J. Austral. Math. Soc. (Series A) 31 (1981), 294–311

IMPERFECT BIFURCATION AND BANACH SPACE SINGULARITY THEORY

LEIF ARKERYD

(Received 1 October 1979; revised 20 October 1980)

Communicated by E. Strzelecki

Abstract

This paper generalizes the theory of imperfect bifurcation via singularity theory as developed by M. Golubitsky and D. Schaeffer to a Banach space setting. Like the parameter-free potential catastrophe theory, where similar generalizations have been discussed in the literature, Banach control spaces allow useful uniform control of function parameters through the universal unfolding. Among the results are tests for various germ properties and a discussion of their reducibility under a Liapunov–Schmidt type splitting, as well as a generalization of the finite dimensional unfolding and germ classification theory.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 58 C 25; secondary 58 F 10.
Key words and phrases: Bifurcation theory, imperfections, singularity theory, elementary catastrophes, versal unfolding, stable unfolding, reducible properties.

0. Introduction

Many applications of catastrophe theory are naturally set in an infinite dimensional Fredholm context. In the parameter-free potential case such a generalization has been considered in [1]–[3], [6], [7]. A similar setting seems to be of interest in the case of imperfect bifurcation via singularity theory as developed in [5] by Golubitsky and Schaeffer.

In an example, the paper [5] analyzes the Brusselator modelled by

$$D_t x = \mu_1 D_\xi^2 X + X^2 Y - (B+1)X + A_0,$$
$$D_t Y = \mu_2 D_\xi^2 Y - X^2 Y + BX$$

on $(0, \pi)$, with Dirichlet boundary data

$$X(0) = X(\pi) = A_0, \qquad Y(0) = Y(\pi) = B/A_0,$$

near a double eigenvalue. Instead of modelling the concentration of the chemical A as a constant A_0, it might be interesting to use the perturbation consisting of the solution $\bar{A}$ of the equation

$$D_\xi^2 \bar{A} - \varepsilon \bar{A} = 0, \qquad \xi \in (0, \pi), \text{ with } \bar{A}(0) = \bar{A}(\pi) = A_0.$$

(This perturbation is used by Golubitsky and Schaeffer in another paper [4], which is dealing with the extra complication of symmetry restrictions on the Brusselator.) For such perturbations (with, as well as without, symmetry restrictions) it would be interesting to obtain a *uniform* control through the versal unfolding of all concentration functions $A(\xi)$, which in some sense are near $\bar{A}(\xi)$. This requires a setting with an infinite dimensional control space B_y. The present paper contains such an infinite dimensional (Banach space) version of the catastrophe analysis of [5], that is for the case of no symmetry restrictions. The main point of the present paper is the proof of the uniformity of the factoring of infinite dimensional perturbations through the universal unfolding.

The paper [5] attacks the versality question directly using induction on the dimension of B_y. Instead we follow the path from [1] with the equivalence of transversal unfoldings as the central theorem. A special feature with the unfolding techniques in a parametrized version is that only certain sub-spaces of the germ tangent spaces are modules over $C^\infty(B_0, R)_0$, thereby allowing a germ to jet reduction by Nakayama's Lemma.

Our strategy is to reduce the analysis as much as possible to the finite-dimensional situation, where the proofs often can be patterned on the parameter-free potential case along the lines of for example the exposition [8]. For the reduction we use a Liapunov-Schmidt type splitting. A suitable version is presented in Section 1. That section also contains some continuity properties of the splitting, as well as most definitions and notations needed in the rest of the paper.

Section 2 is concerned with finitely determined germs. The results can be used to test for various determinacy and codimension properties. The section also contains a discussion of the reducibility of some germ properties under the splitting introduced in Section 1. Finally, unfoldings and germ classification are the topics of Section 3, and it is proved that the finite-dimensional theory generalizes, just as one would expect.

1. Preliminaries

This section contains notations and definitions, as well as some fairly standard results about splitting of functions and germs to be used in the present paper.

Let

$$B = B_x \oplus B_\lambda \oplus B_y$$

be the direct topological sum of three real Banach spaces B_x, B_λ ($\dim B_\lambda < \infty$), and B_y, with norm $|||\cdot|||$. Denote by

$$L^j(B, B')$$

the set of continuous, symmetric, j-linear mappings

$$A: B \to B',$$

where B' is a Banach space with norm $|\cdot|$. Write L instead of L^1 as usual. L^j becomes a complete, normed, linear space under the norm

$$\|A\|_j = \sup|Ax^{(j)}|,$$

where the supremum is taken over all sequences

$$x^{(j)} = (x_1, \ldots, x_j) \quad \text{with } |||x_1||| = \cdots = |||x_j||| = 1.$$

The cartesian product

$$B \times B' \times L^1(B, B') \times \cdots \times L^k(B, B')$$

is the Banach space of k-jets $J^k(B, B')$ with norm $\|\cdot\|_k$, and $J^k_p(B, B')$ is the corresponding Banach space of k-jets at $p \in B$.

The R-linear space of germs at $0 \in B$ of C^∞-functions from B to B' defined on some neighborhood of 0 in B (that is functions in $C^\infty_{\mathrm{loc},0}(B, B')$) is denoted by $E(B, B')$ or $C^\infty(B, B')_0$, or sometimes simply by E when $B' = R$. Whenever suitable, elements of the two spaces will be confused without further comment. The mapping from B to $J^k(B, B')$

$$p \to k\text{-jet of } F \text{ at } p \qquad (F \in C^\infty_{\mathrm{loc}}(B, B'))$$

is denoted by j^kF.

A topology on $C^\infty(U, B')$ (with $U \subset B$ open), relevant for the present local studies, W^∞_w, can be defined using as a basis all sets

$$V(k, \varepsilon, K, F) = \left\{F' \in C^\infty(U, B');\ \sup_{p \in K} \|j^k(F - F')(p)\| < \varepsilon\right\},$$

where $K \subset U$ is closed, $k \geqslant 0$ is an integer, $\varepsilon > 0$, and $F \in C^\infty(U, B')$.

The maximal ideal of E is

$$m = \{f \in E; f(0) = 0\}.$$

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

With

$$M^kE(B, B') = \{F \in E(B, B'); j^{k-1}F(0) = 0\},$$

evidently

$$m^kE(B, B') \subseteq M^kE(B, B').$$

Set $B_0 = B_x \oplus \mathrm{i}B_\lambda$. For $G \in E(B_0, B')$ we introduce

$$\begin{aligned}
\langle G \rangle = \langle G \rangle_{B'} &= C^\infty(B_0, L(B', B'))_0 G,\\
\Delta = \Delta(g) &= D_x G C^\infty(B_0, B_x)_0,\\
\Delta_\lambda = \Delta_\lambda(G) &= D_\lambda G C^\infty(B_\lambda, B_\lambda)_0,\\
\bar{T}G &= \langle G \rangle_{B'} + \Delta(G),\\
TG &= \bar{T}G + \Delta_\lambda,
\end{aligned}$$

together with the following two codimensions

$$\begin{aligned}
\overline{\mathrm{cod}}\ G &= \dim_R E(B_0, B')/\bar{T}G,\\
\mathrm{cod}\ G &= \dim_R E(B_0, B')/TG.
\end{aligned}$$

A germ $F \in E(B, B')$ (sometimes written (F, B_y)) is called an unfolding of G, if

$$F|_{B_0 \oplus \{0\}} = G.$$

Let Γ_{loc} denote the pseudogroup of local diffeomorphisms

$$(B, 0) \to (B, 0),$$

and Γ the group of germs at $0 \in B$ of elements of Γ_{loc}. Define Γ_0 (Γ_x) analogously with respect to B_0 (B_x). Finally $\Gamma(B', B')$ stands for the the group of invertible elements in $L(B', B')$.

We get a category of unfoldings for $G \in E(B_0, B')$, with morphisms

$$(\tau, \phi)\colon (F', B_{y'}) \to (F, B_y)$$

between unfoldings (F, B_y) and $(F', B_{y'})$ of G. Here

$$\tau \in C^\infty(B_0 \oplus B_{y'}, \Gamma(B', B'))_0, \qquad \phi \in C^\infty(B_0 \oplus B_{y'}, B_0 \oplus B_y)_0,$$
$$\phi\colon (x, \lambda, y') \to (\varphi(x, \lambda, y'), \Lambda(\lambda, y'), \psi(y')),$$

with τ, (φ, Λ) reducing to the appropriate identities for $y' = 0$, and F' factors through F, that is

$$F' = \tau \cdot F \circ \phi.$$

An unfolding F is said to be versal, if every unfolding of G factors through F.

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

The relevant equivalences in the present paper are

a) for germs $G, G' \in E(B_0, B')$; $G \sim G'$ if there are

$$\tau \in C^\infty(B_0, \Gamma(B', B'))_0, \quad \text{and} \quad \phi \in \Gamma_0,$$
$$\phi\colon (x, \lambda) \to (\varphi(x, \lambda), \Lambda(\lambda)),$$

such that

$$G' = \tau \cdot G \circ \phi;$$

b) for unfoldings (F, B_y), $(F', B_y) \in E(B, B')$ of $G \in E(B_0, B')$; $F \simeq F'$ if there is a morphism

$$(\tau, \phi)\colon (F, B_y) \to (F', B_y)$$

with $\phi \in \Gamma$.

The germ $G \in E(B_0, B')$ is k-determined, if

$$j^k G(0) = j^k G'(0)$$

implies $G \sim G'$. The least k, if any, such that G is k-determined, is called the determinacy of G, denoted $\det G$. A germ is finitely determined, if it is k-determined for some $k < \infty$.

We shall say that the unfolding F of G satisfies Condition T, if

$$E(B_0, B') = TG + D_y F|_{y=0} B_y.$$

Lemma 1.1. *Suppose that*

$$F \in E(B_0 \oplus B_y, B'), \qquad F' \in E(B_0 \oplus B_{y'}, B'), \qquad F' = \tau \cdot F \circ \phi,$$

where

$$\tau \in C^\infty(B_0 \oplus B_{y'}, \Gamma(B', B'))_0,$$
$$\phi\colon (x, \lambda, y') \to (\varphi(x, \lambda, y'), \Lambda(\lambda, y'), \psi(y')),$$
$$\phi \in C^\infty(B_x \oplus B_\lambda \oplus B_{y'}, B_x \oplus B_\lambda \oplus B_y)_0,$$

with

$$\phi_0 = (\varphi, \lambda)|_{y'=0} \in \Gamma_0, \qquad \psi(0) = 0.$$

Set

$$\tau_0 = \tau|_{y'=0}, \qquad G = F|_{y=0}, \qquad G' = F'|_{y'=0}.$$

Then

$$\left(\tau_0^{-1}\bar{T}G'\right) \circ \phi_0^{-1} = \bar{T}G, \qquad \left(\tau_0^{-1}TG'\right) \circ \phi_0^{-1} = TG,$$
$$\left[\tau_0^{-1}\left(TG' + D_{y'}F'|_{y'=0}B_{y'}\right)\right] \circ \phi_0^{-1} \subseteq TG + D_y F|_{y=0} B_y.$$

In particular F satisfies Condition T, if F' does.

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

The proof is an easy check using the previous definitions.

As in the corresponding potential case ([1], [2]), we reduce most of the proofs in the present paper B_0-wise to a finite-dimensional context using a splitting lemma. Let

$$F \in C^{\infty}_{\mathrm{loc},0}(B, B')$$

be given, such that under the splitting

$$B_x = B_1 \oplus B_2, \qquad B' = B'_1 \oplus B'_2$$

with continuous projections $E_j B' = B'_j$,

$$E_1 D_{x_1} F|_0 \in L(B_1, B'_1)$$

is bijective. Under this hypothesis and with

$$B_3 = B_2 \oplus B_\lambda \oplus B_y,$$

the following lemma holds.

LEMMA 1.2. *If $E_1F(0) = 0$, then there are $\phi \in \Gamma_{\mathrm{loc}}$ with*

$$\phi\colon (x_1, x_3) \to (\varphi(x_1, x_3), x_3),$$

and

$$\tau \in C^{\infty}_{\mathrm{loc},0}(B, \Gamma(B', B')),$$

such that

$$\tau \cdot F \circ \phi = E_1 A x_1 + \tilde{F}(x_3).$$

Here $A = D_x G|_0$, and $\tilde{F}(x_3) = F(h(x_3), x_3)$, with $h(x_3) = \varphi(0, x_3)$. Moreover

$$h(0) = 0,\ E_1\tilde{F} = 0. \tag{1.1}$$

REMARK. a) The map h is uniquely determined by (1.1) in some neighbourhood of zero.

b) If $B'_1 = \operatorname{range} A$, then $\tau = \mathrm{id}_{B'} + \mathcal{O}(x, \lambda, y)$.

c) If $B_2 = \ker A$, then $D_2 h(0) = 0$, and $D_2\tilde{F}(0) = 0$.

PROOF. By the implicit function theorem there is a unique $\phi_1 \in \Gamma_{\mathrm{loc}}$

$$\phi_1\colon (x_1, x_3) \to (x_1 + w(x_3), x_3), \qquad w(0) = 0,$$

such that

$$E_1 F \circ \phi_1|_{x_1 = 0} = 0.$$

Define $\phi_2 \in \Gamma_{\mathrm{loc}}$ by

$$\phi_2^{-1}\colon (x_1, x_3) \to \big((E_1A)^{-1} E_1 F \circ \phi_1(x_1, x_3), x_3\big) = (\bar{x}_1, x_3).$$

Then

$$f \circ \phi_1 \circ \phi_2(\bar{x}_1, x_3) = \big(E_1 A \bar{x}_1, E_2 F \circ \phi_1 \circ \phi_2(\bar{x}_1, x_3)\big).$$

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

Now

$$E_2F \circ \phi_1 \circ \phi_2(\bar{x}_1, x_3) = E_2F \circ \phi_1 \circ \phi_2(0, x_3) + H(\bar{x}_1, x_3)E_1A\bar{x}_1.$$

And so

$$\tau \cdot F \circ \phi_1 \circ \phi_2(\bar{x}_1, x_3) = E_1A\bar{x}_1 + E_2F \circ \phi_1 \circ \phi_2(0, x_3)$$

with

$$\tau = \begin{pmatrix} \mathrm{id}_{B'_1} & 0 \\ -H & \mathrm{id}_{B'_2} \end{pmatrix}.$$

From here the remaining assertions of the lemma and following remarks are easily checked.

The same proof also yields the following refinement, used below to compare two unfoldings (F_1, B_y) and (F_2, B_y) of the same $G \in C^{\infty}_{\mathrm{loc},0}(B_0, B')$ satisfying the hypotheses of the previous lemma.

LEMMA 1.3. *There exist ϕ_{F_j}, τ_{F_j} $(j = 1, 2)$ as in the previous lemma, such that*

$$\tau_{F_j} \cdot F \circ \phi_{F_j} = E_1Ax_1 + \tilde{F}_j(x_3). \tag{1.2}$$

Moreover

$$\begin{gathered} \phi_{F_1}|_{y=0} = \phi_{F_2}|_{y=0}, \qquad \tau_{F_1}|_{y=0} = \tau_{F_2}|_{y=0}. \\ \tilde{F}_1|_{y=0} = \tilde{F}_2|_{y=0} = G(h_F(x_2, \lambda, 0), x_2, \lambda) = \tilde{G}. \end{gathered} \tag{1.3}$$

Enough continuity properties of the above splitting for our purposes, can be derived by easy but tedious checking of the relevant isomorphisms using nothing more exciting than the inverse function theorem. Let F be given as in Lemma 1.2 together with corresponding B_j, B'_j, τ_F, ϕ_F, and h_F, and denote by $U_j = U_j(r)$ a spherical neighbourhood of $0 \in B_j$ of radius r.

LEMMA 1.4. *There are r_j^0, such that for $r_j^1 < r_j^0$ $(j = 1, 3)$*

$$\text{range } \phi \supset U^1 = U_1(r_1^1) \oplus U_3(r_3^1),$$

and such that the following holds;

a) *If $r_1^2 \leqslant r_1^1$, then there is $r_3^2 \leqslant r_3^1$ and $\varepsilon_2 > 0$, such that to*

$$F' \in V(1, \varepsilon_2, \bar{U}^1, F)$$

corresponds a unique

$$p_{F'} = p_{F',1} \oplus 0 \in U^2 = U_1^2(r_1^2) \oplus U_3^2(r_3^2)$$

with $E_1F'(p_{F'}) = 0$. For the corresponding $h_{F'}$ of Lemma 1.2 with $h_{F'}(0) = p_{F',1}$,

$$\text{domain } h_{F'} \supset \bar{U}_3^2,\ h_{F'}(\bar{U}_3^2) \subset U_1^2.$$

b) *For $k \geqslant 1$ and $\varepsilon > 0$, there are $\delta > 0$ and $r_3^2 < r_3^1$, such that*

$$\tilde{F}' \in V\big(k, \varepsilon, \overline{U}_3(r_3^2), \tilde{F}\big),$$

if $F' \in V(k, \delta, \overline{U}^1, F)$. Here $\tilde{F}' = F'(h_{F'}(x_3), x_3)$.

2. Finitely determined germs

In this section we consider tests for various germ properties as well as their reducibility under the splitting defined by Lemma 1.2. Properties are reducible provided they are satisfied for a germ F if and only if they hold for $\tilde{F}$.

LEMMA 2.1. *Let $G^1, G^2 \in E(B_0, B')$ be given as in Lemma 1.2. If*

$$j^k G^1(0) = j^k G^2(0)$$

for some $k \geqslant 1$, then for the corresponding parametrization

$$\tau^j G^j \phi^j = E_1 A x_1 + \tilde{G}^j(x_2, \lambda) \tag{2.1}$$

it follows that

$$j^k \tilde{G}^1(0) = j^k \tilde{G}^2(0). \tag{2.2}$$

PROOF. For convenience we use the parametrization (2.1) with $j = 1$ for the proof. By hypothesis

$$G^2 - G^1 = Q \in M^{k+1} E(B_0, B'). \tag{2.3}$$

Evidently (2.2) follows from

$$\begin{aligned}\tilde{G}^2(\cdot) &= E_2 G_2(h^2(\cdot), \cdot) = E_2 G^1(h^2(\cdot), \cdot) + E_2 Q(h^2(\cdot), \cdot)\\ &= \tilde{G}_1(\cdot) + E_2 Q(h^2(\cdot), \cdot),\end{aligned}$$

since

$$E_2 Q(h^2(\cdot), \cdot) \in M^{k+1} E(B_2 \oplus B_\lambda, B')$$

by (2.3) and (1.1).

We shall only discuss germs $G \in ME(B_0, B')$ and their unfoldings, since $E(B_0, B') = \langle G \rangle$ if $G(0) \neq 0$. Now germs $G \in ME(B_0, B')$ with a Fredholm x-derivative at the origin, and their unfoldings F, satisfy Lemma 1.2 with $B_2 = \ker D_x G(0)$. Our first theorem considers reducibility with respect to such a splitting.

THEOREM 2.2. *The following properties are reducible*;

a) $\underline{\mathrm{cod}}\ G = c$,

b) $\overline{\mathrm{cod}}\ G = c$,

c) $\det G = k$,

d) *F satisfies Condition T*,

e) *F is a versal unfolding of G.*

PROOF. To get an idea of the easy proof of a), b), compare the analogous result in the parameter-free potential case in [1].

As for c) it follows from Lemma 2.1 that k-determinacy of $\tilde{G}$ implies k-determinacy of G. To prove the converse, given

$$\tilde{G}' \in E(B_2 \oplus B_\lambda, B_2')$$

with

$$j^k\tilde{G}'(0) = j^k\tilde{G}(0),$$

we shall construct a $\sim$-equivalence between $\tilde{G}$ and $\tilde{G}'$. We work in the parametrization (2.1) of G, and set

$$G' = E_1Ax_1 + \tilde{G}'(x_2, \lambda).$$

By hypothesis there are germs τ and ϕ giving the $\sim$-equivalence

$$\tau^{-1}G' = G \circ \phi,$$

and so by Remark c of Lemma 1.2, on B_1

$$\tau^{-1}(0)E_1A = D_1\tau^{-1}G'(0) = E_1AD_1\varphi_1(0). \tag{2.4}$$

Using $\dim B_2' < \infty$ we conclude that $\tau(0)E_1 \in \Gamma(B_1', B_1')$. It follows that $E_2\tau(0)E_2 \in \Gamma(B_2', B_2')$. Also by (2.4) $D_1\varphi_1(0)$ is bijective, and so

$$(x_1, x_2, \lambda) \to (\varphi_1, x_2, \lambda), \qquad (\varphi_1, x_2, \lambda) \to (\varphi_1, \varphi_2, \Lambda),$$

as well as

$$(x_2, \lambda) \to (\varphi_2(0, x_2, \lambda), \Lambda)$$

define $\sim$-equivalences. It follows that

$$\tau(x)\tilde{G}(\phi)|_{\varphi_1=0} = \tau(x)G(\phi)|_{\varphi_1=0} = E_1Ax_1 + \tilde{G}'(x_2(\varphi_2, \Lambda), \lambda(\Lambda))$$

is well-defined in terms of the (φ_2, Λ)-variables. Together with an E_2-projection this defines the desired $\sim$-equivalence between $\tilde{G}$ and $\tilde{G}'$.

To prove d) we notice that by Lemma 1.1, Condition T is invariant for transformations as in (1.2). As is easily checked, that parametrization of F satisfies Condition T if and only if $\tilde{F}$ does. Finally e) is proved in Section 3 below.

From here it is easy to deduce a test for k-determinacy in terms of $M^kE(B_0, B')$. As in the previous theorem we only consider $G \in ME(B_0, B')$

with $D_x G(0)$ Fredholm. Take $B_2 = \ker D_x G(0)$ and define

$$M\bar{T}G = -\{\bar{H} = H_1 G + D_x G H_2 \in \bar{T}G;\, H_1(0) = 0,\, H_2(0) = 0\},$$

$$MTG = \{H = \bar{H} + D_\lambda G H_3 \in TG;\, \bar{H} \in M\bar{T}G,\, H_3(0) = 0\}.$$

THEOREM 2.3. a) *If*

$$M^k E(B_0, B') \subseteq M\bar{T}G, \tag{2.5}$$

then G is k-determined.

b) *If G is k-determined, then*

$$M^{k+1}E(B_0, B') \subseteq MTG + M^j E(B_0, B') \tag{2.6}$$

for any j.

REMARK. It seems likely that (2.6) should hold without $M^j E$.

PROOF. a) Write $G = G^1$, and let G^2 be any element in $E(B_0, B')$ with

$$j^k G^2(0) = j^k G^1(0).$$

By Lemma 2.1 it follows that (2.1) and (2.2) hold, and by Lemma 1.1, (2.1), and (2.5) that

$$M^k E(B_2 \oplus B_\lambda, B_2') \subseteq M\bar{T}\tilde{G}'.$$

But this is the condition for the well-known finite-dimensional version of a), so $\tilde{G}^2$ is $\sim$ -equivalent to $\tilde{G}'$ (see [5]). This together with (2.1) proves part a).

b) Just like the parameter-free potential case (see [1]), we use the parametrization (2.1) and Theorem 2.2.c to conclude that it is enough to prove (2.6) in the finite-dimensional case. There the well-known arguments from the corresponding parameter-free potential situation lead to (2.6). Notice, however, that the final touch of Nakayama's lemma to remove $M^j E(B_0, B')$ cannot be applied, due to the Δ_λ-term in TG.

Codimension and determinacy are related as in the following theorem.

THEOREM 2.4. *If $G \in ME(B_0, B')$ has $D_x G(0)$ Fredholm, then*

$$\det G \leqslant 1 + \overline{\operatorname{cod}}\; G. \tag{2.7}$$

PROOF. Using Theorem 2.3.a, the estimate (2.7) follows in the finite-dimensional case by the usual arguments from the corresponding parameter-free potential situation. Together with the reducibility properties of Theorem 2.2b,c, this implies the general case.

Also $\overline{\operatorname{cod}}\; G < \infty$ can be expressed in terms of $M^k E(B_0, B')$.

THEOREM 2.5. *If $G \in ME(B_0, B')$ has $D_x G(0)$ Fredholm, then the following are equivalent;*

a) $\overline{\text{cod}}\, G < \infty$,

b) $M^k E(B_0, B') \subseteq \overline{T}G$ *for some* k.

PROOF. A finite-dimensional proof can be patterned on the corresponding parameter-free potential situation. From here the general case follows, since a) is reducible by Theorem 2.2.b, and b) is also reducible, as is easily proved using Lemma 1.1.

THEOREM 2.6. *If $G \in ME(B_0, B')$ has $D_x G(0)$ Fredholm and $\overline{\text{cod}}\, G = k < \infty$, then there is a neighbourhood N of G in $ME(B_0, B')$ with the $(k+1)$-jet topology, such that $\overline{\text{cod}}\, G' \leqslant k$ if $G' \in N$.*

PROOF. In the finite-dimensional case, a proof can be patterned on the corresponding parameter-free potential situation. From here the general case follows by Theorem 2.2.b together with a suitable germ version of Lemma 1.4.

3. Unfoldings and classification

THEOREM 3.1. *Let $G \in ME(B_0, B')$ be given with $\overline{\text{cod}}\, G < \infty$, and $D_x G(0)$ Fredholm. If the unfoldings (F_1, B_y) and (F_2, B_y) of G both satisfy Condition T, then they are equivalent.*

LEMMA 3.2. *The full Theorem* 3.1 *follows from the special case of*

$$\dim B_0 + \dim B' < \infty, D_x G(0) = 0. \tag{3.1}$$

PROOF. Introduce a splitting as in Lemma 1.3 with $B_2 = \ker D_x G(0)$ and $B_1' = \text{range } D_x G(0)$. By the Fredholm property

$$\dim B_2 + \dim B_2' < \infty,$$

and by Theorem 2.2.d the unfoldings $\tilde{F}_1$ and $\tilde{F}_2$ of $\tilde{G}$ satisfy Condition T. From here $\tilde{F}_1 \simeq \tilde{F}_2$, thus $F_1 \simeq F_2$, provided Theorem 3.1 holds under (3.1).

Let us make a few remarks before we continue the proof of the theorem under (3.1). The condition $\overline{\text{cod}}\, G < \infty$ implies cod $G < \infty$, hence the existence of

$$u_1, \ldots, u_c \in E(B_0, B')$$

projecting onto a c-component basis for E/TG. Evidently

$$H = G + \sum_1^c v_j u_j = G + \langle v, u \rangle \tag{3.2}$$

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

satisfies Condition T. Also $\dim B_y \geqslant c$ for any unfolding (F, B_y) satisfying Condition T.

LEMMA 3.3. *Let* $G \in ME(B_0, B')$ *be given with* $\overline{\text{cod}} < \infty$, *and* $D_x G(0)$ *Fredholm. If the unfolding* (F, G_y) *of* G *is versal, then it satisfies Condition* T.

PROOF. The unfolding H of (3.2) satisfies Condition T and factors through F,

$$H = \tau \cdot F \circ \phi,$$

since F is versal. And so by Lemma 1.1, F satisfies Condition T.

LEMMA 3.4. *The full Theorem* 3.1 *follows from the special case* (3.1) *and*

$$D_y F_1|_{y=0} w = D_y F_2|_{y=0} w \bmod TG \qquad w \in B_y,\ F_2 \simeq H, \tag{3.3}$$

where H *is given by* (3.2).

A proof of Lemma 3.4 can be closely modelled on the corresponding parameter-free potential case (see [1]). As in that paper the proof of Theorem 3.1 will be completed with the help of the flow-method. Set

$$K_t = (1-t)F_2 + tF_1, \qquad I = \{t \in [0, 1];\ K_t \simeq H\}.$$

LEMMA 3.5. *Given* $t_0 \in [0, 1]$, *there is a neighbourhood* Ω_{t_0} *of* t_0, *and germs at* $(0, t_0)$,

a) τ_{t_0} *of a map* $(B \oplus R, \{0\} \oplus \{t_0\}) \to (\Gamma(B', B'), \mathrm{id}_{B'})$,
b) φ_{t_0} *of a map* $(B \oplus R, \{0\} \oplus \Omega_{t_0}) \to (B_x, 0)$,
c) Λ_{t_0} *of a map* $(B_\lambda \oplus B_y \oplus R, \{0\} \oplus \Omega_{t_0}) \to (B_\lambda, 0)$,
d) ψ_{t_0} *of a map* $(B_y \oplus R, \{0\} \oplus \Omega_{t_0}) \to (B_y, 0)$, *such that*

$$\tau_{t_0}(x, \lambda, y, t_0) = \mathrm{id}_{B'}, \qquad (\varphi_{t_0}(\cdot, t_0)), \Lambda_{t_0}(\cdot, t_0), \psi_{t_0}(\cdot, t_0) = \mathrm{id}_B,$$

and

$$\tau_{t_0} K_t(\varphi_{t_0}, \Lambda_{t_0}, \psi_{t_0})|_t = K_{t_0} \qquad (t \in \Omega_{t_0}). \tag{3.4}$$

Moreover

$$\tau_{t_0}|_{t,y=0} H(\varphi_{t_0}, \Lambda_{t_0}, v)|_{t,y=0} \simeq H(x, \lambda, v). \tag{3.5}$$

REMARK. a) Evidently (3.4) does not quite define equivalences between unfoldings for $t \in \Omega_{t_0}$, since

$$\tau_{t_0}(\cdot, t), (\varphi_{t_0}, \Lambda_{t_0})|_{t,y=0}$$

are invertible without necessarily being the required identities.

b) Theorem 3.1 under (3.1) and (3.3) means that $1 \in I$. This follows from $0 \in I$, and I being both open and closed, which in turn is a consequence of (3.3)

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

and Lemma 3.5. The easy argument goes as follows. Set

$$\tilde{\tau}_{t_0}(x, \lambda, y, t) = \tau_{t_0}(x, \lambda, 0, t),$$
$$\chi_{t_0,t}(x, \lambda, y) = \big(\varphi_{t_0}(x, \lambda, 0, t), \Lambda_{t_0}(\lambda, 0, t), y\big).$$

By (3.4)

$$K_t \simeq \tilde{\tau}_{t_0}^{-1} K_{t_0} o \chi_{t_0,t}^{-1},$$

and so by (3.5)

$$K_t \simeq H \quad (t \in \Omega_{t_0}) \quad \text{if } K_{t_0} \simeq H,$$
$$K_{t_0} \simeq H \quad \text{if } K_t \simeq H \quad (t < t_0).$$

LEMMA 3.6. *Given $t_0 \in [0, 1]$, there is a neighbourhood of t_0 and germs at $(0, t_0)$,*
a) *$\tilde{T}$ of a map $B \oplus R \to \Gamma(B', B')$,*
b) *U of a map $B \oplus R \to B_x$,*
c) *L of a map $(B_\lambda \oplus B_y \oplus R, B_\lambda \oplus \{0\} \oplus \Omega_{t_0}) \to (B_\lambda, 0)$,*
d) *S of a map $(B_y \oplus R, \{0\} \oplus \Omega_{t_0}) \to (B_y, 0)$, such that*

$$F_2 - F_1 = \tilde{T}K + D_x KU + D_\lambda KL + D_y KS.$$

We proceed with a proof of Lemma 3.5 and get by standard calculus arguments, that (dropping the index t_0) (3.4) is equivalent to

$$-\tau D_t K = \dot{\tau} K + \tau\big[D_x K\dot{\varphi} + D_\lambda K\dot{\Lambda} + D_y K\dot{\psi}\big], \tag{3.6}$$

and from (3.6) that Lemma 3.5 is a consequence of Lemma 3.6. We sketch a proof for completeness.

To obtain (3.4) as well as a), c), d), solve in the given order

$$\dot{\psi} = S(\psi, t), \quad \psi(t_0) = \mathrm{id}_{B_y},$$
$$\dot{\Lambda} = L(\Lambda, \psi, t), \quad \Lambda(\lambda, y, t_0) = \lambda,$$
$$\dot{\varphi} = U(\varphi, \Lambda, \psi, t), \quad \varphi(x, \lambda, y, t_0) = x,$$
$$\dot{\tau} = \tau\tilde{T}(\varphi, \Lambda, \psi, t), \quad \tau(x, \lambda, y, t_0) = \mathrm{id}_{B'}.$$

For b) we notice that by (3.4) in particular

$$G(x, \lambda) = \tau(x, \lambda, 0, t)G(\varphi, \Lambda)|_{y=0}. \tag{3.7}$$

The assumption $\overline{\mathrm{cod}}\, G < \infty$ implies by Theorem 2.5 that $M^k E(B_0, B') \subset \overline{T}G$ for some k. So the origin is an isolated zero for $(G, D_x G)$, and by (3.7) also for $(G(\varphi, \Lambda), D_\varphi G(\varphi, \Lambda))$. Hence $\varphi(0, t) = 0$, and b) holds.

It remains to prove (3.5). Now

$$\tau u_1(\varphi, \Lambda)|_{y=0}, \ldots, \tau u_c(\varphi, \Lambda)|_{y=0}, \tag{3.8}$$

project onto a basis for E/TG when $t = t_0$, and by continuity also for t in some neighbourhood Ω_{t_0} of t_0, since for some k

$$M^k E \subset \bar{T}G \subset TG.$$

It follows that $\tau H(\varphi, \Lambda, v)$ satisfies Condition T in Ω_{t_0}, and so (3.5) follows by the known finite-dimensional version (see [5]) of Theorem 3.1.

Now a proof of Lemma 3.6 will complete the proof of Theorem 3.1. Contrary to the rest of the paper, throughout that proof germs means germs at $0 \oplus t_0$ (and not at $0 \oplus 0$ as previously). By (3.3) we cand find

$$v_1, \ldots, v_{c'} \in B_y, \qquad \bar{v}_{c'+1}, \ldots, \bar{v}_c \in E(B_\lambda \oplus B_y \oplus R, B_\lambda),$$

with the property that

$$u_j = D_y K|_{y=0, t=t_0} v_j \qquad (j = 1, \ldots, c')$$

project onto a basis for $E(B_0, B')/TG$, and

$$u_j \quad (j = 1, \ldots, c'), \qquad D_\lambda G \bar{v}_j \quad (j = c' + 1, \ldots, c)$$

project onto a basis for $E(B_0, B')/\bar{T}G$. Set

$$\bar{T}K_{t_0} = E(B \oplus R, L(B', B'))K_t + D_x K_t E(B \oplus R, B_x),$$

and introduce the $E(B_y \oplus R, R)$-module V on c variables, with elements

$$Y = (Y_1, \ldots, Y_c) \qquad (Y_1, \ldots, Y_c \in E(B_y \oplus R, R)),$$

as well as the mapping $f\colon V \to E(B \oplus R, B')$ given by

$$fY = \sum_1^{c'} Y_j D_y K v_j + \sum_{c'+1}^{c} Y_j D_\lambda K \bar{v}_j.$$

LEMMA 3.7. $E(B \oplus R, B') = fV + \bar{T}K_{t_0}$.

This follows from the division lemma similarly to the parameter-free potential case (see [1]).

PROOF OF LEMMA 3.6. By Lemma 3.7

$$-D_t K = F_2 - F_1 \in fV + \bar{T}K_{t_0},$$

that is there are germs

$$\tilde{T} \in E(B \oplus R, L(B', B')),$$
$$U \in E(B \oplus R, B_x),$$
$$Y_1, \ldots, Y_c \in E(B_y \oplus R, R),$$

such that

$$F_2 - F_1 = \tilde{T}K_t + D_x K_t U + \sum_1^{c'} Y_j D_y K_t v_j + \sum_{c'+1}^{c} Y_j D_\lambda K_t \bar{v}_j.$$

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

In particular

$$0 = F_2(x, \lambda, 0) - F_1(x, \lambda, 0) = \tilde{T}G + D_x GU + \sum_1^{c'} Y_j D_y K_t|_{y=0} v_j \tag{3.9}$$
$$+ \sum_{c'+1}^{c} Y_j D_\lambda G\bar{v}_j|_{y=0}.$$

Arguing as in (3.8) we obtain that

$$D_y K_t|_{y=0} v_1, \ldots, D_y K_t|_{y=0} v_{c'}, D_\lambda G\bar{v}_{c'+1}, \ldots, D_\lambda G\bar{v}_c$$

project onto a basis for $E(B_0, B')/\overline{TG}$ in a neighbourhood Ω_{t_0} of t_0. This implies in (3.9) that

$$Y_1 = \cdots = Y_c = 0 \qquad (y = 0, t \in \Omega_{t_0}),$$

which completes the proof of Lemma 3.6.

We finally discuss the usual unfolding consequences of the above results for a germ $G \in ME(B_0, B')$ with $\overline{\operatorname{cod}}\, G < \infty$, and $D_x G(0)$ Fredholm.

Proposition 3.8. *Two versal unfoldings (F_1, B_y) and (F_2, B_y) of G are equivalent.*

Proof. This is immediate by Lemma 3.3 and Theorem 3.1.

Proposition 3.9. *The unfolding (F, B_y) of G is versal, if and only if F satisfies Condition T.*

Proof. Versality implies Condition T by Lemma 3.3. For the converse we suppose $(F', B_{y'})$ is another unfolding of G. Then by (3.2)

$$F'' = F' + \sum_1^c v_j u_j$$

satisfies Condition T. By Theorem 3.1 F'' and F are equivalent considered as unfoldings over $B_y \oplus B_{y'} \oplus R^c$. Hence there is an equivalence (τ, ϕ)

$$F' + \sum_1^c v_j u_j = \tau \cdot F \circ \phi$$

holding in particular for $y = 0$, $v = 0$, that is F is versal.

Introduce a splitting parametrization

$$G = E_1 A x_1 + \tilde{G}(x_2, \lambda)$$

with $B_2 = \ker A$, $B_1' = \text{range}\, A$. Then $D_2 G(0) = 0$ by Remark c) of Lemma 1.2. Also

$$\dim B_2 + \dim B_2' < \infty,$$

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

since $D_xG(0)$ is Fredholm. Set

$$\dim B_2 = n, \quad \dim B_2' = n', \quad \operatorname{cod} G = c, \quad K_F = \{(x, \lambda, y);\ F = 0\}.$$

K_F is the catastrophe set of the unfolding (F, B_y) of G.

PROPOSITION 3.10. *There is an unfolding (F, R^c), such that K_F is diffeomorphic to a neighbourhood of zero in $R^n \times R^{c-n'}$. In particular if the Fredholm index of $D_xG(0)$ equals zero, then K_F is diffeomorphic to $B_y = R^c$.*

PROOF. It is evidently enough to unfold $\tilde{G}$. Since $G(0) = 0$, $D_2\tilde{G}(0) = 0$, we can define H of (3.2) taking u_j ($j \leqslant n'$) as a basis for B_2'. We notice that $B_y = R^c$. Under this choice of the u_j', K_H is given as the graph

$$v_j = -P_j\left(\tilde{G} + \sum_{n'+1}^{c} v_j u_j\right),$$

where P_j denotes the u_j-coordinate in the $u_1, \ldots, u_{n'}$-basis. The graph K_H is diffeomorphic to its source $B_2 \times R^{c-n'}$, and thus to $R^n \times R^{c-n'}$.

This proposition implies that K_F is a manifold if (F, B_y) is versal. The corresponding catastrophe map χ_F is defined by the composition of the embedding $K_F \subseteq B$ and the natural projection $B \to B_y$. In particular if $D_xG(0)$ is Fredholm of index zero then χ_F can be parametrized as a map $R^c \to R^c$.

PROPOSITION 3.11. *The equivalence class of χ_F depends only on the equivalence class of $\tilde{G}$.*

This can be proved as in the parameter-free potential case (see [8]).

Finally a few words about stability. For this we define F_1 and F_2 in $E(B, B')$ to be $\cong$ equivalent, if

$$F_2 = \tau \cdot F_1 \circ \phi$$

for some

$$\tau \in C^\infty(B, \Gamma(B', B'))_0, \qquad \varphi \in \Gamma,$$
$$\phi\colon (x, \lambda, y) \to (\varphi(x, \lambda, y), \Lambda(\lambda, y), \psi(y)).$$

Functions are in the present stability context considered on open neighbourhoods U of the origin in B. Suppose F' is a representative on U of the germ $F \in E(B, B')$. Set

$$G = F|_{y=0}, \qquad G' = F'|_{y=0}.$$

As always in this paper we assume $G \in ME(B_0, B')$, $\overline{\operatorname{cod}}\, G < \infty$, and that $D_xG(0)$ is Fredholm.

https://doi.org/10.1017/S1446788700019443 Published online by Cambridge University Press

DEFINITION. F is strongly stable if for every open neighbourhood U of 0, and for every representation F' of F on U, there is a neighbourhood $V(k, \lambda, K, F')$ of F' with $k \leqslant \overline{\text{cod}}\, G + 1$, such that for every function $F'' \in V$, there is a point $(x, \lambda, y) \in U$, such that F'' at (x, λ, y) is $\cong$ -equivalent to F' at 0.

REMARK, This stability is a property of the $\cong$ -equivalence classes, as is easily checked.

PROPOSITION 3.12. *Under the above hypotheses F is strongly stable, if F satisfies Condition T.*

PROOF. By the previous remark and Lemma 1.1, we can work in the splitting parametrization of Lemma 1.2. The proof can be reduced to the case of $\dim B_x < \infty$, using Theorem 2.2 and the continuity of the splitting given by Lemma 1.4. When $\dim B_x < \infty$, the proof can be carried through close to the corresponding one in [9], where the parameter-free potential situation is considered. The methods and results of Sections 2 and 3 above are used in the proof in the relevant places. The steps in [9] employing compact subsets of the control variables are actually consequences of only the continuity properties in our B_y-context. We leave the somewhat lengthy details to the reader.

REMARK. a) Also the (F, U)-stability properties of [5] can be obtained in our infinite dimensional context.

b) The converse of Proposition 3.12 is in the context of [9] proved using Thom's transversality theorem. Without extra restrictions on B_y, it is not evident to the present author how to adapt that proof to the present situation.

References

[1] L. Arkeryd, 'Catastrophe theory in Hilbert space', Tech. Report, Math. Dept., University of Gothenburg, Gothenburg (1977).

[2] L. Arkeryd, 'Thom's Theorem for Banach spaces', *J. London Math. Soc.* (2) **19** (1979), 359–370.

[3] D. Chillingworth, 'A global genericity theorem for bifurcation in variational problems', *J. Funct. Anal.* **35** (1980), 251–278.

[4] M. Golubitsky, D. Schaeffer, 'Bifurcation analysis near a double eigenvalue of a model chemical reaction', Tech. Report 1859, Math. Research Center, University of Wisconsin (1978).

[5] M. Golubitsky, D. Schaeffer, 'A theory for imperfect bifurcation via singularity theory', *Comm. Pure Appl. Math.* **32** (1979), 21–98.

[6] R. Magnus, 'Determinacy in a class of germs on a reflexive Banach space', Math. Proc. *Cambridge Philos. Soc.* **84** (1978), 293–302.

[7] R. Magnus, 'Universal unfoldings in Banach spaces: reduction and stability', Battelle Research Report (Mathematics) 107, Geneva (1977).

[8] D. Trotman, C. Zeeman, 'The classification of elementary catastrophes of codimension < 5, (Springer Lecture Notes 525 (1976) 263–327).

[9] G. Wassermann, 'Stability of unfoldings' (Springer Lecture Notes 393 (1974)).

Department of Mathematics
Chalmers University of Technology
and The University of Göteborg
S-412 96 Göteborg, Sweden