $(\mathcal{E}_t)_{t\leq m+1}$-stopping times. However, when we are above time m, the already observed process $(X_t)_{t\leq m}$ provides us with some information which could be potentially helpful, i.e. we suspect that the X-part information could be used to find a better stopping time than $\bar{\tau}$ for the Y-part. In this section we show that this is not the case, that is, the information provided for the X-part does not influence our optimal decision when stopping in the Y-part.

Let us recall the backwards induction theorem (see, e.g. Theorem 3.2 of [1]).

Theorem 3.1. *Let $(Y_i)_{i\leq n}$ be a process adapted to a filtration $(\mathcal{A}_i)_{i\leq n}$. Let $L_n = Y_n$ and, inductively backwards, $L_i = \max(Y_i, \mathbb{E}(L_{i+1} \mid \mathcal{A}_i))$. Then $\bar{\gamma} = \inf\{i : Y_i = L_i\}$ is a stopping time maximizing $\mathbb{E}Y_\gamma$ over all $(\mathcal{A}_i)$-stopping times γ.*

The following lemma says that independent information does not change the optimal stopping time. (We introduce Ω_1 below to aid further technical arguments.)

Theorem 3.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\mathcal{L}_i)_{i\le n}$ be a filtration, $\mathcal{L}_n \subseteq \mathcal{F}$, $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra independent of $\mathcal{L}_n$, and $\Omega_1 \in \mathcal{G}$. Let $\mathcal{H}_i = \sigma(\mathcal{L}_i \cup \mathcal{G})$. Let $(Y_i)_{i\le n}$ be a stochastic process adapted to the filtration $(\mathcal{L}_i)_{i\le n}$, and let $\bar\tau$ be an $(\mathcal{L}_i)$-stopping time maximizing $\mathbb{E}Y_\tau$ over all $(\mathcal{L}_i)$-stopping times τ. Let $Y_i^* = \mathbf{1}_{\Omega_1} Y_i$, $1 \le i \le n$. Then $\bar\tau$ is also an optimal stopping time maximizing $\mathbb{E}Y^*_\gamma$ over all $(\mathcal{H}_i)$-stopping times γ.*

Proof. If Λ is a random variable independent of $\mathcal{G}$ and $\Lambda^* = \mathbf{1}_{\Omega_1}\Lambda$, then $\mathbb{E}(\Lambda^* \mid \sigma(\mathcal{G} \cup \mathcal{L}_i)) = \mathbf{1}_{\Omega_1}\mathbb{E}(\Lambda \mid \mathcal{L}_i)$. Let L_i^* be the random variables constructed for the process $(Y_i^*, \mathcal{H}_i)_i$ in the backwards induction procedure. By the fact stated above we can easily prove (inductively defining the L_i^*) that $L_i^* = \mathbf{1}_{\Omega_1} L_i$, where the L_i are the random variables constructed for the process $(Y_i, \mathcal{L}_i)_i$ in the backwards induction procedure. Thus, the optimal stopping time γ^* maximizing $\mathbb{E}Y^*_\xi$ over all $(\mathcal{H}_i)$-stopping times ξ is equal on Ω_1 to a stopping time $\bar\gamma$ maximizing $\mathbb{E}Y_\mu$ for all $(\mathcal{L}_i)$-stopping times μ. Thus, we have

$$\mathbb{E}Y^*_{\gamma^*} = \mathbb{E}\mathbf{1}_{\Omega_1} Y_{\gamma^*} = \mathbb{E}\mathbf{1}_{\Omega_1} Y_{\bar\gamma} = \mathbb{P}(\Omega_1)\mathbb{E}Y_{\bar\gamma} = \mathbb{P}(\Omega_1)\mathbb{E}Y_{\bar\tau} = \mathbb{E}\mathbf{1}_{\Omega_1} Y_{\bar\tau} = \mathbb{E}Y^*_{\bar\tau}.$$

The next theorem is the main result of this section and states that if we concatenate two independent processes then we can find an optimal stopping time maximizing the expected value of the concatenated process by replacing the second process with the trace left on it by an optimal stopping time for this second process.

Theorem 3.3. *With the above assumptions about the processes X, Y, C, and Z, if $\overline{\zeta}$ is a stopping time maximizing $\mathbb{E}Z_\zeta$ over all $(\mathcal{E}_t)$-stopping times ζ then the stopping time $\overline{\delta} = \overline{\delta}_{\overline{\zeta},\overline{\tau}}$, defined as $\overline{\delta} = \overline{\zeta}$ if $\overline{\zeta} \le m$ and $\overline{\delta} = m + \overline{\tau}$ when $\overline{\zeta} = m + 1$, maximizes $\mathbb{E}C_\delta$ over all $(\mathcal{M}_t)$-stopping times δ.*

Proof. From Theorem 3.2 applied to $\mathcal{G} = \mathcal{K}_m$, $\mathcal{H}_i = \mathcal{M}_{m+i}$, and $\Omega_1 = [\delta > m]$, we have

$$\int_{[\delta>m]} Y_{\overline{\tau}} \,\mathrm{d}P \ge \int_{[\delta>m]} C_\delta \,\mathrm{d}P = \int_{[\delta>m]} Y_{\delta-m} \,\mathrm{d}P.$$

Let us now define the following stopping time $\alpha\colon \Omega \to \{1, \dots, m+1\}$:

$$\alpha(\omega) = \begin{cases} \delta(\omega) & \text{if } \delta(\omega) \le m, \\ m+1 & \text{if } \delta(\omega) > m. \end{cases}$$

Note that α is a stopping time with respect to the filtration $(\mathcal{E}_t)_{t\le m+1}$. We have

$$\begin{aligned} \mathbb{E}C_\delta &= \int_{[\delta\le m]} X_\delta \,\mathrm{d}P + \int_{[\delta>m]} Y_{\delta-m} \,\mathrm{d}P \\ &\le \int_{[\delta\le m]} X_\delta \,\mathrm{d}P + \int_{[\delta>m]} Y_{\overline{\tau}} \,\mathrm{d}P \\ &= \mathbb{E}Z_\alpha \\ &\le \mathbb{E}Z_{\overline{\zeta}} \\ &= \mathbb{E}C_{\overline{\delta}}, \end{aligned}$$

where the last inequality follows from the optimality of the stopping time $\overline{\zeta}$.

4. MIA process

In this section we show that, for processes in secretary-type problems, (2.6) implies that these processes satisfy the monotone case theorem. In particular, this concerns both the no-information and the full-information secretary problems.

In what follows we consider the optimal best-choice problem on posets discussed in, for instance, [7] and [13]. (The best-choice problem on posets with restricted a priori knowledge about an underlying partial order was considered in a number of papers; for further bibliography and results, see [2], [3], [5], [8], [11], and [14].) In this paper we always assume that a poset has the greatest element (see the remark below).

Let $E = \{e_1, \dots, e_t\}$ and $F = \{f_1, \dots, f_t\}$ be labeled posets. If the function $e_i \mapsto f_i$, $i \le t$, establishes an order isomorphism between E and F, then we say that E and F are order isomorphic and we write $E \cong F$.

Let $V = \{(V_t, \mathcal{H}_t),\ 1 \le t \le m\}$ be a stochastic process whose values are labeled subposets which grow in time, $V_t \subseteq V_{t+1}$ and $|V_{t+1} \setminus V_t| = 1$. An element of a poset is labeled with the time of its arrival. Let $v_t = v_t(\omega)$ be the element that arrives at time t. Thus, label t is assigned to both the element v_t that appeared at time t and to the poset V_t consisting of all elements $v_1, v_2, \dots, v_t$ that have appeared up to time t. In particular, $|V_t(\omega)| = t$. We can assume that the structure of a poset V_t consists of all the information we have at time t, but it is also possible that V_t is a particular poset; for instance, V_t is a sequence of t numbers from $[0, 1]$ and these numbers are known at time t. We assume that, for each ω in the set $\{v_1(\omega), \dots, v_m(\omega)\}$, there always exists the greatest element $M(\omega)$. If V is a fixed poset from which we draw elements then $M(\omega)$ is the greates element of V (with no dependence on a particular ω). This happens for instance in the classical no-information secretary problem, but in the full-information case the largest element $M(\omega)$ of V_m is not a priori known (and here it does depend on ω). For the secretary problem on a given poset R, $|R| = m$, $\mathcal{H}_t$ is generated by events of the form

$$A_F = \{\omega = (v_1, \dots, v_t, \dots, v_m) \colon (v_1, \dots, v_t) \cong F\}$$

for all possible labeled posets F, where the ω are permutations of elements of R. For the full-information search given by a sequence of independent, identically distributed random variables with uniform distribution in $[0, 1]$,

$$X_1, \dots, X_t, \dots, X_m,$$

we assume that $\sigma(X_1, \dots, X_t) = \mathcal{H}_t$. As these are our main examples, we do not provide a formalism to describe more general situations.

We now consider the secretary-type problem of maximizing the probability $\mathbb{P}[v_\tau = M]$ over all stopping times τ. Let us rescale the time in the following way. We recursively define the sequence $\{\xi_i \colon 1 \le i \le m\}$ of stopping times by

$$\xi_1(\omega) = 1, \qquad \xi_{i+1}(\omega) = \min\{j > \xi_i(\omega) \colon \mathbb{P}[v_j = M] > 0\}, \tag{4.1}$$

using the convention that if the set under min is empty then $\xi_{i+1}(\omega) = m$. Let $\mathcal{F}_t = \mathcal{H}_{\xi_t}$. We have

$$\mathbb{P}[v_{\xi_t} = M] > 0 \quad \text{or} \quad \mathbb{P}[v_{\xi_t} = M] = \cdots = \mathbb{P}[v_{\xi_m} = M] = 0. \tag{4.2}$$

In a standard way (cf., e.g. [1]) we can reformulate the problem to deal again with a process X_i, $1 \le i \le m$, satisfying (2.1) for which maximizing the expected value $\mathbb{E}X_\tau$ over all $(\mathcal{F}_t)$-stopping times τ is equivalent to maximizing the probability $\mathbb{P}[v_\rho = M]$ over all $(\mathcal{H}_t)$-stopping

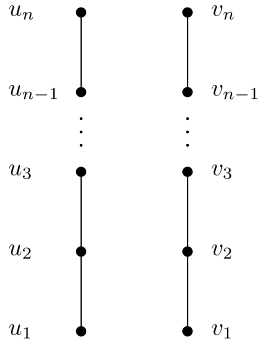

FIGURE 1.

times ρ. The process X is related to V in the following way. Let $W_t = \mathbf{1}_{[v_{\xi_t}=M]}$. As, in general, W_t is not $\mathcal{F}_t$-measurable, we define $X_t = \mathbb{E}(W_t \mid \mathcal{F}_t)$.

We call X the MIA for the process V. By (4.2), X satisfies (2.1).

Theorem 4.1. *If $X = \{(X_t, \mathcal{F}_t),\ 1 \le t \le m\}$ is the MIA for some process V and $X_t(\omega) > 0$, then*

$$\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega) = X_t(\omega)$$

for a.e. ω. In particular, for an MIA, (2.5) is equivalent to (2.6).

Proof. For $X_t(\omega) > 0$, we have

$$\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega) = \mathbb{E}(\mathbf{1}_{[v_{\xi_t}=M]} \mid \mathcal{F}_t)(\omega) = \mathbb{E}(W_t \mid \mathcal{F}_t)(\omega) = X_t(\omega).$$

Remark. We can generalize the above considerations to the secretary-type problem on posets with multiple maximal elements defining MIAs for consecutive stopping times ξ_t ensuring that $\mathbb{P}[v_{\xi_t}$ is maximal$] > 0$. However, posets with more than one maximal element do not usually satisfy the monotone case theorem. Below we consider an example of a poset consisting of two independent linear orders in which we highlight the rather universal phenomenon of MIAs for naturally appearing posets with more than one maximal element not satisfying the monotone case theorem.

Let $P = (\{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}, \le)$, where $u_1 \le u_2 \le \cdots \le u_n$, $v_1 \le v_2 \le \cdots \le v_n$, and all pairs u_i, v_j, $1 \le i, j \le n$, are incomparable (see Figure 1). Let $\xi_1(\omega) = 1$, and recursively define $\xi_{i+1}(\omega) = \min\{j > \xi_i(\omega) \colon \omega_j$ is maximal in $\omega_1, \dots, \omega_j$. The counterpart of the MIA process for the search on P is the process $X_t = \mathbb{E}(\mathbf{1}_{[\omega_{\xi_t} \text{ is maximal}]} \mid \mathcal{H}_{\xi_t})$, where, as usual, the σ-algebras $\mathcal{H}_t$ describe all possible situations that can happen up to time t when the selector at time t knows only a relative order of the elements that have appeared up to time t and the order in which these elements appeared. As before, let $\mathcal{F}_t = \mathcal{H}_{\xi_t}$.

Let $\omega = (u_{n-1}, u_{n-2}, \dots, u_1, u_n, v_1, v_2, \dots, v_n)$. Then $\xi_1(\omega) = 1$, $\xi_2(\omega) = n$, $\xi_3(\omega) = n+1$, and $\xi_4(\omega) = n+2$. For $t = 2$, $X_t(\omega) = 1$, $X_{t+1}(\omega) = 1/n$, $\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) = 1/n$, $\mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega) = (1/n)(1 + \frac{1}{2} + \cdots + 1/(n-1))$, and, of course, the implication

$$X_t(\omega) \ge \mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) \quad \Longrightarrow \quad X_{t+1}(\omega) \ge \mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega)$$

does not hold here.

A similar nasty counterexample can be constructed for posets R whose longest chain(s) is (are) orderwise independent of a substantial portion of R (even if the Hasse diagram of R is connected) and this portion of R is not very shallow.

5. Classical secretary searches and their concatenations

It turns out that the most classical examples of MIAs for the no-information and the full-information secretary problems satisfy (2.6).

In the most classical no-information secretary problem there are n linearly ordered elements that are observed by a selector in a random order. At a given moment, the selector knows only the order formed by these elements. The aim is to maximize the probability of selecting the best element.

Proposition 5.1. *The MIA for the no-information secretary problem satisfies (2.6).*

Proof. In this case $V_t = \{v_1, \dots, v_t\}$ is a labeled, linearly ordered set, where the sequence $v_1, \dots, v_t$ is equally probable with all possible sequences of length t of pairwise different elements chosen from the set $\{1, 2, \dots, m\}$. Let the stopping times ξ_t be defined by (4.1), and let X be the MIA for V. We have, for $\xi_t(\omega) = l$ and $\xi_{t+1}(\omega) = l'$, where $m > l' > l$ (the case $l' = m$ is trivial),

$$X_t(\omega) = \frac{l}{m}, \qquad X_{t+1}(\omega) = \frac{l'}{m},$$

and

$$\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) = \frac{l}{m} \sum_{i=l}^{m-1} \frac{1}{i}, \qquad \mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega) = \frac{l'}{m} \sum_{i=l'}^{m-1} \frac{1}{i}.$$

Hence, (2.6) is equivalent to

$$\frac{l'}{m} \frac{l}{m} \sum_{i=l}^{m-1} \frac{1}{i} \geq \frac{l}{m} \frac{l'}{m} \sum_{i=l'}^{m-1} \frac{1}{i},$$

which obviously holds.

In the full-information secretary problem a selector is presented with n numbers, one at a time, a priori chosen with uniform distribution on the interval $[0, 1]$ and, as in the no-information secretary problem, the aim is to maximize the probability of selecting the best number.

Proposition 5.2. *The MIA for the full-information secretary problem satisfies (2.6).*

Proof. Let $S_1, S_2, \dots, S_m$ be independent random variables of uniform distribution on $[0, 1]$. (Of course, considering an ordered set-valued process $(\{S_1, \dots, S_t\})_t$ is, from the point of view of searching for the absolute maximal element, equivalent to considering the process S itself.) Let $\mathcal{H}_t = \sigma\{S_1, \dots, S_t\}$. For $v_i(\omega) = S_i(\omega)$, let the stopping times ξ_t be defined by (4.1). Let $\mathcal{F}_t = \mathcal{H}_{\xi_t}$. Let X be the MIA for S.

Assume that $\xi_t(\omega) = l$. We have

$$\begin{aligned}
\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) &= \sum_{j=l+1}^{m} S_l^{j-l-1}(\omega) \int_{S_l(\omega)}^{1} t^{m-j} \, \mathrm{d}t \\
&= \sum_{j=l+1}^{m} S_l^{j-l-1}(\omega) \frac{1}{m-j+1} (1 - S_l^{m-j+1}(\omega)) \\
&= \sum_{j=l+1}^{m} S_l^{j-l-1}(\omega) \frac{1}{m-j+1} - S_l^{m-l}(\omega) \sum_{j=l+1}^{m} \frac{1}{m-j+1}.
\end{aligned}$$

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Assume that $\xi_{t+1}(\omega) = k > l$ and $S_k(\omega) > S_l(\omega)$. Now (2.6) is equivalent to

$$b^{m-k}\Bigg(\sum_{j=l+1}^{m} a^{j-l-1}\frac{1}{m-j+1} - a^{m-l}\sum_{j=l+1}^{m}\frac{1}{m-j+1}\Bigg)$$
$$\geq a^{m-l}\Bigg(\sum_{j=k+1}^{m} b^{j-k-1}\frac{1}{m-j+1} - b^{m-k}\sum_{j=k+1}^{m}\frac{1}{m-j+1}\Bigg),$$

where $a = S_l(\omega)$ and $b = S_k(\omega)$.

The above inequality is equivalent to

$$b^{m-k}\sum_{j=l+1}^{m} a^{j-l-1}\frac{1}{m-j+1} - a^{m-l}\sum_{j=k+1}^{m} b^{j-k-1}\frac{1}{m-j+1}$$
$$\geq a^{m-l}b^{m-k}\sum_{j=l+1}^{k}\frac{1}{m-j+1}.$$

Multiplying both sides of this inequality by $a^{-m+l}b^{-m+k}$, we obtain

$$\sum_{j=l+1}^{m} a^{j-1-m}\frac{1}{m-j+1} - \sum_{j=k+1}^{m} b^{j-1-m}\frac{1}{m-j+1} \geq \sum_{j=l+1}^{k}\frac{1}{m-j+1}.$$

As $j-1-m \leq 0$, we have

$$\sum_{j=l+1}^{m} a^{j-1-m}\frac{1}{m-j+1} - \sum_{j=k+1}^{m} b^{j-1-m}\frac{1}{m-j+1}$$
$$\geq \sum_{j=l+1}^{m} b^{j-1-m}\frac{1}{m-j+1} - \sum_{j=k+1}^{m} b^{j-1-m}\frac{1}{m-j+1}$$
$$= \sum_{j=l+1}^{k} b^{j-1-m}\frac{1}{m-j+1}$$
$$\geq \sum_{j=l+1}^{k}\frac{1}{m-j+1}.$$

In the following four examples we consider all possible concatenations of the processes $V^{(m)} = (V_i)_{i\leq m}$ (for the no-information secretary problem) and $S^{(n)} = (S_j)_{j\leq n}$ (for the full-information secretary problem). We know the optimal stopping times and the probability of success when using optimal stopping times for $V^{(m)}$ (see [1]) and $S^{(n)}$ (see [6]). Let $P_v(n)$ and $P_s(n)$ be the probabilities of success when using optimal stopping times for $V^{(n)}$ and $S^{(n)}$, respectively.

Let $X = (X_i, \mathcal{F}_i)_i$ and $Y = (X_i, \mathcal{G}_i)_i$ be MIAs for either the no-information or full-information search, depending on which pair, $(V^{(m)}, V^{(n)})$, $(V^{(m)}, S^{(n)})$, $(S^{(m)}, S^{(n)})$, or $(S^{(m)}, V^{(n)})$, we concatenate. Let $Z = (Z_i, \mathcal{E}_i)_{i\leq m+1}$ be the merge of X and Y (defined at the beginning of Section 3).

In view of Propositions 5.1 and 5.2 and Theorems 4.1, 2.2, and 3.3 applied to X and Y, for a given ω, it will be crucial to find the time t of the first ith relative record such that

$$Z_{\rho_i}(\omega) \geq \mathbb{E}(Z_{\rho_{i+1}} \mid \mathcal{E}_i)(\omega) \tag{5.1}$$

(recall that the ρ_i are defined in (2.2)).

Example 5.1. Consider the process A which is the concatenation of $V^{(m)}$ and $V^{(n)}$. Here (5.1) for the time t of the ith relative record takes the form

$$\frac{t}{m} \geq \frac{t}{m}\bigg(\frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{m-1} + P_v(n)\bigg),$$

which is equivalent to

$$1 - P_v(n) \geq \frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{m-1}. \tag{5.2}$$

From Proposition 5.1 we know that the MIA for $V^{(m)}$ satisfies (2.6). By Theorem 4.1, for an MIA, (2.5) is equivalent to (2.6). By Theorem 2.2 we know that the process Z satisfies the monotone case theorem. By Theorem 3.3, maximizing $\mathbb{E}Z_\zeta$ is equivalent to maximizing the probability of the best choice in our concatenation problem. Then

$$\tau = \begin{cases} \min\{t \geq t^*\colon v_t = \max\{v_1, \dots, v_t\},\ t \leq m\} & \text{if the set under min is nonempty,} \\ m + \overline{\tau} & \text{otherwise,} \end{cases}$$

where t^* is the smallest t satisfying inequality (5.2) and $\overline{\tau}$ is the optimal stopping time for $V^{(m)}$. It is easy to see that, when $n, m \to \infty$,

$$\lim_{m\to\infty} \frac{t^*}{m} = \exp(\lim_{n\to\infty} P_v(n) - 1) = \exp(\mathrm{e}^{-1} - 1) \approx 0.53$$

(see [10]).

Example 5.2. Consider the process B which is the concatenation of $V^{(m)}$ and $S^{(n)}$. Here (5.1) for the time t of the ith relative record takes the form

$$\frac{t}{m} \geq \frac{t}{m}\bigg(\frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{m-1} + P_s(n)\bigg),$$

which is equivalent to

$$1 - P_s(n) \geq \frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{m-1}. \tag{5.3}$$

Analogously as in Example 5.1, it follows from Proposition 5.1 and Theorems 4.1, 2.2, and 3.3 that the optimal stopping time for the process B, where $V^{(m)}$ is followed by $S^{(n)}$, is

$$\tau = \begin{cases} \min\{t \geq t^*\colon v_t = \max\{v_1, \dots, v_t\},\ t \leq m\} & \text{if the set under min is nonempty,} \\ m + \overline{\tau} & \text{otherwise,} \end{cases}$$

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where t^* is the smallest t satisfying inequality (5.3) and $\overline{\tau}$ is the optimal stopping time for $S^{(n)}$. It is easy to prove that, when $n, m \to \infty$,

$$\lim_{m\to\infty} \frac{t^*}{m} = \exp(\lim_{n\to\infty} P_s(n) - 1) \approx 0.657,$$

where

$$\lim_{n\to\infty} P_s(n) = \mathrm{e}^{-c} + (\mathrm{e}^{c} - c - 1) \int_1^{\infty} \mathrm{e}^{-ct} t^{-1} \, \mathrm{d}t = 0.580\,164\,223\,9\ldots$$

($c = 0.804\,352\,262\,8\ldots$ is the unique real solution of the equation

$$\int_{-\infty}^{c} \mathrm{e}^{t} t^{-1} \, \mathrm{d}t - \gamma - \ln(c) = 1,$$

where γ is the Euler–Mascheroni constant; see [6] and [15]).

Example 5.3. Consider the process C which is the concatenation of $S^{(m)}$ and $S^{(n)}$. Here (5.1) for the time t of the ith relative record takes the form

$$S_t^{m-t} \geq \sum_{j=t+1}^{m} S_t^{j-t+1} \int_{S_t}^{t} t^{m-j} \, \mathrm{d}x + P_s(n) S_t^{m-t},$$

which is equivalent to

$$1 - P_s(n) \geq \sum_{i=1}^{m-t} \frac{1}{i} (S_t^{-i} - 1). \tag{5.4}$$

Analogously as in Example 5.1, it follows from Proposition 5.2 and Theorems 4.1, 2.2, and 3.3 that the optimal stopping time for the process C is

$$\tau = \begin{cases} \min\{t \colon S_t = \max\{S_1, \ldots, S_t\},\ S_t \text{ satisfies (5.4)}, \\ \qquad t \leq m\} & \text{if the set under min is nonempty,} \\ m + \overline{\tau} & \text{otherwise,} \end{cases}$$

where $\overline{\tau}$ is the optimal stopping time for $S^{(n)}$.

Example 5.4. Consider the process D which is the concatenation of $S^{(m)}$ and $V^{(n)}$. Here (5.1) for the time t of the ith relative record takes the form

$$S_t^{m-t} \geq \sum_{j=t+1}^{m} S_t^{j-t+1} \int_{S_t}^{t} t^{m-j} \, \mathrm{d}x + P_v(n) S_t^{m-t},$$

which is equivalent to

$$1 - P_v(n) \geq \sum_{i=1}^{m-t} \frac{1}{i} (S_t^{-i} - 1). \tag{5.5}$$

Analogously as in Example 5.1, it follows from Proposition 5.2 and Theorems 4.1, 2.2, and 3.3 that the optimal stopping time for the process D is

$$\tau = \begin{cases} \min\{t \colon S_t = \max\{S_1, \ldots, S_t\},\ S_t \text{satisfies (5.5)}, \\ \qquad t \leq m\} & \text{if the set under min is nonempty,} \\ m + \overline{\tau} & \text{otherwise,} \end{cases}$$

where $\overline{\tau}$ is the optimal stopping time for $V^{(n)}$.

Even if we think that these formulae should hold, the formalism to justify them is rather unpleasant and intuitions from similar cases may be misleading, but, for the processes whose MIAs satisfy (2.6), we can apply Theorem 4.1 as in the above examples.

Remark. Using Theorem 2.2, we can find an optimal stopping time for the search consisting of n consecutive searches on posets where the MIAs for the first $n-1$ searches satisfy (2.5). For instance, instead of concatenating two searches of the form $V^{(k)}$ and $S^{(k)}$ we can concatenate any finite number of them. Such concatenations of no-information secretary searches were considered in [10] and of full-information secretary searches in [9].

6. Two special examples

In this section we present two examples that go beyond MIAs for secretary searches on posets with one maximal element. While in the first example we still consider an MIA-like process for a search on a very special poset with two maximal elements (considered in [4]), in the second example we consider a process in which information is spread via a random walk (considered in [12]).

Example 6.1. Consider a process consisting of two chains, $V_1 = \{x_1, \dots, x_n\}$ and $V_2 = \{y_1, \dots, y_n\}$ with x_n and y_n being maximal elements. For every i, $1 \le i \le n$, the elements x_i and y_i are incomparable and $x_i, y_i \le x_j, y_j$ for $i < j$. We call the elements x_i and y_i twins occurring on level i. Let $V = V_1 \cup V_2$ (see Figure 2). The elements of V are observed one at a time in the order $v_1(\omega), \dots, v_{2n}(\omega)$ given by a random permutation ω, i.e. $\omega_t = v_t(\omega)$, and we assume that all $(2n)!$ permutations of V are equiprobable.

At every time t, $1 \le t \le 2n$, we observe the partial order induced by the set $\{\omega_1, \omega_2, \dots, \omega_t\}$. Our goal is to maximize the probability of selecting the maximal twin x_n or y_n.

We recursively define the following stopping times beyond which it does not make sense to stop. Let $\xi_0(\omega) = 0$ and

$$\xi_t(\omega) = \min\{i > \xi_{t-1}(\omega) \colon \omega_i \text{ is a maximal element of } \{\omega_1, \omega_2, \dots, \omega_i\} \\ \text{and } \{\omega_1, \omega_2, \dots, \omega_{i-1}\} \text{ contains the twin of } \omega_i\},$$

with the assumptions that if the set under the minimum is empty then its minimum is $2n$.

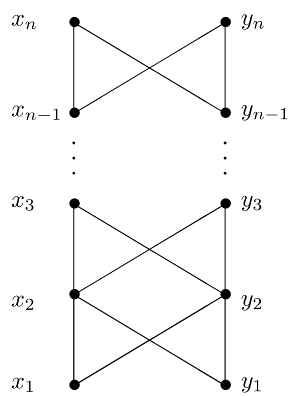

FIGURE 2.

As is usual for a search on a fixed poset, let the σ-algebras $\mathcal{H}_t$ describe all possible situations that can happen up to time t when the selector at time t knows only a relative order of the elements which have appeared up to time t and the order in which these elements appeared. Let $\mathcal{F}_t = \mathcal{H}_{\xi_t}$.

It is known (see [4]) that the optimal stopping time τ maximizing the probability of choosing one of the best twins is given by

$$\tau(\omega) = \min\{\xi_t(\omega)\colon \text{the number of levels occupied by } \omega_1, \omega_2, \dots, \omega_{\xi_t} \text{ is at least } k_n\},$$

where $k_n = \min\{k\colon 5 - 2n/k \geq \sum_{i=k}^{n-1}(1/i)\}$. Let $X_t(\omega) = \mathbb{E}(\mathbf{1}_{[v_{\xi_t} \in \{x_n, y_n\}]} \mid \mathcal{F}_t)(\omega)$.

Let $N(t,k)$ denote the event that the number of levels occupied by the elements $\omega_1, \omega_2, \dots, \omega_{\xi_t}$ is equal to k. We have $X_t|_{N(t,k)}(\omega) = k/n$ and

$$\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)|_{N(t,k)}(\omega) = \sum_{j=1}^{n-k} \frac{2j+1}{3j} \binom{n-j-1}{k-1} \binom{n}{k}^{-1}.$$

The conclusion of Theorem 4.1 holds here and in this case inequality (2.6) is equivalent to

$$\frac{l}{n} \sum_{j=1}^{n-k} \frac{2j+1}{3j} \binom{n-j-1}{k-1} \binom{n}{k}^{-1} \geq \frac{k}{n} \sum_{j=1}^{n-l} \frac{2j+1}{3j} \binom{n-j-1}{l-1} \binom{n}{l}^{-1}$$

for $1 \leq k < l < n$, which is easy to prove using the following combinatorial identity:

$$\sum_{j=1}^{n-k} \frac{1}{j} \binom{n-j-1}{k-1} = \binom{n-1}{k-1} \sum_{i=k}^{n-1} \frac{1}{i}$$

for all $1 \leq k \leq n-1$ (see Proposition 3 of [4]).

Now we present an example that goes beyond MIAs for the best-choice problems. The following problem was considered in [12].

Example 6.2. A messenger that carries information moves according to a symmetric random walk on the set of integers $\mathbb{Z}$ starting at 0. The random walk is not visible to the observer. The points from the integer interval $V_f = \{-n+1, \dots, -1, 0, 1, \dots, n-1\}$ are considered friendly points and all the remaining integers are considered hostile points. If our messenger reaches a new point then this point receives the information and a signal is sent to an observer that controls the game; if a newly reached point is a friendly point then the messenger scores 1. At this point the messenger's account consists of the number of all informed friends unless an enemy was visited. If a hostile point is visited then the information is leaked to the enemies, a different signal is sent to the observer, and the game is over with the messenger's account equal to 0. The clock here does not count according to the moves of the random walk but according to new informed points; in other words, the observer's clock strikes t if the tth signal about a newly informed point arrives. Let L_k be the event that at the time when the kth point has been informed, all k informed points belong to V_f. Let $\mathcal{F}_t = \sigma(L_1, L_2, \dots, L_t)$. The payoff function is $X_t = t$ if all t points which have been informed are in V_f and $X_t = 0$ otherwise; the convention is that the starting point 0, though it is a friendly point, does not count as far as the number of informed points is concerned. Let us fix a time horizon at $m = 2n-2$, which is sufficient for full generality here (after this time we always lose).

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Let Y be an independent copy of the process X. Let C be the concatenation of X and Y. This corresponds to the game that consists either in stopping in the X-part and collecting the gain or, if we do not stop at this part, starting the Y-part from scratch. To find an optimal stopping time for this game, it is enough by Theorem 3.3 to find an optimal stopping for a merge Z of X and Y (defined at the beginning of Section 3), or, equivalently, for the process $(Z_{\rho_i}, \mathcal{F}_{\rho_i})_{i\le m+1}$, where the stopping times ρ_i are defined in (2.2).

We will show that the process $(Z_{\rho_i}, \mathcal{F}_{\rho_i})_{i\le m+1}$ satisfies the monotone case theorem. To this end, by Theorem 2.2, it is enough to show that (2.5) holds for X and $Q = Y_{\tau^*}$, where τ^* is the optimal stopping time of maximizing $\mathbb{E}Y_\tau$ over all stopping times τ.

Assume that for an ω at a time t we are still in the friendly zone. From [12, Section 2] we know that

$$\tau^* = n-1, \qquad X_t(\omega) = t,$$
$$\mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) = (t+1)\biggl(1 - \frac{2n}{(t+2)(2n-t-1)}\biggr), \qquad \mathbb{E}Q = n-1,$$
$$\mathbb{E}(\mathbf{1}_{[X_{t+1}=0]} \mid \mathcal{F}_t)(\omega) = \frac{2n}{(t+2)(2n-t-1)}.$$

Thus, we want to show that

$$\begin{aligned}(t+1)^2\biggl(1 - \frac{2n}{(t+2)(2n-t-1)}\biggr) &+ (t+1)(n-1)\frac{2n}{(t+2)(2n-t-1)} \\ &\ge t(t+2)\biggl(1 - \frac{2n}{(t+3)(2n-t-2)}\biggr) + t(n-1)\frac{2n}{(t+3)(2n-t-2)}.\end{aligned}$$

This inequality is equivalent to

$$1 + \frac{2n(t+1)}{(t+2)(2n-t-1)}(n-t-2) \ge \frac{2nt}{(t+3)(2n-t-2)}(n-t-3).$$

Hence, it is enough to show that

$$\frac{n-t-2}{2n-t-1} \ge \frac{n-t-3}{2n-t-2},$$

which is equivalent to

$$(n-t-2)(2n-t-2) \ge (n-t-3)(2n-t-1),$$

which holds because $1 \ge -n$.

7. Some poset examples and counterexamples

In this section we will present several examples that answer some natural questions related to the main results of this paper. They all arise from the poset whose Hasse diagram is presented in Figure 3 for an appropriate choice of the parameters m, n, and r.

Example 7.1. In this example we consider the MIA of a poset that satisfies the monotone case theorem but fails to satisfy (2.6). We also find an optimal stopping time for this case.

Let D be a poset whose Hasse diagram is the tree given in Figure 4. Let $X = (X_t)_{t\le 8}$ be the MIA for this case.

We consider situations in which the last observed element is the best and we determine in which of these situations stopping is the optimal decision. Thus, we find the optimal stopping time.

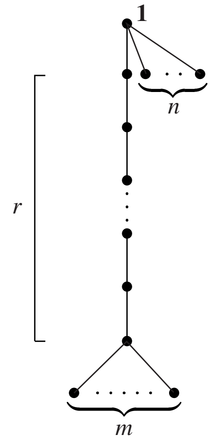


FIGURE 3.

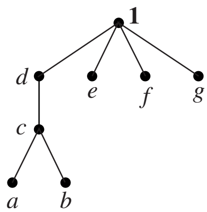
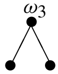


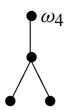
FIGURE 4.

(a) $\bullet^{\omega_1}$: $X_1(\omega) = \frac{1}{8} < \mathbb{E}(X_2 \mid \mathcal{F}_1)(\omega)$ and $\mathbb{E}(X_2 \mid \mathcal{F}_1)(\omega) > \frac{1}{2}$; we do not stop.

(b)  $^{\omega_2}$: $X_2(\omega) = \frac{7}{12} > \frac{1}{2} > \mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega)$; we stop.

(c)  $^{\omega_3}$: $X_2(\omega) = \frac{5}{7} > \frac{1}{2}$ ($t = 2$ or 3); we stop.

(d)  $^{\omega_3}$: $X_2(\omega) = \frac{8}{9} > \frac{1}{2}$; we stop.

(e)  $^{\omega_4}$: $X_2(\omega) = \frac{2}{3} > \frac{1}{2}$; we stop.

Of course, we always stop in situations where the absolute best element is determined uniquely.

Using the optimal stopping time, the probability of success equals $p = \frac{23}{30} = 0.7(6)$. (We do not choose the best element when the best element is the first element because we do not stop at this point, or when the first moment we stop in situations (b)–(e) we observe an element that is the best but not the absolute best. In all other cases we win. Thus, we have $1 - p = \frac{1}{8} + 5 \cdot 6!/8! + 2 \cdot 5!/8! + 4 \cdot 5!/8! + 2 \cdot 4!/8! = \frac{7}{30}$.)

It is not difficult to check that X satisfies the monotone case theorem.

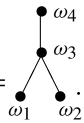
Now consider the permutation $\omega = (a, b, c, d, e, f, g, \mathbf{1})$. Again, it is a simple task to check that $X_2(\omega) = \frac{8}{9}$, $X_3(\omega) = \frac{2}{3}$, and $\mathbb{E}(X_4 \mid \mathcal{F}_3)(\omega) = \frac{1}{3}$. From $X_2(\omega) = \frac{8}{9}$, it immediately follows that $\mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega) \le \frac{1}{9}$, whence (2.6) does not hold.

Example 7.2. In this example we consider the process which is the double iteration of the best-choice search on a finite poset D whose Hasse diagram is the tree given in Figure 4. The MIA for a single search satisfies the monotone case theorem but fails to satisfy (2.6), as noted in Example 7.1. Let Z be a merge (defined at the beginning of Section 3) of two consecutive copies of MIAs for a single search on D. The process $(Z_{\rho_t})_{t\le 9}$, as we will show, does not satisfy the monotone case theorem. Nevertheless, stopping the first time the inequality

$$Z_{\rho_t}(\omega) \ge \mathbb{E}(Z_{\rho_{t+1}} \mid \mathcal{F}_{\rho_t})(\omega) \tag{7.1}$$

holds is optimal.

We first show that this process does not satisfy the monotone case theorem. Let us assume that in the first search we have a permutation $\omega = (\omega_1, \dots, \omega_8)$ such that

$\{\omega_1, \omega_2, \omega_3, \omega_4\}|_{\le} =$ 

.

Then $Z_{\rho_2}(\omega) = \frac{8}{9}$, $\mathbb{E}(Z_{\rho_3} \mid \mathcal{F}_{\rho_2})(\omega) = \frac{1}{9}(\frac{1}{2} + \frac{1}{2}\frac{4}{5}) + \frac{8}{9}\frac{23}{30} = \frac{211}{270} \approx 0.781\,481\,48$, $Z_{\rho_3}(\omega) = \frac{2}{3}$, and $\mathbb{E}(Z_{\rho_4} \mid \mathcal{F}_{\rho_3})(\omega) = \frac{1}{3} + \frac{2}{3}\frac{23}{30} = \frac{38}{45} = 0.8(4)$, and we have

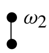
$$Z_{\rho_2}(\omega) > \mathbb{E}(Z_{\rho_3} \mid \mathcal{F}_{\rho_2})(\omega) \quad \text{and} \quad Z_{\rho_3}(\omega) < \mathbb{E}(Z_{\rho_4} \mid \mathcal{F}_{\rho_3})(\omega).$$

We now show that the optimal stopping time $\bar{\tau}$ in this process maximizing the probability of choosing a maximal element in one of the two searches, is given by the condition

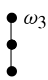
$$\bar{\tau} = \min\{t \colon Z_{\rho_t} \ge \mathbb{E}(Z_{\rho_{t+1}} \mid \mathcal{F}_{\rho_t})\}.$$

We consider situations in which the last observed element is the best. We determine in which of these situations stopping is the optimal decision. Thus, we find the optimal stopping time.

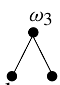
(a) $\bullet^{\omega_1}$: $X_1(\omega) = \frac{1}{8} < \frac{23}{30}$; we do not stop.

(b) 

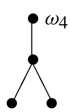
: $X_2(\omega) = \frac{7}{12} < \frac{23}{30}$; we do not stop.

(c) 

: $X_t(\omega) = \frac{5}{7} < \frac{23}{30}$ ($t = 2$ or 3); we do not stop.

(d) 

: $X_2(\omega) = \frac{8}{9}$ and $\frac{8}{9} > \frac{1}{9} \cdot \hat{p} + \frac{8}{9} \cdot \frac{23}{30}$, where $\hat{p}$ is the probability of choosing the best element in the first search using the optimal strategy; we stop.

(e) 

: $X_2(\omega) = \frac{2}{3} < \frac{23}{30}$; we do not stop.

Of course, we always stop in situations where the absolute best element in the first search is determined uniquely.

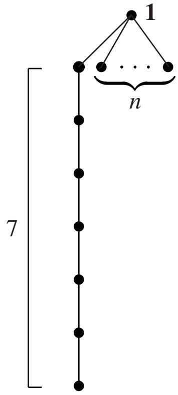


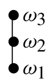
FIGURE 5.

Then the probability of success equals $\tilde{p} = q + r \cdot p \approx 0.9217$, where $q = \frac{591}{840}$ is the probability that we win in the first search, $r = \frac{239}{840}$ is the probability that we do not stop in the first search, and p is the maximal probability of success in a single search (see Example 7.1).

Example 7.3. In this example we consider a poset for which the MIA of the secretary-type search on this poset does not satisfy the monotone case theorem, i.e. for some t and some ω, $X_t(\omega) > \mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega)$ and $X_{t+1}(\omega) < \mathbb{E}(X_{t+2} \mid \mathcal{F}_{t+1})(\omega)$.

Let us consider the best-choice search on a finite poset whose Hasse diagram is the tree given in Figure 5, and let $X = (X_t)_{t \le n+8}$ be the MIA for this case.

Let ω be a permutation such that

$\{\omega_1, \omega_2, \omega_3\}|_{\le} =$  .

Then we have

$$X_2(\omega) = \frac{n+7}{n+\binom{8}{2}}, \qquad \mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega) = \frac{n+1}{n+\binom{8}{2}}\left(\frac{6}{n+1} + \frac{5}{n+2} + \cdots + \frac{1}{n+6}\right),$$

$$X_3(\omega) = \frac{3}{8}$$

$$\text{and} \quad \mathbb{E}(X_4 \mid \mathcal{F}_3)(\omega) = \frac{5(n+1)}{8\binom{7}{3}}\left(\frac{\binom{6}{2}}{n+1} + \frac{\binom{5}{2}}{n+2} + \frac{\binom{4}{2}}{n+3} + \frac{\binom{3}{2}}{n+4} + \frac{\binom{2}{2}}{n+5}\right).$$

For $n \ge 12$, we obtain

$$X_2(\omega) > \mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega) \quad \text{and} \quad X_3(\omega) < \mathbb{E}(X_4 \mid \mathcal{F}_3)(\omega).$$

Assume that $n > 14$. We will show that, although X does not satisfy the monotone case theorem, we stop optimally the first moment the inequality

$$X_t(\omega) > \mathbb{E}(X_{t+1} \mid \mathcal{F}_t)(\omega) \tag{7.2}$$

holds. Of course, we should consider only the cases where $t > 1$. Note that if we see the tth relative record at the moment ξ_t and the set $\{\omega_1, \ldots, \omega_{\xi_t}\}$ is not linearly ordered then $X_t(\omega) = 1$,

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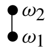
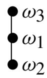
and, thus, we should consider only the situations when relative records appear for chains. If we see a chain



at $t = \xi_t = 2$ then, as is easy to check, we have $X_t(\omega) > \frac{1}{2}$, (7.2) holds, and we obviously stop. If at $t = 3$ we see a chain



then we have $\xi_2 = 3$ and

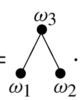
$$X_2(\omega) = \frac{3}{8} < \mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega) = \frac{5(n+1)}{8\binom{7}{3}}\biggl(\frac{\binom{6}{2}}{n+1} + \frac{\binom{5}{2}}{n+2} + \frac{\binom{4}{2}}{n+3} + \frac{\binom{3}{2}}{n+4} + \frac{\binom{2}{2}}{n+5}\biggr),$$

and we do not stop. If we see a chain of length greater than three with a present relative record, the probability of success is at least $\frac{1}{2}$ and, of course, (7.2) holds, and we stop.

Example 7.4. In this example we consider a poset for which the MIA of the secretary-type search on this poset satisfies (7.2) for some ω and t, but, according to the optimal strategy, we do not stop at the moment t for this ω.

Let us consider the best-choice search on a finite poset whose Hasse diagram is the tree given in Figure 3, and let $X = (X_t)_{t \le n+m+r+1}$ be the MIA for this case.

Consider now a permutation ω such that

$\{\omega_1, \omega_2, \omega_3\}|_{\le} =$  .

In this case we have

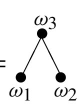
$$X_2(\omega) = \frac{\binom{n}{2} + \binom{m}{2} + mn + rn}{\binom{n}{2} + \binom{m}{2} + mn + rn + r\binom{m}{2}},$$

$$\mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega) = \Biggl(1 - \frac{\binom{n}{2} + \binom{m}{2} + mn + rn}{\binom{n}{2} + \binom{m}{2} + mn + rn + r\binom{m}{2}}\Biggr)\frac{n+1}{r}\sum_{k=1}^{r}\frac{1}{n+k}.$$

For $n = 1$, $r = 31$, and $m = 3$, we have $X_2(\omega) = \frac{37}{130} \approx 0.284\,615\,384\,6$, $\mathbb{E}(X_3 \mid \mathcal{F}_2)(\omega) = \frac{93}{130}\frac{2}{31}(H_{32} - 1) \approx 0.141\,161\,316\,7$, where H_n is the nth harmonic number ($H_{32} \approx 4.058\,495\,195$). Thus, (7.2) holds, but we do not stop at the moment $t = 3$ because even the simple strategy of waiting for the first element incomparable with the previous elements gives a probability of success of $\frac{1}{2}(1 - \frac{37}{130}) > \frac{37}{130}$.

Example 7.5. In this example we consider a double secretary-type search: the first search is on the poset considered in Example 7.4 and the second search is on a chain consisting of two elements, i.e.  . In this example (7.1) holds for some ω and t, but, according to the optimal strategy, we do not stop at the moment t.

Consider a permutation ω such that

$\{\omega_1, \omega_2, \omega_3\}|_{\le} =$  .

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In this case, for Z being a merge (defined at the beginning of Section 3) of the MIAs for the secretary searches on our two posets, we have

$$Z_{\rho_2}(\omega) = \frac{\binom{n}{2} + \binom{m}{2} + mn + rn}{\binom{n}{2} + \binom{m}{2} + mn + rn + r\binom{m}{2}},$$

$$\mathbb{E}(Z_{\rho_3} \mid \mathcal{F}_{\rho_2})(\omega) = \left(1 - \frac{\binom{n}{2} + \binom{m}{2} + mn + rn}{\binom{n}{2} + \binom{m}{2} + mn + rn + r\binom{m}{2}}\right) \frac{n+1}{r} \sum_{k=1}^{r} \frac{1}{n+k} + \frac{1}{2} \frac{\binom{n}{2} + \binom{m}{2} + mn + rn}{\binom{n}{2} + \binom{m}{2} + mn + rn + r\binom{m}{2}}.$$

For $n = 1, r = 31$, and $m = 3$, we have $Z_{\rho_2}(\omega) = \frac{37}{130} \approx 0.284\,615\,384\,6$ and $\mathbb{E}(Z_{\rho_3} \mid \mathcal{F}_{\rho_2})(\omega) = \frac{93}{130}\frac{2}{31}(H_{32} - 1) + \frac{1}{2}\frac{37}{130} \approx 0.283\,469\,009$. Thus, inequality (7.1) holds, but we do not stop at the moment $t = 3$ because $Z_{\rho_2}(\omega) < 0.5$ and the probability of success in the second search equals 0.5.

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References

[1] Chow, Y. S., Robbins, H. and Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping*. Houghton Mifflin, Boston, MA.

[2] Freij, R. and Wästlund, J. (2010). Partially ordered secretaries. *Electron. Commun. Prob.* **15,** 504–507.

[3] Garrod, B. and Morris, R. (2013). The secretary problem on an unknown poset. *Random Structures Algorithms* **43,** 429–451.

[4] Garrod, B., Kubicki, G. and Morayne, M. (2012). How to choose the best twins. *SIAM J. Discrete Math.* **26,** 384–398.

[5] Georgiou, N., Kuchta, M., Morayne, M. and Niemiec, J. (2008). On a universal best choice algorithm for partially ordered sets. *Random Structures Algorithms* **32,** 263–273.

[6] Gilbert, J. P. and Mosteller, F. (1966). Recognizing the maximum of a sequence. *J. Amer. Statist. Assoc.* **61,** 35–73.

[7] Kaźmierczak, W. (2013). The best choice problem for a union of two linear orders with common maximum. *Discrete Appl. Math.* **161,** 3090–3096.

[8] Kozik, J. (2010). Dynamic threshold strategy for universal best choice problem. In *21st International Meeting on Probabilistic, Combinatorial, and Asymptotic Methods in the Analysis of Algorithms*, Association of Discrete Mathematics and Theoretical Computer Science, Nancy, pp. 439-451.

[9] Kuchta, M. (2014). Iterated full information secretary problem. Submitted.

[10] Kuchta, M. and Morayne, M. (2014). A secretary problem with many lives. *Commun. Statist. Theory Meth.* **43,** 210–218.

[11] Kumar, R., Lattanzi, S., Vassilvitskii, S. and Vattani, A. (2011). Hiring a secretary from a poset. In *Proc. ACM Conf. Electron. Commerce*, ACM, New York, pp. 39–48.

[12] Kurpisz, A. and Morayne, M. (2014). Inform friends, do not inform enemies. *IMA J. Math. Control Inf.* **31,** 435–440.

[13] Morayne, M. (1998). Partial-order analogue of the secretary problem: the binary tree case. *Discrete Math.* **184,** 165–181.

[14] Preater, J. (1999). The best-choice problem for partially ordered objects. *Operat. Res. Lett.* **25,** 187–190.

[15] Samuels, S. M. (1991). Secretary problems. In *Handbook of Sequential Analysis*, Dekker, New York, pp. 381–405.