QUOTIENTS OF PSEUDO GROUPS BY INVARIANT FIBERINGS

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Introduction Let Γ be a continuous pseudo group acting on a manifold M. Denote by (M, M', ρ) a fibered manifold preserved by transformations in Γ. Then any f in Γ locally induces a local transformation f' of M'. Denote by Γ/ρ the set of all such f'. Then it might seem natural to expect that Γ/ρ is a continuous pseudo group acting on M'. However the matter is not so simple. For instance, take f' and g' in Γ/ρ such that the composition $f' \circ g'$ can be defined. Then they can be lifted to local transformations f and g belonging to Γ. But there is no guarantee to the effect that they can be lifted in such a way that the composition of g and f can be defined, i.e. the image by g has non-empty intersection with the domain of f. So we can not conclude that Γ/ρ is a pseudo group. Thus, what we can expect is, roughly speaking, as follows: There is a unique pseudo group Γ' acting on M' such that Γ/ρ forms a substantial part of Γ'. We call such Γ' the quotient of Γ by (M, M', ρ) if it exists (cf. 5.2). The main purpose of the present paper is to show the existence and continuity of the quotient pseudo group for transitive Γ. Even if Γ is not transitive, the argument used for transitive case implies the existence and continuity of the quotient, provided we remove a proper subvarieties of M. Moreover it seems reasonable to conjecture that the quotient exists for any intransitive continuous pseudo group Γ. However we can not expect that the quotient is continuous. This will be shown by an example (cf. §6). Denote by $J^r\Gamma$ the set of all r-jets of local transformations belonging to Γ. In the present paper Γ is defined to be continuous when any x in M satisfies the following conditions: (1) we can find a neighborhood U of x and a fibered manifold (U, U_i, ρ) such that fibers coincide with orbits of the restriction of Γ to U, (2) for sufficiently large r, there is a neighborhood $\mathscr{V}_r$ of the identity jet $I^r(x)$ at x in the space of r-jets such that the component $\mathscr{U}_r$ of $\mathscr{V}_r \cap J^r\Gamma$ containing

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$\Gamma^r(x)$ is a submanifold, (3) we can choose $\mathscr{U}_{r+1}$ and $\mathscr{U}_r$ such as in (2) so that $(\mathscr{U}_{r+1}, \mathscr{U}_r, \rho_r^{r+1})$ is a fibered manifold where ρ_r^{r+1} is the canonical projection of $(r+1)$-jets to r-jets, (4) we can find a neighborhood U of x in M and an integer r_0 such that a local transformation f of U is in Γ if and only if $j^{r_0}(f)$ is contained in $J^{r_0}\Gamma$. Submanifold in the present paper is not assumed to be closed. Throughout the paper we restrict ourselves in the category of real analyticity. So we usually omit the adjective "real analytic". Except in § 6, all the pseudo groups considered are transitive. So we also omit the adjective "transitive".

§ **1. Derived Space.** Let E, E' be vector spaces over a field K. K will be fixed throughout this section and we omit "over K". Denote by $L(E, E')$ the vector space of linear mappings of E into E'. Let F be a vector subspace of $L(E, E')$.

1.1. Definition. *By the derived space of F, denoted by $\mathfrak{D}(F)$, we mean the subspace of $L(E, F)$ consisting of all b in $L(E, F)$ such that for any u and u' in E*

$$b(u)\,u' = b(u')\,u$$

where $b(u)$ is the image of u by b and $b(u)u'$ is the image of u' by $b(u)$.

1.2. We set

$$\delta(F) = \text{Codimension of } \mathfrak{D}(F) \text{ in } L(E, F).$$

For $u_1, \ldots, u_r$ in E denote by $F(u_1, \ldots, u_r)$ the subspace in the r times direct product of E' consisting of all vectors $(q(u_1), \ldots, q(u_r))$ with q in F. Let $\sigma_r(F)$ be the maximum of the dimension of $F(u_1, \ldots, u_r)$ for all choices of $u_1, \ldots, u_r$ in E.

1.3. Definition. *F is said to be an involutive subspace of $L(E, E')$, or simply involutive, when*

$$\delta(F) = \sigma_1(F) + \cdots + \sigma_{n-1}(F)$$

where n is the dimension of E.

The notion of involutive subspaces was introduced by E. Cartan in connection with that of exterior differential systems in involution. Namely, an exterior differential system of certain type is in involution if and only if F such as above associated with the system is an involutive subspace. A special case of this theorem will be used later (cf. Prop. 2, 3.).

1.5. PROPOSITION. *If F is an involutive subspace of $L(E, E')$, $\mathfrak{D}(F)$ is an involutive subspace of $L(E, F)$.*

1.6. PROPOSITION. *Let F_j, $j=0, 1, \ldots$, be a sequence of vector spaces such that $F_0 \subseteq L(E, E')$ and $F_{j+1} \subseteq \mathfrak{D}(F_j) \subseteq L(E, F_j)$ for all j, where $\mathfrak{D}(F_j)$ is the derived space of F_j as a subspace of $L(E, F_{j-1})$. Then there exists an integer j_0 such that $F_{j+1} = \mathfrak{D}(F_j)$ and F_j is an involutive subspace of $L(E, F_{j-1})$ for any $j \geq j_0$.*

This proposition is a special case of the prolongation theorem in the case when our field K is the field of real numbers or of complex numbers. An algebraic proof of our proposition was obtained recently by S. Sternberg. The following two propositions are easy to check.

1.7. Let E' be a subspace of E''. We have the canonical injection i of $L(E, E')$ into $L(E, E'')$. For a subspace F of $L(E, E')$

$$\mathfrak{D}(F) = \mathfrak{D}(iF) \subseteq L(E, F).$$

1.8. Let j be a surjective homomorphism: $E_1 \rightarrow E$. Then j induces a canonical injective homomorphism j (resp. j') of $L(E, E')$ into $L(E_1, E')$ (resp. of $L(E, F)$ into $L(E_1, jF)$). For a subspace F of $L(E, E')$

$$\mathfrak{D}(j(F)) = j'(\mathfrak{D}(F)).$$

1.9. Use the notation in 1.7 and 1.8, except we write $F \circ j$ in stead of $j(F)$. Then F is an involutive subspace of $L(E, E')$ if and only if $iF \circ j$ is an involutive subspace of $L(E_1, E'')$.

1.10. The following reformulation of 1.6 will be used later.

PROPOSITION. *Let G_l, $l=0, 1, \ldots$, be a sequence of vector spaces such that*

$$G_l \subseteq L(E + G_0 + \cdots + G_{l-1},\ E + G_0 + \cdots + G_{l-1})$$

where $G_{-1} = 0$. Let j_l be the canonical projection of $E + G_0 + \cdots + G_l$ onto $E + G_0 + \cdots + G_{l-1}$ and i_l be the canonical injection of G_l into $E + G_0 + \cdots + G_l$. Assume that there is a subspace G_l' of $L(E + G_0 + \cdots + G_{l-2}, G_{l-1})$ such that

(1) $$G_l = i_{l-1} G_l' \circ j_l \qquad \textit{for } l \geq 1.$$

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Assume further that $G_{l+1} \subseteq i_l \mathfrak{D}(G_l) \circ j_l$. *Then there exists an integer* l_0 *such that* $G_{l+1} = i_l \mathfrak{D}(G_l) \circ j_l$ *and* G_l *is an involutive subspace of* $L(E + G_0 + \cdots + G_{l-1}, E + G_0 + \cdots + G_{l-1})$.

Proof. (1) establishes an isomorphism k_l of G_l onto G'_l. Then by 1.7 and 1.8 our last assumption can be written as

$$k_l G'_{l+1} \subseteq \mathfrak{D}(G'_l) \circ j_{l-1}.$$

1.7 and 1.8 together with the above inclusion relation for $l-1, l-2, \ldots$ implies that $G'_{l+1} \subseteq \mathfrak{D}^{l-1}(G'_1) \circ j_1 \circ \cdots \circ j_{l-1}$, where in the notation $\mathfrak{D}^{l-1}$ we omit writting several identifications. Then there is a subspace F_l in $L(E, G_{l-1})$ such that

$$F_l \circ j_1 \circ \cdots j_{l-2} = G'_l.$$

Using the above isomorphism of F_l and G'_l, we can consider F_l as a subspace of $L(E, F_{l-1})$. Then our assumption implies that F_l is in $\mathfrak{D}(F_{l-1})$. Then our contention follows from 1.6 and 1.9.

§ 2. Cartan systems

2.1. Definition. *A Cartan system on a manifold* M *is a finite dimensional vector subspace* Ω *of the space of linear differntial forms on* M *such that we can find a basis* $\omega^1, \ldots, \omega^n$ *of* Ω *and linear differential forms* $\tilde{\omega}, \ldots, \tilde{\omega}^m$ *on* M *satisfying the following conditions:*

1) $$d\omega^i = \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k + a^i_{j\lambda} \omega^j \wedge \tilde{\omega}^\lambda$$

where c^i_{jk} *and* $a^i_{j\lambda}$ *are constants and* $c^i_{jk} + c^i_{jk} = 0$,

2) $\omega^1, \ldots, \omega^n, \tilde{\omega}^1, \ldots, \tilde{\omega}^m$ *are linearly independent at each point of* M,

3) *the matrices* $a_\lambda = \|a^i_{j\lambda}\|$, $1 \leqq \lambda \leqq m$, *are linearly independent.*

A Cartan system is called reduced when $\omega^1, \ldots, \omega^n, \ldots, \tilde{\omega}^m$ form a basis of linear differential forms over the ring of functions.

If $\theta^1, \ldots, \theta^n$ is another basis of Ω, the condition 1) is satisfied for the forms θ^i, $\tilde{\omega}^\lambda$ with a possibly different set of constants. $\tilde{\omega}^1, \ldots, \tilde{\omega}^m$ such as in the definition is called an auxiliary set of forms for Ω.

2.2. Keeping the above notations, denote respectively by Ω^* and by $e_1, \ldots,$

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e_n the dual vector space of Ω and the dual basis of $\omega^1, \ldots, \omega^n$. Let $\underline{a}_\lambda$ be the linear transformation of Ω^* defined by

$$\underline{a}_\lambda(e_j) = a^i_{j\lambda} e_i.$$

Denote by $L(\Omega)$ the vector subspace of $L(\Omega^*, \Omega^*)$ generated by $\underline{a}_1, \ldots, \underline{a}_m$. It is easy to check that $L(\Omega)$ thus defined is independent of the choice of forms $\omega^1, \ldots, \omega^n, \tilde{\omega}^1, \ldots, \tilde{\omega}^m$. We say that a Cartan system Ω is involutive when $L(\Omega)$ is an involutive subspace of $L(\Omega^*, \Omega^*)$.

2.3. Denote by ρ_1 (resp. ρ_2) the projection of $M \times M$ onto the first factor M (resp. the second factor M). Let $\sum(\Omega)$ be the exterior differential system generated by $\rho_1^* \omega^i - \rho_2^* \omega^i$. By a theorem of E. Cartan we have the following.

Proposition. *The exterior differential system $\sum(\Omega)$ on $(M \times M, M, \rho_1)$ is involution if and only if Ω is involutive.*

2.4. Definition. *A pseudo group operating on a manifold M is called a Cartan pseudo group when there exists an involutive Cartan system Ω such that the pseudo group consists of all local transformations which leave each member of Ω invariant.*

2.4. Proposition. *Let $\tilde{\omega}^1, \ldots, \tilde{\omega}^m$ and $\xi^1, \ldots, \xi^q$ be two sets of auxiliary forms of a Cartan system Ω. Then $m = q$ and there are constants b^μ_λ and functions h^λ_i defined on M such that*

$$\xi^\lambda = b^\lambda_\mu \tilde{\omega}^\mu + h^\lambda_i \omega^i$$

Proof. Write

$$d\omega^i = \frac{1}{2} c'^i_{jk} \omega^j \wedge \omega^k + a'^i_{j\lambda} \omega^j \wedge \xi^\lambda$$

Then comparing with 2.1.1), we see easily that the form $a^i_{j\lambda}\tilde{\omega}^\lambda - a'^i_{j\lambda}\xi^\lambda$ is a linear combination of $\omega^1, \ldots, \omega^n$. Since the matrices $a'_\lambda = \|a'^i_{j\lambda}\|$ are linearly independent, the resulting linear system can be solved with respect to ξ^λ and we have the required relations among ξ^λ, $\tilde{\omega}^\lambda$, and ω^i. Since we can also express $\tilde{\omega}^\lambda$ as a linear combination of ξ^μ and ω^i, we find that m is equal q.

2.6. Fix a basis ω^i and auxiliary set of forms $\tilde{\omega}^\lambda$ of Ω. An element b in $L(\Omega^*, L(\Omega))$ will be expressed by a matrix b^λ_i where

$$b(e_i) = b^\lambda_i a_\lambda$$

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where a_λ is as in 2.1.3) and e_i is as in 2.2. We recall that $\mathfrak{D}(L(\Omega))$ is a subspace of $L(\Omega^*, L(\Omega))$. We say that an auxiliary set of forms $\xi^1, \ldots, \xi^m$ of Ω is restricted (with respect to $\tilde{\omega}^\lambda$) when $a^i_{j\lambda} = a'^i_{j\lambda}$ where $a'^i_{j\lambda}$ is as in the proof in 2.5. Then we have the following.

PROPOSITION. *A set $\xi^1, \ldots, \xi^m$ of linear differential forms on M is a restricted auxiliary set of forms of Ω if and only if we have the expression*

$$\xi^\lambda = \tilde{\omega}^\lambda + h^\lambda_i \omega^i$$

where h^λ_i is a function on M such that $h^\lambda_i(x)$ is the expression of an element $h(x)$ in $\mathfrak{D}(L(\Omega))$ for each x in M.

Proof. If $\xi^1, \ldots, \xi^m$ is a restricted auxiliary sets of forms of Ω, then b^λ_μ in 2.5 must be equal to δ^λ_μ because a_λ are linearly independent. Then it is a matter of checking to see that $\|h^\lambda_i(x)\|$ belongs to $\mathfrak{D}(L(\Omega))$.

2.7. PROPOSITION. *Let Ω be a Cartan system on a manifold M. Given a point p in M, we can find a neighborhood U on p, a fibered manifold (U, U', ρ), and a reduced Cartan system Ω' on U' such that ρ^* induces an isomorphism of Ω' onto Ω. Moreover such (U, U', ρ) and Ω' are unique up to obvious isomorphism provided we replace U and U' by smaller neighborhoods of p and $\rho(p)$.*

Proof. Take a basis $\omega^1, \ldots, \omega^n$ of Ω and write the structure equation as

$$d\omega^i = \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k + \omega^j \wedge \tilde{\omega}^i_j,$$

where c^i_{jk} are constants. We will show that the equation $\omega^1 = \cdots = \omega^n = \cdots = \tilde{\omega}^i_j = \cdots = 0$ is completely integrable. Taking the exterior derivative of the both sides and using the structure equation, we find that $\omega^j \wedge d\tilde{\omega}^i_j$ is a linear combination of forms of the type $\omega^i \wedge \omega^j \wedge \omega^k$, $\omega^i \wedge \omega^j \wedge \tilde{\omega}^k_h$, and $\omega^i \wedge \tilde{\omega}^j_k \wedge \tilde{\omega}^h_l$. Then it is easy to check that the above equation is completely integrable. Take a small neighborhood U of p and let (U, U', ρ) be the fibered manifold of the maximal integrals of the above equation restricted to U. If $x^1, \ldots, x^{n+m}$, $y^1, \ldots, y^s$ is a local coordinate of (U, U', ρ), then

$$\omega^i = w^i_r(x, y)\, dx^r,$$
$$\tilde{\omega}^i_j = v^i_{jr}(x, y)\, dx^r \qquad (r = 1, \ldots, n+m).$$

By observing the structure equation 2.1.1), we see easily that the functions

w_r^i do not depend on y. Hence there are forms θ^i on U' such that $\omega^i = \rho^*\theta^i$. Let Ω' be the vector space generated by $\theta^1, \ldots, \theta^n$. In order to see that Ω' is a Cartan system, take a cross-section g of (U, U', ρ) and observe the image of structure equation by g^*. This finishes the proof of the first part of our proposition. By 2.5, the equation

$$\omega^1 = \cdots = \omega^n = \cdots = \tilde{\omega}_j^i = \cdots = 0$$

is uniquely determined by Ω. Moreover, any fibered manifold (U, U', ρ) satisfying our conditions must be a fibered manifold of maximal integrals of the above equation. Therefore (U, U', ρ) and Ω' are unique up to isomorphism and schrinking of neighborhoods.

2.8. DEFINITION. *A vector subspace Ω_1 of a Cartan system on M is said to be a Cartan subsystem when Ω_1 itself is a Cartan system on M.*

2.9. *Let Ω_1 be a vector subspace of a Cartan system Ω. Then Ω_1 is a Cartan subsystem of Ω if and only if the equation $\Omega_1 = 0$ is completely integrable.*

Proof. Assume that the equation $\Omega_1 = 0$ is completely integrable. Take a basis $\omega^1, \ldots, \omega^n$ of Ω such that the first n' members form a basis of Ω_1. Let $\tilde{\omega}^\lambda$ be an auxiliary set of forms of Ω. Then

$$d\omega^s = \frac{1}{2} c_{tr}^s \, \omega^t \wedge \omega^r + \omega^t \wedge \tilde{\omega}_t^s$$

where $s, t, r = 1, \ldots, n'$,

$$\tilde{\omega}_t^s = a_{t\lambda}^s \tilde{\omega}^\lambda + b_{tj}^s \, \omega^j \qquad (j = n'+1, \ldots, n),$$

and where a_t^s and b_{tj}^s are constants. Let $q_\sigma = \|q_{t\sigma}^s\|$, $\sigma = 1, \ldots, m'$, be a maximal subset of linear independent matrices in the set $a_\lambda = \|a_{t\lambda}^s\|$ and $b_j = \|b_{tj}^s\|$. Write $a_\lambda = q_\sigma u_\lambda^\sigma$ and $b_j = q_\sigma v_j^\sigma$. Setting $\pi^\sigma = u_\lambda^\sigma \tilde{\omega}^\lambda + v_j^\sigma \omega^j$, we find that

$$d\omega^s = \frac{1}{2} c_{tr}^s \, \omega^t \wedge \omega^r + q_{t\sigma}^s \omega^t \wedge \pi^\sigma.$$

Then it is easy to check that Ω_1 is a Cartan subsystem. The converse is trivial.

§ 3. **Prolongations of Cartan systems.** In this section we fix a Cartan system Ω on a manifold M. We also fix a basis ω^i and an auxiliary set of

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forms $\tilde{\omega}^\lambda$ of Ω. However these will play only a secondary role, and all the concept introduced will be independent of the choice of ω^i and $\tilde{\omega}^\lambda$. We fix a reference point x_0 in M.

3.1. Let $p(M, x_0)$ be the space of 1-jets of local transformations of M with target x_0. We have the source mapping α of $p(M, x_0)$ onto M. For a differential form θ on an open subset of M and for a point x in the subset, denote by $(\theta)_x$ the multi-linear function on the tangent vector space at x assigned by θ. Take a 1-jet $X = j^1_x(f)$ where f is a local transformation of M. For a given differential form ξ, $(f^*\xi)_x$ is independent of the choice of a representative f of X. We denote this multi-linear function by $X^*\xi$. X is said to be an invariant 1-jet (with respect to Ω) when

$$X^*\omega = (\omega)_x \text{ and } X^*(d\omega) = (d\omega)_x$$

for every ω in Ω. Denote by $p(M, x_0)_\Omega$ the set of invariant 1-jets with source x_0.

3.2. Proposition. *$p(M, x_0)_\Omega$ is a submanifold of $p(M, x_0)$, $(p(M, x_0)_\Omega, M, \alpha)$ is a fibered manifold, and the dimension of fibers is equal to the dimension of $\mathfrak{D}(L(\Omega))$.*

Proof. Choose linear differential forms $\eta^1, \ldots, \eta^r$ defined on a neighborhood U of a given point x such that ω^i, $\tilde{\omega}^\lambda$, η^v form a basis of linear differential forms on U. For X in $p(M, x_0)$ with source y in U, set

$$X^*\omega^i = p^i_j(X)(\omega^j)_y + {}'p^i_\lambda(X)(\tilde{\omega}^\lambda)_y + {}''p^i_v(X)(\eta^v)_y,$$
$$X^*\tilde{\omega}^\lambda = q^\lambda_j(X)(\omega^j)_y + {}'q^\lambda_\mu(X)(\tilde{\omega}^\mu)_y + {}''q^\lambda_v(X)(\eta^v)_y.$$

Then p^i_j, ${}'p^i_\lambda$, ${}''p^i_v$, q^λ_j, q^λ_μ, ${}''q^\lambda_v$ can be completed to a chart of $p(M, x_0)$ defined on $\alpha^{-1}(U)$. $p(M, x_0)_\Omega \cap \alpha^{-1}(U)$ is defined by the following equation (cf. 2.6.):

$$(1) \qquad {}'p^i_\lambda = {}''p^i_v = {}''q^i_v = p^i_j - \delta^i_j = {}'q^\lambda_\mu - \delta^\lambda_\mu = 0,$$

$$(2) \qquad a^i_{k\lambda} q^\lambda_j - a^i_{j\lambda} q^\lambda_k = 0,$$

where a's are as in 2.1.1). Let $q^\lambda_{j\sigma}$, $\sigma = 1, \ldots, m'_1$, be a basis of the space of solutions of the equation (2). This equation is the equation for q in $L(\Omega^*, L(\Omega))$ to be in $\mathfrak{D}(L(\Omega))$ expressed in terms of basis (cf. 2.6.). Hence m'_1 is equal to the dimension of $\mathfrak{D}(L(\Omega))$. The equation (1) together with

$$q^\lambda_j = q^\lambda_{j\sigma} u^\sigma$$

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with arbitrary u gives a parameterization for elements in $p(M, x_0)_\Omega$. Then it is trivial to confirm our assertion.

3.3. PROPOSITION. *If X and Y are invariant 1-jets and if $\alpha(X)=\alpha(Y)$, then $X \circ Y^{-1}$ is an invariant 1-jet.*

Proof is obvious from the definition.

3.4. DEFINITION. *Denote by $\Gamma(\Omega)$ the pseudo group of local transformations f of M such that $f^*\omega=\omega$ for any ω in Ω.*

3.5. Let f be a local transformation of M. Denote by $U(f)$ the domain of definition of f. Let $p(f)$ be a local transformation of $p(M, x_0)$ defined by the following formula: For X in $\alpha^{-1}(U(f))$

$$p(f)(X) = X \circ j^1_x(f)^{-1}$$

where x is the source of X. Assume that f is in $\Gamma(\Omega)$. Then by 3.5. $p(f)$ preserves the submanifold $p(M, x_0)_\Omega$. Denote by $p(f)$ the restriction of $p(f)$ to $p(M, x_0)_\Omega$.

3.6. PROPOSITION. *Let (x) be a chart in M. Then we have the chart (x, u) introduced in the proof of 3.2. Take an element f in $\Gamma(\Omega)$ defined on the domain v' of the chart. Then we can find functions $\underline{u}^\sigma(x)$ such that for any X in $p(M, x_0)_\Omega$ with source in U'*

$$u^\sigma(p(f)(X)) = u^\sigma(X) + \underline{u}^\sigma(\alpha(X)).$$

Proof. Since f^* keeps the structure equation 2.1.1), $f^*\tilde{\omega}^\lambda = \tilde{\omega}^\lambda + k^\lambda_j \omega^j$, where k^λ_j is a function. By the defintion of q^λ_j in 3.2, there is a function $\underline{u}^\sigma$ such that $k^\lambda_j = q^\lambda_{j\sigma}\underline{u}^\sigma$. Then we can easily check our assertion by going back to the definition of the function q^λ_j in 3.2.

3.7. DEFINITION. *Define a linear differential form $\omega^{n+\lambda}$ on $p(M, x_0)_\Omega$ for each $\lambda=1, \ldots, m$ by the following formula:*

$$(\omega^{n+\lambda})_X = \alpha^* \circ (X^*\tilde{\omega}^\lambda) \tag{1}$$

where α^* is the linear mapping induced by α on tangent vector spaces. By the definition

$$\omega^{n+\lambda} = \alpha^*\tilde{\omega}^\lambda + q^\lambda_{j\sigma} u^\sigma \alpha^* \omega^j.$$

Denote by $p(\Omega)$ the vector space generated by $\alpha^*\omega^j$ and $\omega^{n+\lambda}$ over constants. $p(\Omega)$ is a vector subspace of the space of linear differential forms on $p(M, x_0)_\Omega$. If we take another auxiliary set of forms $\xi^1, \ldots, \xi^m$ for Ω and if we denote by $\theta^{n+\lambda}$ the form obtained from ξ^λ by the same process as we obtained $\omega^{n+\lambda}$ from $\tilde{\omega}^\lambda$, then by 2.5 we have $\theta^{n+\lambda} = b^\lambda_\mu \omega^{n+\mu} + h^\lambda_j(x_0)\,\omega^j$. Therefore $p(\Omega)$ is independent of choices of auxiliary set of forms for Ω.

3.8. Proposition. *Let f be an element in $\Gamma(\Omega)$. Then for each ω in $p(\Omega)$. $(p^1(f))^*\omega = \omega$.*

Proof. By the definition of $\omega^{n+\lambda}$

$$\begin{aligned}(p^1(f))^*(\omega^{n+\lambda})_{p(f)(X)} &= (p^1(f))^*(\alpha)^*(X\circ j^1_x(f)^{-1})^*\tilde{\omega}^\lambda) \\ &= (p^1(f))^*(\alpha)^*((f^{-1})^*(X^*\tilde{\omega}^\lambda) \\ &= \alpha^*(X^*\tilde{\omega}^\lambda) = (\omega^{n+\lambda})_X\end{aligned}$$

because $f^{-1}\circ\alpha = \alpha\circ p(f^{-1})$.

3.9. Definition. *Let H be a locally closed submanifold of $p(M, x_0)_\Omega$. We say that H is an admissible submanifold (with respect to Ω) when the following conditions are satisfied:*

1) There is an open neighborhood U of x_0 in M such that (H, U, α) is a fibered manifold.

2) for any X_1 and X_2 in H we can find a neighborhood W_i of X_i in H, $i = 1, 2$, and an element f of $\Gamma(\Omega)$ such that $p^1(f)$ is defined on W_1 and maps W_1 (resp. X_1) into W_2 (resp. to X_2).

3.10. Proposition. *Let H be an admissible submanifold of $p(M, x_0)_\Omega$. Denote by τ the canonical injection of H into $p(M, x_0)_\Omega$. Fix a point X_1 in H. Take an element ω in $p(\Omega)$. Then $(\tau)^*\omega = 0$ if and only if $((\tau)^*\omega)_{X_1} = 0$.*

Proof. Take an arbitrary element X_2 in H. Choose f such as in 3.9.2). Then by 3.8 and 3.9.2) we see easily that the assumption $((\iota)^*\omega)_{X_1} = 0$ implies $((\iota)^*\omega)_{X_2} = 0$.

3.11. Proposition. *Let H be an admissible submanifold of $p(M, x_0)_\Omega$. Denote by $p(\Omega, H)$ the restriction of $p(\Omega)$ to H. If Ω is reduced (cf. 2.1), $p(\Omega, H)$ is a reduced Cartan system on H.*

Proof. Fix an element X_1 in H. Using the independent functions $u^1, \ldots,$

$u^{m'}$ on $p(M, x_0)_\Omega$ introduced in 3.2, we may assume because of 3.9.1) that the tangent vector space of H at X_1 is defined by the condition: $du^{m_1+1} = \cdots = du^{m'} = 0$. Then by 3.9.2) and 3.6 the tangent vector space of H at any point of H is defined by the same equation. In the following we observe everything on H. Since Ω is reduced and since $\omega^{n+\lambda}$ has the expression 3.7.(1), we can write

$$(1) \qquad d\omega^{n+\lambda} = \frac{1}{2} \underline{c}^{n+\lambda}_{k'k''} \omega^{k'} \wedge \omega^{k''} + q^{\lambda}_{j\tau} du^{\tau} \wedge \omega^{j}$$

($k', k'' = 1, \ldots, n+m, \tau = 1, \ldots, m_1$), where $\underline{c}$'s are functions skew-symmetric in k' and k''. Set $c^{n+\lambda}_{k'k''} = \underline{c}^{n+\lambda}_{k'k''}(X_1)$. Choose functions v^{τ}_{k} ($k = 1, \ldots, n+m$) and set

$$\theta^{\tau} = du^{\tau} + v^{\tau}_{k} \omega^{k}.$$

Then we have the equality

$$(2) \qquad d\omega^{n+\lambda} = \frac{1}{2} c^{n+\lambda}_{k'k''} \omega^{k'} \wedge \omega^{k''} + q^{\lambda}_{j\tau} \theta^{\tau} \wedge \omega^{j}$$

if and only if

$$(3) \qquad \underline{c}^{n+\lambda}_{kk'}(X) = c^{n+\lambda}_{kk'} - q^{\lambda}_{k\tau} v^{\tau}_{k'}(X) + q^{\lambda}_{k'\tau} v^{\tau}_{k}(X)$$

for any X in H, where we set $q^{\lambda}_{k\tau} = 0$ for $k > n$. Consider this as an equation on unknown v^{τ}_{k} for each fixed X. We claim that this equation has solution. Namely, take f such as in 3.9.2) and such that $p(f)(X) = X_1$. Set

$$(p(f)^*(du^{\tau}))_X = (du^{\tau})_X + {}'v^{\tau}_{k}(\omega^{k})_X$$

(cf. 3,6) Then by applying $(p(f))^*$ to (1), we see easily that the above ${}'v^{\tau}_{k}$ is a solution of the equation. Hence (3) has solution for each fixed X. Therefore we can find functions v^{τ}_{k} which satisfies the equation (3) for arbitrary X. Then θ^{τ} with this choice of v^{τ}_{k} satisfies the equality (2). Then it is easy to check that $p(\Omega, H)$ is a reduced Cartan system.

3.12. The following is well-known:

Proposition. *If Ω is involutive, $p(M, x_0)_\Omega$ itself is an admissible submanifold of $p(M, x_0)_\Omega$,* and *$p(\Omega)$ is an involutive Cartan system.*

§ 4. Quotient by a Cartan subsystem.

4.1. Definition. *A vector subspace Ω' of a Cartan system Ω on M is called*

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a Cartan subsystem when Ω' is a Cartan system on M.

4.2. A CONSTRUCTION. Let Ω be a reduced involutive Cartan system on a manifold M. Take a reference point x_0 in M. Let Ω' be a Cartan subsystem of Ω. By Proposition 2.7, replacing M by an open neighborhood of x_0 if necessary, we can find a fibered manifold (M, M', π) and a reduced Cartan system Π on M' such that π^* induces an isomorphism of Π onto Ω'. π induces a fibered manifold $(p(M, x_0), p(M', w_0), j^1\pi)$ where $w_0 = \pi(x_0)$. Denote by $p(M', x_0)^{\Omega}_{\Omega'}$ the image by $j^1\pi$ of $p(M, x_0)_{\Omega}$. It will be shown that $p(M', x_0)^{\Omega}_{\Omega'}$ is an admissible submanifold of $p(M', w_0)_{\Pi}$. Therefore we have a reduced Cartan system $p(\Pi, p(M', x_0)^{\Omega}_{\Omega'})$ by 3.11. Denote by $p(\Omega'; \Omega)$ the image by $(j^1\pi)^*$ of $p(\Pi, p(M', x_0)^{\Omega}_{\Omega'})$. It will be shown that $p(\Omega'; \Omega)$ is a Cartan subsystem of $p(\Omega)$. For simplicity we set

$$H = p(M', x_0)^{\Omega}_{\Omega'}, \quad p(\Pi, H) = \Pi_1,$$
$$M_1 = p(M, x_0)_{\Omega}, \quad \pi_1 = j^1(\pi).$$

We have the following commutative diagram:

$$\begin{array}{ccccc} M & \xleftarrow{\ \alpha\ } & p(M, x_0) & \supseteq & M_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\pi_1} \\ M' & \xrightarrow{\ \alpha\ } & p(M', w_0) & \supseteq & H \end{array}$$

4.3. PROPOSITION. *H is an admissible submanifold of $p(M', w_0)_{\Pi}$. (M_1, H, π_1) and (H, M', α) are fibered manifolds.*

Proof. We will show first that H satisfies the condition 3.9.2). Let W_1 and W_2 be two elements in H. Take X_r in M_1 such that $W_r = \pi_1(X_r)$. Since Ω is involutive, there is f in $\Gamma(\Omega)$ such that $p(f)(X_1) = X_2$. f is locally a prolongation of an element f' in $\Gamma(\Omega')$. Then it is clear that f' satisfies the condition 3.9.2) for Π. For a generic point X in M_1, H is a submanifold on a neighhborhood of $W = \pi_1(X)$ and (M_1, H, π_1) is a fibered manifold on a neighborhood of X. Then because of the homogenity condition 3.9.2), the same conclusion follows for arbitrary element in M_1. Similar argument also shows that (H, M', α) is a fibered manifold.

4.4. PROPOSITION. *$p(\Omega'; \Omega)$ is a Cartan subsystem of $p(\Omega)$.*

Proof. It is clear by the construction that elements in $p(\Omega'; \Omega)$ are invariant differential forms of $\Gamma(p(\Omega))$. Then by a theorem proved in [2] $p(\Omega'; \Omega)$ is contained in $p(\Omega)$. Hence it is a Cartan subsystem of $p(\Omega)$.

4.5. Take a basis $\theta^1, \ldots, \theta^{n'}$ of Π and an auxiliary set of forms $\xi^1, \ldots, \xi^{m'}$ for Π. Then Π_1 has a basis $\theta^s, \theta^{n'+\kappa}$ $(s=1, \ldots, n', \kappa=1, \ldots, m')$ (cf. 3.7). We have the structure equation

$$d\theta^{n'+\kappa} = \frac{1}{2} c_{tt'}^{n'+\kappa} \theta^t \wedge \theta^{t'} + b_{s\nu}^{\kappa} \theta^s \wedge \eta^{\nu}$$

$(t, t'=1, \ldots, n'+m')$, where $\{\eta^\nu\}$ is an auxiliary set of forms of Π_1. Each ξ^κ induces an element y_κ in $L(\Pi)$ (cf. 2.2). Let e_s be the dual basis of θ^s. Denote by $L'(\Pi_1)$ the vector subspace of $L(\Pi^*, L(\Pi))$ generated by b_ν defined by the formula:

$$b_\nu(e_s) = b_{s\nu}^{\kappa} y_\kappa.$$

As was remarked in the section 3,

$$L'(\Pi_1) \subseteq \mathfrak{D}(L(\Pi)).$$

Let i be the canonical injection of $L(\Pi)$ into the direct sum $\Pi^* + L(\Pi)$. Denote by j the canonical projection of $\Pi^* + L(\Pi)$ onto Π^*. θ^t form a basis of Π_1. Let f_t be the basis of Π_1^* dual to θ^t. Denote by q the isomorphism of $\Pi^* + L(\Pi)$ onto Π_1^* which sends e_s (resp. y_κ) to f_s (resp. to $f_{n'+\kappa}$). Then by the definition

$$i \circ L'(\Pi_1) \circ j = q^{-1} L(\Pi_1) q \subseteq i \circ \mathfrak{D}(L(\Pi)) \circ j.$$

4.6. Starting from Ω, Ω', (M, M', π), and Π as in 4.2, we carry out the construction in 4.2 successively as follows: We already defined M_1, Π_1, and π_1 in 4.2. We set $M_1' = H$, $\Omega_1' = p(\Omega'; \Omega)$, and $x_1 =$ the identity jet at x_0. Then Ω_1' is a Cartan subsystem of $\Omega_1 = p(\Omega)$ and π_1^* induces an isomorphism of Π_1 onto Ω_1'. Morever by 3,12 Ω_1 is reduced and involutive. Assume now that we constructed a fibered manifold (M_r, M_r', π_r), a reference point x_r in M_r, a reduced involutive Cartan system Ω_r on M_r, a Cartan subsystem Ω_r' of Ω_r, and a reduced Cartan system Π_r on M_r' such that π_r^* induces an isomorphism of Π_r onto Ω_r', for each $r=1, \ldots, l-1$. We set

$$M_l = p(M_{l-1}, x_{l-1}), \quad \Omega_l = p(\Omega_{l-1}),$$
$$\pi_l = j^1(\pi_{l-1}), \quad M_l' = p(M_{l-1}', x_{l-1})_{\Omega'_{l-1}}^{\Omega_{l-1}},$$
$$\Pi_l = p(\Pi_{l-1}, M_l'), \quad \Omega_l' = \pi_l^*(\Pi_l),$$

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and x_l = the identity jet at x_{l-1}. Thus we defined the above objects for each $r = 1, 2, \dots$.

4.7. Introduce the mappings i, j, and q as in 4,5 at each stage of the above inductive construction, say i_r, j_r, q_r for $r = 0, 1, \dots$. Set $E = \Pi^*$, $G_0 = L(\Pi)$, and $G_r = q_r'^{-1} L(\Pi_r) q_r'$ where $q_r' = q_{r-1} \cdots q_0$. Then by 4.5 we find that they satisfy the conditions in 1.10. Therefore there exists an integer r_0 such that for $r \geq r_0$ $L(\Pi_r)$ is involutive and $\mathfrak{D}(L(\Pi_r))$ can be canonically identified with $L(\Pi_{r+1})$ (cf. 1.7 and 1.8). Hence for $r \geq r_0$ Π_r is a reduced involutive Cartan system and $M'_{r+1} = p(M'_r, w_r)$.

4.8. We set

$$N = M_{r_0}, \quad N' = M'_{r_0}, \quad N_r = M_{r_0+r}, \quad N'_r = M'_{r_0+r},$$
$$\underline{\Omega} = \Omega_{r_0}, \quad \underline{\Pi} = \Pi_{r_0}, \quad \underline{\Omega}_r = \Omega_{r_0+r}, \quad \underline{\Pi}_r = \Pi_{r_0+r}.$$

We continue to use the same reference points, but omit them in notations when there is no possibility of confusion. We have the fibered manifolds

$$(N_r, N, \alpha_r) \text{ and } (N_r, N'_r, \rho_r).$$

Take the vector subspace $\Omega_r^\sharp$ of $\underline{\Omega}$ such that

$$\underline{\Omega}_r \supseteq \alpha_r^*(\underline{\Omega}) \cap \rho_r^*(\underline{\Pi}_r) = \alpha_r^*(\Omega_r^\sharp).$$

4.9. Proposition. *$\Omega_r^\sharp$ is a Cartan subsystem of $\underline{\Omega}$.*

Proof. By 2.9 it is sufficient to show that the equation $\Omega_r^\sharp = 0$ is completely integrable. Take linearly independent elements ω^s, θ^t, ξ^u in $\underline{\Omega}_r$ such that ω^s, θ^t (resp. ω^s, ξ^u) form a basis of $\alpha_r^*(\underline{\Omega})$ (resp. of $\rho_r^*(\underline{\Pi}_r)$). Then we easily check that

$$d\omega^s \equiv z_{tu} \xi^u \wedge \theta^t \pmod{\Omega_r^\sharp}.$$

Take tangent vectors L and K to N such that L is tangent to the fibers of (N_r, N'_r, ρ_n) and $\langle \omega^s, K \rangle = 0$. Since ω^s belongs to the image of ρ_r^* it follows that

$$0 = z_{tu} \langle \xi^u, K \rangle \langle \theta^t, L \rangle$$

If our choices of L and K have enough freedom so that $\langle \theta^t, L \rangle$ and $\langle \xi^u, K \rangle$ can take arbitrary values, it follows that z's are zero and so the equation $\Omega_r^\sharp = 0$ is completely integrable. The freedom for $\langle \xi^u, K \rangle$ is obvious. If for a

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X in $N(a_t\theta^t)_x=0$ on the tangent to the fiber, then because of the transitivity of the pseudo group $\Gamma(\underline{\Omega})$ it follows that $a_t\theta^t$ vanishes on the tangent to any fibers. This contradicts the choice of θ^t unless $a_t=0$.

4.10. By the construction $\Omega_r^\#$ is an increasing sequence of subspaces of $\underline{\Omega}$. Hence there exists an integer r_1 such that $\Omega_r^\#=\Omega_{r+1}^\#$ for $r\geq r_1$. Set $\Omega^\#=\Omega_{r_1}^\#$.

PROPOSITION. *We can find an auxiliary set of forms $\tilde{\omega}^\lambda$ of $\underline{\Omega}$ such that for a basis ω^s of $\Omega^\#$*

$$d\omega^s=\frac{1}{2}c^s_{s's''}\omega^{s'}\wedge\omega^{s''}+a^s_{s'\lambda}\omega^{s'}\wedge\tilde{\omega}^\lambda$$

Proof. Choose forms θ^t on N so that ω^s, θ^t form a basis of $\underline{\Omega}$. Take an auxiliary set of forms $\tilde{\omega}^\lambda$ of $\underline{\Omega}$. Write

$$d\omega^s=\frac{1}{2}c^s_{s's''}\omega^{s'}\wedge\omega^{s''}+\omega^{s'}\wedge\xi^s_{s'}$$

where $\xi^s_{s'}$ is a linear combination of θ^t and $\tilde{\omega}^\lambda$. Take a basis ω^s, ξ^u of $\underline{\Pi}_{r_1}$ and an auxiliary set of form η^τ of $\underline{\Pi}_{r_1}$. Then on N_{r_1} $\xi^s_{s'}$ is a linear combination of ω^s, ξ^u, and η^τ. Hence on N_{r_1+1} $\xi^s_{s'}$ is a linear combination of invariant forms since $\Pi_{r_1+1}=P(\underline{\Pi}_{r_1})$. Let E (resp. E') be the subspace generated by θ^t (resp. by $\xi^s_{s'}$). Then the above shows that any element θ in $E\cap E'$ is a linear combination of elements in $\alpha^*_{r_1+1}(\underline{\Pi}_{r_1+1})$. Since θ is an invariant form, this implies that θ is in $\alpha^*_{r_1+1}(\underline{\Omega}_{r_1+1})$. Hence if θ is not zero, $\Omega^\#_{r_1+1}$ is actually larger than $\Omega^\#_{r_1}$. Hence $E\cap E'=0$. Therefore we can rechoose our auxiliary forms $\tilde{\omega}^\lambda$ so that our contention holds.

4.11. In this and the next paragraph we will replace manifolds involved by open submanifolds containing the reference points whenever necessary without mentioning it and keep the same notation for the schrinked manifolds.

By 2.7 we can find a fibered manifold $(N, N^\#, \delta)$ and a reduced Cartan system $\underline{\Omega}^\#$ on $N^\#$ such that δ^* induces an isomorphism of $\underline{\Omega}^\#$ onto $\Omega^\#$. Since $\Omega^\#$ can be considered as a subsystem of $\underline{\Pi}_r$ and also contains $\underline{\Pi}$, it follows that we have the commutative diagram for any large r

(1)
$$\begin{array}{ccccc} N & & \longleftarrow & & N_r \\ & \searrow & & \swarrow & \\ \downarrow & & N^\# & & \downarrow \\ & \swarrow & & \nwarrow & \\ N' & & \longleftarrow & & N'_r \end{array}$$

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where by the arrows we indicate that the Cartan system on the target manifold is mapped in the Cartan system on the source manifold. $N_r \to N^\#$, $N'_r \to N^\#$ induce mappings

$$(2) \qquad \begin{aligned} N_{r+1} &\longrightarrow p(N^\#, x^\#)_{\underline{\Omega}^\#} \\ N'_{r+1} &\longrightarrow p(N^\#, x^\#)_{\underline{\Omega}^\#} \end{aligned}$$

Then by extending the diagram (1) we see easily that the images of the two mappings in (2) are equal.

4.12. THEOREM. *Let Ω be a reduced involutive Cartan system on M. Take a reference point x_0 in M. Let Ω' be a Cartan subsystem of Ω. Replacing M by an open submanifold containing x_0 if necessary, we take a fibered manifold (M, M', π) and a reduced Cartan system Π on M' such that π^* induces an isomorphism of Π onto Ω'. Then the following holds when we replace M and M' by small neighborhoods of reference points. Using the notations in this section, denote by Γ' the pseudo group on M' defined as follows: The equation of Γ' is of order r_0+r and the equation on a neighborhood of the space of identity (r_0+r)-jets is described as $p^{r_0+r}(f) \subseteq \Gamma(\underline{\Pi}_n)$. Then Γ' is continuous. Any element h in $\Gamma(\Omega)$ is locally a prolongation of an element in Γ'. Moreover, for any element f in Γ' such that $j_y^{r_0+r}(f)$ for any y in $U(f)$ is sufficiently near to the space of identity jets and for any x in M such that $\gamma(x)$ is in $U(f)$, f can be lifted locally to an element in $\Gamma(\Omega)$ defined on a neighborhood of x.*

Proof. Since $\underline{\Pi}_r$ is involutive, Γ' is continuous. Take h in $\Gamma(\Omega)$. Then h is locally a product of h_i such that $j^{r'}(h_i)$ is sufficiently near to identity jets, where $r' = r_0 + r$. Each h_i (resp. h) is a prolongation of f_i (resp. of f), where f_i and f are local transformations of M'. By the construction of M', f_i is in Γ'. Hence f is in Γ'. It remains to see the last assertion in our theorem.

Take f in Γ' such that its r'-jets are sufficiently near to identity jets. Set $g = p^{r_0}(f)$. Then g belongs to $\Gamma(\underline{\Pi})$. Hence by 4.13 g can be lifted to $g^\#$ in $\Gamma(\underline{\Omega}^\#)$. By the last conclusion of 4.12, $j^1(g^\#)$ is in the image by $j^1\gamma$ of the space of invariant 1-jets of N. Then $g^\#$ can be locally lifted to an element $h^\sim$ in $\Gamma(\underline{\Omega})$ by the following lemma in the theory of exterior differential systems. $h^\sim$ induces an element h in $\Gamma(\Omega)$ which is a lifting of f.

4.14. LEMMA. *Let $\underline{\Omega}$ be a reduced involutive Cartan system on N. Take a*

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fibered manifold $(N, N^{\#}, \gamma)$ *and a Cartan system* $\underline{\Omega}^{\#}$ *on* $N^{\#}$ *such that* δ^{*} *induces an injection of* $\underline{\Omega}^{\#}$ *into* $\underline{\Omega}$. *We assume that the proposition in* 4.10 *holds for* $\Omega^{\#}=\gamma^{*}\underline{\Omega}^{\#}$ *and* $\underline{\Omega}$. *Take* $g^{\#}$ *in* $\Gamma(\underline{\Omega}^{\#})$ *such that* $j^{1}(g^{\#})$ *is in the image by* $j^{1}\gamma$ *of the space of invariant* 1*-jets of* N. *Take* x *in* N *such that* $\gamma(x)$ *is in the domain of* $g^{\#}$. *Then* $g^{\#}$ *can be lifted locally to an element in* $\Gamma(\underline{\Omega})$ *defined on a neighborhood of* x.

Proof of the lemma is obtained by actually constructing the lifting by following the recipe of E. Cartan for the construction of general solution of involutive exterior differential systems. We note that the assumption that 4.10 holds for $\Omega^{\#}$ is essential to guarantee that the polar functions of $\Sigma(\underline{\Omega})$ satisfy conditions necessary to carry out the lifting construction.

5. Quotient pseudo group

5.1. Definition. *Let* Γ *be a pseudo group acting on a manifold* V. *A fibered manifold* (V, V', ρ) *is called an invariant fibering of* V *(with respect to* Γ*) when every element in* Γ *sends fibers into fibers.*

When this is the case, for any f in Γ and for any point x in the domain of f there is a neighborhood U of x such that the restriction of f to U is a prolongation of a local transformation f' of V'. Such f' is called a reduction of f.

5.2. Definition. *Keeping the above notation, denote by* Γ' *a pseudo group acting on* V'. *We say that* Γ' *is the quotient pseudo group of* Γ *with respect to the invariant fibering when the following conditions are satisfied:* (1) *for any* f *in* Γ, *every reduction of* f *is in* Γ', (2) *for any* x_0 *in* V, *we can find a neighborhood* U *of* x_0 *and a neighborhood* $\mathfrak{W}$ *of the identity* r*-jet* $I^{r}(\rho(x_0))$ *of* V' *for an integer* r *such that, for any* g *in* Γ' *with* $j^{r}(g)$ *in the component of* $\mathfrak{W}\cap J^{r}\Gamma'$ *containing* $I^{r}(\rho(x_0))$ *and for any* x *in* U, g *can be locally lifted to an element in* Γ *defined on a neighborhood of* x, (3) Γ' *is minimum with respect to the properties* (1) *and* (2).

5.3. Proposition. *Let* Ω *be a reduced Cartan system on a manifold* M. *Replacing* M *by an open submanifold if necessary, let* (M, V, ρ) *be the fibered manifold of maximal integrals of the completely integrable equation* $\Omega=0$. *Assume that* $\Gamma(\Omega)$ *is transitive. Then the quotient of* $\Gamma(\Omega)$ *exists. If* $\Gamma(\Omega)$ *is* *continuous, the quotient is also continuous.*

Proof. Fix a reference point w_0 in M. For an arbitrary point w of M choose f in $\Gamma(\Omega)$ such that $f(w)=w_0$. Let f' be a reduction of f defined on a neighborhood of $x=\rho(w)$. Set $\tau(w)=j^1_x(f')$. We claim that $\tau(w)$ is independent of the choice of f and that τ is an injective and analytic mapping of M into $p(V, x_0)=$ the manifold of invertible 1-jets of V with target $x_0=\rho(w_0)$. In order to see this, take a basis w^i and an auxiliary set of forms $\tilde{\omega}^\lambda$ of Ω. Take coordinate (u, v) (resp. (x, y)) defined on a neighborhood of w_0 (resp. of w) for the fibered manifold (M, V, ρ), where v and y are fiber coordinates, such that $(\omega^i)_{w_0}=(du^i)_{w_0}$, $(\tilde{\omega}^\lambda)_{w_0}=(dv^\lambda)_{w_0}$. We have the expression $\omega^i=\omega^i_j(x,y)dx^j$. By writing down the expression of $((f^{-1})^*\omega^i)_{w_0}$, we find that $\omega^i_j(x_1, y_1)=(\partial f^i/\partial x^j)_{x=x_j}$ where $w=(x_1, y_1)$. Hence τ is well defined and analytic. By the structure equation and by the condition $(\omega^i)_w=(dx^i)_w$, $(\tilde{\omega}^\lambda)=(dy^\lambda)_w$ for a fixed w, we see easily that $(\partial\omega^i_j/\partial y^\lambda)_w=a^i_{j\lambda}$. Then it is easy to check that τ is injective.

By the definition of τ, it is a simple matter to see that $\tau\circ f\circ\tau^{-1}=p(f')$ for any f in $\Gamma(\Omega)$. Then the equation of $\Gamma(\Omega)$ induces an equation for a pseudo group of V. Denote this pseudo group by Γ'. Then it is clear that Γ' is the quotient of $\Gamma(\Omega)$. If the equation of $\Gamma(\Omega)$ is involutive then the equation of Γ' is involutive and hence Γ' is continuous.

5.4. Proposition. *Let Γ be a transitive pseudo group acting on V. Take an invariant fibered manifold (V, V', ρ). Assume that for each point x in V there is an open neighborhood of x, say U_x, such that there is the quotient pseudo group of the restriction of Γ to U_x. Assume that each fiber is connected. Then the quotient of Γ exists.*

Proof. Take a point y in V. Let u and x be two points in V such that $\rho(u)=\rho(x)=y$. Denote by Γ_1 (resp. Γ_2) the quotient of $\Gamma|U_u$ (resp. $\Gamma|U_x$). It is sufficient to show that there is a neighborhood U' of y such that $\Gamma_1|U'=\Gamma_2|U'$. Since the fibers are connected we easily reduce to the case when $\rho(U_u\cap U_x)$ is a neighborhood of y. Then $U'=\rho(U_u\cap U_x)$ satisfies our condition.

5.5. Theorem. *Let Γ be a continuous transitive pseudo group acting on a manifold V. Take an invariant fibered manifold (V, V', ρ). Assume that each fiber is connected. Then the quotient of Γ with respect to the fibering exists* *and is a continuous pseudo group.*

Proof. By 5.4, we can afford to schrink our manifold to a small neighborhood of a point in V if necessary. In the following we will omit mentioning such schrinking. By taking the standard prolongation a number of times, we find a Cartan system Ω on a manifold M and a fibered manifold structure (M, V, ρ') such that $\Gamma(\Omega)$ is an isomomrphic prolongation of Γ. By taking one more standard prolongation if necessary, we can assume that, for each X in M, $(\Omega)_X$ contains the image by $(\rho')^*$ of the cotangent vector space of V at $\rho'(X)$. Denote by $\Omega(X)$ the set of all ω in Ω such that $(\omega)_X$ is equal to zero on the tangents to fibers of $(M, V', \rho\circ\rho')$. By the transtivity of $\Gamma(\Omega)$ and the invariance of the above fibering, it follows that $\Omega(X)$ is independent of X. Thus $\Omega' = \Omega(X)$ is a vector subspace of Ω, and $(\Omega')_X$ is equal to the image by $(\rho\circ\rho')^*$ of the cotangent vector space of V' at $\rho\circ\rho'(X)$. Hence Ω' is completely integrable and so Ω' is a Cartan subsystem of Ω. Take a fibered manifold (M, M', π) and a Cartan system Π on M' such that π^* induces an isomorphism of Π onto Ω'. By 4.14 there is the quotient pseudo group Γ_1 acting on M'. Clearly Γ_1 is a pseudo subgroup of $\Gamma(\Pi)$.

By the construction of Ω', it follows that there is a fibered manifold structure (M', V', π') such that $\rho\circ\rho' = \pi'\circ\pi$. Moreover (M', V', π') is the fibered manifold of maximal integrals of the equation $\pi = 0$. Hence by 5.3 there exists the quotient continuous pseudo group Γ'' of $\Gamma(\Pi)$. By recalling the proof of 5.3, we see easily that continuous pseudo subgroup Γ_1 of $\Gamma(\Pi)$ induces a continuous pseudo subgroup Γ' of Γ''. It is clear that Γ' is the quotient pseudo group Γ.

6. An example. In this section we give an example to show that theorem 5.5 is not true for intransitive continuous pseudo groups.

For a given real number t let $G(t)$ be the group of all matrices

$$\left\|\begin{matrix} 1 & a & b \\ 0 & 1 & ta \\ 0 & 0 & 0 \end{matrix}\right\|$$

where a, b are arbitrary real numbers. Let (x^i), $1 \le i \le 4$, be the coordinates in R^4. Denote by Γ the pseudo group of all local transformations $y^i = F^i(x_1,$

$\ldots, x^4)$, $1 \leq i \leq 4$, of R^4 such that

1) the matrix $\|\partial F^i/\partial x^j\|$, $1 \leq i, j \leq 3$, belongs to $G(x^4)$ at each point $(x^1, \ldots, x^4)$ of the domain of F,

2) $F^4(x^1, \ldots, x^4) = x^4$.

Denote by (x^i, y^j, p^i_j) the coordinates in the space of 1-jets of local transformations of R^4. Then elements of Γ are solutions of

$$(1)\qquad \begin{aligned} &y^4 - x^4 = 0, \quad p^1_1 - 1 = 0, \\ &p^2_1 = p^2_2 - 1 = p^2_3 - x^4 p^1_2 = 0, \\ &p^3_1 = p^3_2 = p^3_3 - 1 = 0. \end{aligned}$$

Let S be the system of equations defined by the equation (1) together with

$$(2)\qquad \begin{aligned} &p^1_{1i} = p^1_{22} = p^2_{1i} = p^2_{22} = 0, \\ &p^2_{33} - x^4 p^1_{23} = p^2_{34} - p^1_2 - x^4 p^1_{24} = 0 \end{aligned}$$

S is contained in the first prolongation of (1). Computing the successive prolongation of S we can easily check that the conditions of the prolongation theorem [1] are satisfied. Hence Γ is continuous.

The general transformation of Γ is given by

$$\begin{aligned} &y^1 = x^1 + \varphi_1(x^3, x^4)x^2 + \varphi_2(x^3, x^4), \\ &y^2 = x^2 + x^4\varphi_1(x^3, x^4) + \varphi_3(x^4), \\ &y^3 = x^3 + \varphi_4(x^4), \\ &y^4 = x^4, \end{aligned}$$

where φ_i are arbitrary functions of their arguments. Consider the fibration $R^4 \to R^3$ defined by $(x^1, \ldots, x^4) \to (x^2, x^3, x^4)$; clearly it is invariant under the action of Γ. The pseudo gronp Γ' induced by Γ in R^3 is not continuous. This can be checked by computing the space of r-jets belonging to Γ' and observing that they degenerate at points $x^4 = 0$.

References

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