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COMMUTATIVE SELF-INJECTIVE RINGS

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1. Introduction. All rings considered here are commutative containing at least two elements, but may not have identity. A ring R is said to be self-injective if R as an R-module is injective. A ring R is said to be pre-self-injective if every proper homomorphic image of R is self-injective [**9**]. Study of pre-self-injective rings was initiated by Levy [**10**], who established a characterization of Noetherian pre-self-injective rings with identity in terms of other well-known types of rings. Recently Klatt and Levy [**9**] have characterized all pre-self-injective rings with identity. In this paper we are mainly interested in Noetherian rings. For the sake of convenience we shall call a pre-self-injective ring an (I)-ring. A ring R will be said to be a (PMI)-ring if for each proper prime ideal P with $P^2 \neq 0$, the ring R/P^2 is self-injective. Clearly, an (I)-ring is a (PMI)-ring. However, Example 2 and the remark following it show that a (PMI)-ring need not be an (I)-ring. In § 3, Theorem 2 gives a characterization of all those Noetherian rings R with identity, having the property that for a prime ideal P, R_P is an (I)-ring. Theorem 3 generalizes Levy's theorem to Noetherian rings without identity. In § 4, we study (PMI)-rings. Theorem 5, on (PMI)-rings, is analogous to Levy's theorem. Theorem 6 gives a characterization of all Noetherian (PMI)-rings.

2. Preliminaries. A commutative ring R with identity $1 \neq 0$ is called a special primary ring if it has a unique prime ideal $P \neq R$ such that $P^n = (0)$ for some n and the only ideals of R are $R, P, P^2, \ldots, P^{n-1}$, (0) [**1**, p. 267]. In fact, a ring R with identity is a special primary ring if and only if R is a local, principal ideal ring (PIR) with descending chain condition (DCC). A domain J (may not have identity) is called a primary domain if (0) and J are its only prime ideals [**5**, p. 263]. An ideal A of a ring R is said to be semi-primary if $\sqrt{A}$ is a prime ideal. A ring R is said to satisfy the (*)-condition if every semi-primary ideal of R is primary [**4**, p. 73]. By a proper prime ideal of a ring R we understand any prime ideal different from R. Here we emphasize that the zero ideal of a domain will be treated as a proper prime ideal. A ring R is said to have dimension n if there exists a strictly ascending chain $P_0 < P_1 < P_2 < \ldots < P_n$ of proper prime ideals of R, but no such chain of $(n + 2)$-proper prime ideals exists in R [**4**, p. 73]. A ring R is said to be a u-ring if R is the only ideal of R with radical R. All other terms and terminologies are those of Zariski and Samuel [**11**; **12**], unless otherwise stated.

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3. (I)-rings. Levy [**10**] characterized Noetherian (I)-rings with identity. The following theorem is due to Levy [**10**].

THEOREM 1. *A Noetherian ring R with identity is an* (I)*-ring if and only if one of the following cases holds:*
(i) *R is a Dedekind domain, or*
(ii) *R is a PIR with DCC, or*
(iii) *R is a local ring, whose maximal ideal M has composition length* 2 *and satisfies $M^2 = (0)$.*

As an immediate corollary of the above theorem, we obtain the following lemma.

LEMMA 1. *Let R be a local ring. If R is an* (I)*-ring, then one of the following holds:*
(i) *If R is a domain, then R is a local, Dedekind domain and has at most two proper prime ideals, namely* (0) *and the maximal ideal;*
(ii) *If R is not a domain, then R has exactly one proper prime ideal.*

LEMMA 2. *Let R be a Noetherian ring with identity such that for each proper prime ideal P, the quotient ring R_P is an* (I)*-ring. If P_1 and P_2 are two proper prime ideals of R such that $P_1 < P_2$, then every P_2-primary ideal Q_2 contains P_1, and P_1 is only a P_1-primary ideal.*

Proof. Consider R_{P_2}. For any ideal A of R, let A^e denote its extension in R_{P_2}. Then $P_1^e < P_2^e$ are two proper prime ideals of R_{P_2}. Thus Lemma 1 yields $P_1^e = (0)$. Let Q_2 be any P_2-primary ideal of R; then Q_2^e is a P_2^e-primary ideal of R_{P_2}. Thus $(0) < Q_2^e$. Since there is one-to-one inclusion-preserving correspondence between primary ideals of R contained in P_2 and all primary ideals of R_{P_2} contained in P_2^e given by $Q \leftrightarrow Q^e$ [**11**, p. 225, Corollary 2], we obtain $P_1 < Q_2$ and P_1 is only a P_1-primary ideal of R.

THEOREM 2. *Let R be any Noetherian ring with identity. Then for every proper prime ideal P of R, the quotient ring R_P is an* (I)*-ring if and only if R is a direct sum of a finite number of* (I)*-rings.*

Proof. Let (for each proper prime ideal P) the quotient ring R_P be an (I)-ring. Let $(0) = \bigcap_{i=1}^{n} Q_i$ be an irredundant decomposition of (0) into primary ideals. By Lemma 2, (0) does not have any embedded component. Let $P_i = \sqrt{Q_i}$ for $1 \leqq i \leqq n$. Suppose that $P_i + P_j \neq R$ for some $i \neq j$. We can find a proper prime ideal P containing $P_i + P_j$. Then in R_P, P^e, P_i^e, and P_j^e become three distinct prime ideals. However, this is not possible in view of Lemma 1. Hence $P_i + P_j = R$ whenever $i \neq j$, consequently by [**11**, p. 177, Theorem 31], $Q_i + Q_j = R$ for $i \neq j$ and by [**11**, p. 178, Theorem 32],

$$R \cong \oplus \sum_{i=1}^{n} \frac{R}{Q_i}.$$

For each i, let $R_i = R/Q_i$. Suppose that $P_i = Q_i$ for some i. Let P be any maximal ideal of R containing P_i. Then for $\bar{P} = P/Q_i$, $(R_i)_{\bar{P}} \cong R_P/P_iR_P$. By Lemma 1, R_P/P_iR_P is a local, Dedekind domain. Hence $(R_i)_{\bar{P}}$ is a Dedekind domain for each of its maximal ideals $\bar{P}$ and consequently R_i is a Dedekind domain by [**2**, Theorem 8]. By Theorem 1, R_i is an (I)-ring. Suppose that $P_i \neq Q_i$. By Lemma 2, P_i is a maximal ideal of R. In this case R_i is a local ring isomorphic to $R_{P_i}/Q_iR_{P_i}$. Again R_i is an (I)-ring. Hence R is a direct sum of a finite number of (I)-rings.

Conversely, let $R = \bigoplus \sum_{i=1}^{n} R_i$, where each R_i is an (I)-ring. By Theorem 1, each R_i is either a Dedekind domain or a ring satisfying the DCC. However, every ring with identity satisfying the DCC is a direct sum of a finite number of local rings, each of which also satisfies the DCC [**11**, p. 205, Theorem 3]. Thus we can suppose that each R_i is either a Dedekind domain or a local ring satisfying the DCC. Let P be any proper prime ideal of R. Then there exists a positive integer $i \leqq n$ such that $P = P_i + \sum_{j \neq i} R_j$, where P_i is a proper prime ideal of R_i. Further, R_P is isomorphic to $(R_i)_{P_i}$. If R_i is a Dedekind domain, then $(R_i)_{P_i}$ is also a Dedekind domain and by Theorem 1, $(R_i)_{P_i}$ is an (I)-ring. If R_i is a local ring satisfying the DCC, then $(R_i)_{P_i} \cong R_i$ is an (I)-ring by hypothesis. Hence R_P is an (I)-ring. This completes the proof.

LEMMA 3. *Any self-injective ring has identity.*

Proof. Let R be a self-injective ring. The set $R_1 = \{(a, n) \colon a \in R,\ n$ an integer$\}$ is a ring with identity with addition and multiplication defined by:

$$(a, n) + (b, m) = (a + b, n + m), \qquad (a, n) \cdot (b, m) = (ab + ma + nb, nm),$$

and R is embeddable as an ideal into R_1 by the mapping $a \to (a, 0)$ for all $a \in R$. Each R-module M can be made into a unital R_1-module by defining $x(a, n) = xa + nx$ for every $x \in M$ and $(a, n) \in R_1$. It was proved by Faith and Utumi [**3**, Corollary (1.4)] that any R-module M is injective as an R-module if and only if M satisfies Baer's condition as an R_1-module. Consequently, R is injective as an R_1-module and is a direct summand of R_1. Hence there exists an idempotent $e \in R_1$ such that $R = eR_1$. Then $e \in R$, and e is the identity of R.

The following theorem, which we state without proof, is due to Gilmer and Mott [**6**, Theorem 3].

THEOREM 3. *Let R be a* u-*ring in which the set of proper prime ideals is inductive. Then for all $\pi \in R$, $\pi \in \pi R$. If, in addition, the zero ideal is a product of a finite number of prime ideals, then R has identity.*

Trivially, in a Noetherian ring the family of all proper prime ideals is inductive and the zero ideal is a product of a finite number of prime ideals. Hence we see that any Noetherian u-ring always has identity. Now we prove a theorem which generalizes Levy's theorem to Noetherian rings without identity.

THEOREM 4. *Let R be a Noetherian ring without identity. Then R is an* (I)*-ring if and only if R is a trivial simple ring* (*i.e. a simple ring whose square is zero*).

Proof. Let R be an (I)-ring which is not simple. Then there exists a proper ideal A of R, and R/A is self-injective. By Lemma 3, R/A has identity and therefore $\sqrt{A} \neq R$. Thus R is a Noetherian u-ring, and hence R has identity. This is a contradiction. Hence R is a simple ring. Further, R is trivial, since it has no unity. The converse is obvious.

4. (PMI)-rings. Let us recall that a ring R is said to be a (PMI)-ring if for each proper prime ideal P, R/P^2 is a self-injective ring whenever $P^2 \neq (0)$. The following lemma is due to Gilmer and Mott [**6**, Lemma 3].

LEMMA 4. *If A is an ideal of a ring R such that there is no ideal properly between A and A^2, then the only ideals between A and A^n are $A, A^2, \ldots, A^n$.*

LEMMA 5. *In any ring R, if A is a nil ideal, then any idempotent modulo A in R can be lifted to an idempotent in R.*

Proof. Let $\bar{e} = e + A$ be an idempotent in $\bar{R} = R/A$. Then $(e^2 - e) \in A$ and $(e^2 - e)^n = 0$. By [**7**, Lemma 1.12], $e^n = e^{2n}b$ for some $b \in R$. Let $f = e^n b$; then f is an idempotent in R and $fR = e^n R$. Thus $\bar{f}\bar{R} = \bar{e}^n \bar{R} = \bar{e}\bar{R}$. Hence $\bar{f} = \bar{e}$, since $\bar{R}$ is a commutative ring. This proves the lemma.

LEMMA 6. *Let R be a Noetherian ring and P a proper prime ideal of R such that R/P^2 is a self-injective ring. Then every ideal of R with radical P is a power of P, and is primary. Further, there is no ideal properly between P and P^2.*

Proof. Let Q be any ideal of R such that $\sqrt{Q} = P$. If $P = (0)$, then trivially Q is primary. Suppose that $P \neq (0)$. Now R/P^2 is self-injective and by Lemma 3, R/P^2 has identity. But any Noetherian self-injective ring is a quasi-Frobenius ring and satisfies the DCC [**8**, p. 77, Lemma 2]. Consequently, P is a maximal ideal of R. Then by the lemma proved by Levy [**10**, p. 149] there is no ideal properly between P and P^2. Now for some n, $P^n \subset Q \subset P$. By Lemma 4, $Q = P^k$ for some k. This proves the first part of the lemma. If $k = 1$, then $Q = P$ and trivially Q is primary. Let $k > 1$.

Let $\bar{R} = R/Q = R/P^k$. The Jacobson radical $J(\bar{R})$ of $\bar{R}$ is $\bar{P} = P/P^k$; $J(\bar{R})$ is nilpotent. Hence by Lemma 5 every idempotent modulo $\bar{P}$ in $\bar{R}$ can be lifted to an idempotent in $\bar{R}$. Thus we can find $e \in R$ such that $\bar{e} = e + P^k$ is an idempotent in $\bar{R}$ and e is the identity modulo P. Then $\bar{R} = \bar{e}\bar{R} \oplus (1 - \bar{e})\bar{R}$, where $(1 - \bar{e})\bar{R} = \{\bar{a} - \bar{e}\bar{a}: \bar{a} \in \bar{R}\}$ is contained in $J(\bar{R})$ and is nilpotent. Now $\bar{P} = \bar{A} \oplus (1 - \bar{e})\bar{R}$, for some maximal ideal $\bar{A}$ of $\bar{e}\bar{R}$. Then

$$R/P^2 \cong \bar{R}/\bar{P}^2 \cong [\bar{e}\bar{R}/\bar{A}^2] \oplus [(1 - \bar{e})\bar{R}/((1 - \bar{e})\bar{R})^2].$$

Since $(1 - \bar{e})\bar{R}/[(1 - \bar{e})\bar{R}]^2$ is nilpotent, it cannot have a non-zero idempotent. The above isomorphism together with the fact that R/P^2 has identity yield $(1 - \bar{e})\bar{R} = (\bar{0})$. This implies that $\bar{R} = \bar{e}\bar{R}$ has identity. In a ring with identity, every ideal whose radical is a maximal ideal is primary [**11**, p. 153, Corollary 1]. This implies that $(\bar{0})$ is primary in $\bar{R}$, and hence Q is primary in R. The last part is now immediate.

LEMMA 7. *Let R be a Noetherian (PMI)-ring and P_1, P_2 $(P_1 < P_2)$ any two proper prime ideals of R. Then $P_1 < P_2^n$ for every positive integer n and $P_1 = \bigcap_n P_2^n$.*

Proof. Since $P_1 < P_2$, $P_2^2 \neq (0)$, R/P_2^2 is self-injective. By Lemma 6, there is no ideal properly between P_2 and P_2^2. Since R/P_1 is a Noetherian domain, we have, by Krull's theorem [**11**, p. 216, Theorem 12], $\bigcap_n (P_2/P_1)^n = (\bar{0})$, i.e. $P_1 = \bigcap_n (P_2^n + P_1)$. Thus $P_2^{n+1} + P_1 < P_2^n + P_1$ for every $n \geqq 1$. Since $P_2^n \subset P_2^n + P_1 \subset P_2$, we have, by Lemma 4, $P_2^n + P_1 = P_2^{t_n}$ for some $t_n \geqq 1$. Clearly $t_n \to \infty$ as $n \to \infty$. Hence $P_1 = \bigcap_n P_2^{t_n} = \bigcap_n P_2^n$. This proves the lemma.

The following theorem is analogous to Theorem 1.

THEOREM 5. *Let R be a Noetherian ring with identity. Then R is a (PMI)-ring if and only if it is one of the following types:*

(i) *R is a Dedekind domain, or*

(ii) *R is a PIR satisfying the DCC, or*

(iii) *R is a local ring such that the square of its maximal ideal is zero.*

Proof. Let R be a (PMI)-ring. Suppose that R has a proper prime ideal P such that $P^2 = (0)$. If P is a maximal ideal of R, then R is of type (iii). Suppose that P is not maximal. Consider $\bar{R} = R/P$ and let $\bar{P}' = P'/P$ be any non-zero proper prime ideal of $\bar{R}$. By Lemma 7, $P < P'^2$. Thus $R/P'^2 \cong \bar{R}/\bar{P}'^2$ implies that $\bar{R}/\bar{P}'^2$ is self-injective, and $\bar{R}$ is a (PMI)-domain. By Lemma 6, $\bar{R}$ is a Noetherian domain such that for any non-zero proper prime ideal $\bar{P}'$ there is no ideal properly between $\bar{P}'$ and $\bar{P}'^2$. Hence by [**2**, Theorem 8] $\bar{R}$ is a Dedekind domain. Consider $R_{P'}$; its unique maximal ideal is P'^e, the extension of P' in R_P, and $P^e < (P'^e)^n$ for every n. By Krull's theorem, $\bigcap_n (P'^e)^n = (0)$. Thus $P^e = (0)$. Consider any $x \in P$ and let M be any maximal ideal of R. Then $P < M$ and in R_M, $P^e = (0)$ implies that there exists $y \in R \setminus M$ such that $xy = 0$. Thus $(0)\colon (x) \not\subset M$. Hence $(0)\colon (x) = R$ and $x = 0$. This yields $P = (0)$. Hence R itself is a Dedekind domain and is of type (i).

Now suppose that R has no proper prime ideal P with $P^2 = (0)$. Then for every proper prime ideal P, R/P^2 is self-injective and as seen in Lemma 6, P is maximal. Thus by [**2**, Theorem 1], R satisfies the DCC. Hence $R = \oplus \sum_{i=1}^n R_i$, a direct sum of a finite number of local rings R_i satisfying the DCC. For each i, let P_i be a maximal ideal of R_i. Then $Q_i = P_i + \sum_{j \neq i} R_j$

is a proper prime ideal of R and $R/Q_i^2 \cong R_i/P_i^2$ is self-injective. Further, $P_i^{k_i} = (0)$ for some k_i. By Lemmas 4 and 6, R_i is a special primary ring and thus a PIR. Hence R is a PIR and is of type (ii).

Conversely, if R is of type (i) or (ii), then by Theorem 1, R is an (I)-ring and in particular R is a (PMI)-ring. If R is of type (iii), then the condition defining a (PMI)-ring holds vacuously for R. Hence again R is a (PMI)-ring. This proves the theorem.

We now establish the main theorem of this section.

THEOREM 6. *Let R be a Noetherian ring. Then R is a (PMI)-ring if and only if it is one of the following types:*

(i) *R has a proper prime ideal P such that $P^2 = (0)$ and R/P is a primary domain, or*

(ii) *R is a nilpotent ring, or*

(iii) *R is a Dedekind domain, or*

(iv) *R is a PIR satisfying the DCC.*

Proof. Let R be a (PMI)-ring. Since any field is a primary domain, we have, by Theorem 5, R is of type (iii), (iv) or (i), whenever R has identity. Let R be without identity. If R has no proper prime ideal, then R is nil and hence nilpotent and is of type (ii). Suppose that R has at least one proper prime ideal. Then the following two cases arise.

Case I. *R has a proper prime ideal P with $P^2 = (0)$.* Consider $\bar{R} = R/P$. It follows on the same lines as in the proof of Theorem 5 that $\bar{R}$ is a (PMI)-domain. By Lemma 6, every semi-primary ideal of $\bar{R}$ is primary, i.e. $\bar{R}$ satisfies the (*)-condition. Hence by [**4**, Theorem 7], $\bar{R}$ is either a primary domain or a u-domain of dimension $\leqq 1$. Thus if $\bar{R}$ is a primary domain, then R is of type (i). Let $\bar{R}$ be a u-domain. Since $\bar{R}$ is Noetherian, we see, by Theorem 3, that $\bar{R}$ has identity. Therefore we can find an element $e \in R$ such that $\bar{e} = e + P$ is the identity of $\bar{R}$. Because of Lemma 5, we can suppose that e itself is an idempotent. Now $R = eR \oplus (1 - e)R$, where

$$(1 - e)R = \{x - ex: x \in R\}$$

is contained in P. Now $P = A \oplus (1 - e)R$ for some proper prime ideal A of the ring eR. If P is maximal, then trivially R is of type (i). Suppose that P is not maximal; then A is not maximal in eR and we can find a maximal ideal A' of the ring eR containing A. Further, $M = A' \oplus (1 - e)R$ is a maximal ideal of R, which is prime and satisfies $M^2 \neq (0)$. Thus R/M^2 is self-injective and it has identity. However,

$$\begin{aligned} R/M^2 &\cong [eR/A'^2] \oplus [(1 - e)R/((1 - e)R)^2] \\ &\cong [eR/A'^2] \oplus (1 - e)R, \end{aligned}$$

since $[(1 - e)R]^2 = (0)$. This yields $(1 - e)R = (0)$. Thus $R = eR$ implies

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that R has identity. This is a contradiction. Hence P is a maximal ideal of R, and R is of type (i).

Case II. *For every proper prime ideal* P, $P^2 \neq (0)$. In this case every proper prime ideal of R is maximal; $J(R)$, the Jacobson radical of R, is nilpotent and is an intersection of a finite number of proper prime ideals say $P_1, P_2, \ldots, P_n$. As each P_i is maximal we obtain

$$\frac{R}{J(R)} \cong \oplus \sum_{i=1}^{n} \frac{R}{P_i}.$$

Thus $R/J(R)$ has identity. By Lemma 5, we can find an idempotent e in R such that e is the identity modulo $J(R)$. We obtain $R = eR \oplus T$, where T is a nilpotent ideal and $e \neq 0$. Then $P = P' \oplus T$ for some proper prime ideal P' of the ring eR, and

$$R/P^2 \cong eR/P'^2 \oplus T/T^2.$$

Since R/P^2 has identity and T is nilpotent, the above isomorphism yields $T = (0)$. Hence $R = eR$, and R has identity. This again contradicts the assumption that R has no identity. Hence in any case R is of type (i).

The converse follows immediately from Theorems 1 and 5.

We conclude this paper with some examples and a remark.

Example 1. Let Z be the ring of integers, p any prime number, and n any integer greater than one. The rings $pZ_{(p)}$, $(p)/(p^n)$, Z, and $Z/(n)$ are rings of types (i), (ii), (iii), and (iv), respectively, in Theorem 6.

Example 2. Let K be any field and V a finite-dimensional vector space over K of dimension greater than two. Let $R = \{(\alpha, u): \alpha \in K, u \in V\}$. In R define $(\alpha, u) + (\beta, v) = (\alpha + \beta, u + v)$; $(\alpha, u) \cdot (\beta, v) = (\alpha\beta, \alpha v + \beta u)$. Under these compositions R becomes a local ring with

$$M = \{(0, u): u \in V\}$$

as its maximal ideal such that $M^2 = (0)$. Thus R is a (PMI)-ring. However, the composition length of M is greater than two. Consequently, Theorem 1 shows that R is not an (I)-ring.

Remark. A simple comparison of Theorems 5 and 1 shows that any Noetherian (PMI)-domain with identity is an (I)-ring. However the following brief discussion points out that a non-Noetherian (PMI)-domain with identity need not be an (I)-domain. As defined in [**9**], a ring R with identity is called a valuation ring if for every pair of its elements, one divides the other. A valuation ring R is said to be maximal if every family of pairwise solvable congruences $x \equiv x_\alpha \pmod{J_\alpha}$ (where each $x_\alpha \in R$ and each J_α is an ideal of R) has a simultaneous solution x. A valuation ring R whose every proper homomorphic image is maximal is called an almost maximal valuation ring [**9**, p. 408]. It follows from [**9**, Theorem (3.5)] that a valuation domain D of rank one is an (I)-domain if and only if D is almost maximal. Thus consider any rank-one valuation domain R, which is not almost maximal. R cannot be an (I)-ring and R will not be Noetherian (since a Noetherian rank-one valuation ring is a Dedekind domain and is an (I)-ring). But in a valuation domain R of rank one, which is not Noetherian, the maximal ideal M is idempotent. Thus $R/M^2 = R/M$ is self-injective. This shows that R is a (PMI)-domain, which is not an (I)-domain.

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