Canad. J. Math. Vol. 75 (5), 2023 pp. 1516–1539
http://dx.doi.org/10.4153/S0008414X22000414
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# Global existence of the strong solution to the 3D incompressible micropolar equations with fractional partial dissipation

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*Abstract.* In this paper, we considered the global strong solution to the 3D incompressible micropolar equations with fractional partial dissipation. Whether or not the classical solution to the 3D Navier–Stokes equations can develop finite-time singularity remains an outstanding open problem, so does the same issue on the 3D incompressible micropolar equations. We establish the global-in-time existence and uniqueness strong solutions to the 3D incompressible micropolar equations with fractional partial velocity dissipation and microrotation diffusion with the initial data $(\mathbf{u}_0, \mathbf{w}_0) \in H^1(\mathbb{R}^3)$.

## 1 Introduction and main results

In 1965, Eringen [11] first introduced the micropolar equations in order to model micropolar fluids. Micropolar fluids are fluids with microstructure. They belong to a class of fluids with nonsymmetric stress tensor (called polar fluids) and include, as a special case, the classical fluids modeled by the Navier–Stokes equations (see, e.g., [4, 10, 12, 21]). The system of the micropolar equations is a significant generalization of the Navier–Stokes equations covering many more phenomena such as fluids consisting of particles suspended in a viscous medium (see, e.g., [21, 23, 24]). The micropolar equations have been extensively studied and applied by many engineers and physicists.

The 3D micropolar equations can be stated as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = (\nu + \kappa)\Delta \mathbf{u} + 2\kappa \nabla \times \mathbf{w}, \\ \partial_t \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{w} + 4\kappa \mathbf{w} = \gamma \Delta \mathbf{w} + \mu \nabla \nabla \cdot \mathbf{w} + 2\kappa \nabla \times \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{1.1}$$

Here, $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^3$ denotes the fluid velocity, $\mathbf{w} = \mathbf{w}(x,t) \in \mathbb{R}^3$ the field of microrotation representing the angular velocity of the rotation of the fluid particles, $\pi(x,t)$ the scalar pressure, and the positive parameter $\nu$ denotes the kinematic viscosity, $\kappa$ the microrotation viscosity, and $\gamma, \mu$ the angular viscosities.

The micropolar equations are not only important in physics, but also mathematically significant. The well-posedness problem on the micropolar and closely

Received by the editors May 31, 2022; accepted September 6, 2022.
Published online on Cambridge Core September 13, 2022.
Liu is supported by the Panzhihua University Foundation (Grant No. 035200075).
AMS subject classification: **76B03, 76D03.**
Keywords: Incompressible micropolar, gractional, partial dissipation.

related equations, such as the magneto-micropolar equations, have been extensively investigated (see, e.g., [2, 6, 13, 15, 19–21, 29, 30, 33, 34]). For the initial boundary-value problem, Galdi and Rionero [15] obtained the weak solution. Lukaszewicz [19] established the global existence of weak solutions with sufficiently regular initial data. The existence and uniqueness of strong solutions to the micropolar equations either local for large data or global for small data are considered in [2, 14, 20, 29] and the references therein. However, whether or not the smooth solutions of micropolar equations (1.1) can develop finite-time singularities remains open. Generally speaking, the global regularity problem for the micropolar equations is easier than that for the corresponding incompressible magnetohydrodynamic equations and harder than that for the corresponding incompressible Boussinesq equations. The global existence of weak solutions and strong solutions with initial data small for 3D micropolar equations were obtained in [15, 21].

In the 2D case, the global well-posedness problem on the 2D micropolar equations with full dissipation can be obtained similarly as that for the 2D Navier–Stokes equations (see, e.g., [3, 5, 7, 22, 26]). Recently, a lot of works are focused on the 2D micropolar equations with partial dissipation (see, e.g., [8, 9, 28]). We will apologize for not addressing exhaustive reference in this paper.

When $\mathbf{w} = 0$ and $\kappa = 0$, the system (1.1) is reduced to the 3D incompressible Navier–Stokes equations.

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \nu \Delta \mathbf{u}, \quad x \in \mathbb{R}^3, \ t > 0, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{1.2}$$

Whether or not the classical solutions of the 3D incompressible Navier–Stokes equations (1.2) can develop finite-time singularities remains an outstanding open problem. The Millennium prize problem is supercritical in the sense that the standard Laplacian dissipation in (1.2) may not provide sufficient regularization. Some works (see, e.g., [16, 18]) proved that the following generalized Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = -\nu(-\Delta)^\alpha \mathbf{u}, \quad x \in \mathbb{R}^3, \ t > 0, \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1.3}$$

has a unique global-in-time solution with $\alpha \geq \frac{5}{4}$ and any smooth initial data $\mathbf{u}_0$ which has finite energy. The following reference [27] is also relevant on the generalized Navier–Stokes equations. It gives a very simple proof on the global well-posedness for $\alpha \geq \frac{5}{4}$. Here, the fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

However, some scholars devoted to consider whether or not the global existence and regularity can be constructed for any $\alpha < \frac{5}{4}$. Tao [25] obtained the global regularity for the system which just replace the operator $(-\Delta)^\alpha$ by $\frac{(-\Delta)^{\frac{5}{4}}}{\sqrt{\log(2-\Delta)}}$ in

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(1.3). Replacing the operator $\sqrt{\log(2-\Delta)}$ by $\log(2-\Delta)$, Barbato, Morandin, and Romito [1] improved those result. All of these results imply that it is extremely difficult to reduce $\alpha$ lower than $\frac{5}{4}$. For the system (1.3), $\alpha = \frac{5}{4}$ may be thought as the critical index of the natural energy functional. More precisely, if we assume

$$E(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|_{L^2}^2 + \int_0^t \|\nabla \mathbf{u}\|_{L^2}^2 dt,$$

and inserting $\mathbf{u}_\lambda(x,t) = \lambda^{2\alpha-1}\mathbf{u}(\lambda x,\ \lambda^{2\alpha}t)$ into the above equation to obtain

$$E(\mathbf{u}_\lambda) = \lambda^{4\alpha-5}E(\mathbf{u}),$$

and the natural energy functional is invariant just when $\alpha = \frac{5}{4}$.

Very recently, Yang, Jiu, and Wu [31] studied the global regularity problem on 3D Navier–Stokes equations with partial hyperdissipation. They obtained the global existence and uniqueness of strong solutions.

In this paper, we consider the 3D incompressible micropolar equations with hyperdissipation as follows:

(1.4)
$$\begin{cases} \partial_t u_1 + \mathbf{u}\cdot\nabla u_1 = -\partial_1\pi - (\nu+\kappa)\Lambda_2^{\frac{5}{2}}u_1 - (\nu+\kappa)\Lambda_3^{\frac{5}{2}}u_1 + 2\kappa\varepsilon_{1jk}\partial_j \mathrm{w}_k, \\ \partial_t u_2 + \mathbf{u}\cdot\nabla u_2 = -\partial_2\pi - (\nu+\kappa)\Lambda_1^{\frac{5}{2}}u_2 - (\nu+\kappa)\Lambda_3^{\frac{5}{2}}u_2 + 2\kappa\varepsilon_{2jk}\partial_j \mathrm{w}_k, \\ \partial_t u_3 + \mathbf{u}\cdot\nabla u_3 = -\partial_3\pi - (\nu+\kappa)\Lambda_1^{\frac{5}{2}}u_3 - (\nu+\kappa)\Lambda_2^{\frac{5}{2}}u_3 + 2\kappa\varepsilon_{3jk}\partial_j \mathrm{w}_k, \\ \partial_t \mathrm{w}_1 + \mathbf{u}\cdot\nabla \mathrm{w}_1 + 4\kappa\mathrm{w}_1 = -\gamma\Lambda_2^{\frac{5}{2}}\mathrm{w}_1 - \gamma\Lambda_3^{\frac{5}{2}}\mathrm{w}_1 + \mu\partial_1(\nabla\cdot\mathbf{w}) + 2\kappa\varepsilon_{1jk}\partial_j u_k, \\ \partial_t \mathrm{w}_2 + \mathbf{u}\cdot\nabla \mathrm{w}_2 + 4\kappa\mathrm{w}_2 = -\gamma\Lambda_1^{\frac{5}{2}}\mathrm{w}_2 - \gamma\Lambda_3^{\frac{5}{2}}\mathrm{w}_2 + \mu\partial_2(\nabla\cdot\mathbf{w}) + 2\kappa\varepsilon_{2jk}\partial_j u_k, \\ \partial_t \mathrm{w}_3 + \mathbf{u}\cdot\nabla \mathrm{w}_3 + 4\kappa\mathrm{w}_3 = -\gamma\Lambda_1^{\frac{5}{2}}\mathrm{w}_3 - \gamma\Lambda_2^{\frac{5}{2}}\mathrm{w}_3 + \mu\partial_3(\nabla\cdot\mathbf{w}) + 2\kappa\varepsilon_{3jk}\partial_j u_k, \\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}(x,0) = \mathbf{u}_0(x),\ \mathbf{w}(x,0) = \mathbf{w}_0(x). \end{cases}$$

Here, $\mathbf{u} = (u_1,\ u_2,\ u_3)$ denotes the velocity field and $\mathbf{w} = (\mathrm{w}_1,\ \mathrm{w}_2,\ \mathrm{w}_3)$ the microrotation field. $\varepsilon_{ijk}, (i,j,k) \in \{1,2,3\}$ is Levi-Civita alternating tensor defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is an even permutation}, \\ -1, & \text{if } (i,j,k) \text{ is an odd permutation}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.5}$$

Here, $\Lambda_k^\alpha$ with $\alpha > 0$ and $k = 1,2,3$ denote the directional fractional operators defined via the Fourier transform

$$\widehat{\Lambda_k^\alpha f}(\xi) = |\xi_k|^\alpha \widehat{f}(\xi),\ k = 1,2,3,$$

where $\xi = (\xi_1,\ \xi_2,\ \xi_3)$ and $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator.

The main results of this paper are stated as follows.

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***Theorem 1.1*** *Assume* $(\mathbf{u}_0,\ \mathbf{w}_0) \in H^1(\mathbb{R}^3)$. *Then, system* (1.4) *has a global strong solution* $(\mathbf{u},\ \mathbf{w})$ *satisfying*

$$
\begin{aligned}
&(\mathbf{u},\ \mathbf{w}) \in L^\infty(0,\infty;\ H^1),\\
&\Lambda_2^{\frac{5}{4}} u_1,\ \Lambda_3^{\frac{5}{4}} u_1,\ \Lambda_2^{\frac{5}{4}} \nabla u_1,\ \Lambda_3^{\frac{5}{4}} \nabla u_1 \in L^2(0,\infty;\ L^2),\\
&\Lambda_1^{\frac{5}{4}} u_2,\ \Lambda_3^{\frac{5}{4}} u_2,\ \Lambda_1^{\frac{5}{4}} \nabla u_2,\ \Lambda_3^{\frac{5}{4}} \nabla u_2 \in L^2(0,\infty;\ L^2),\\
&\Lambda_1^{\frac{5}{4}} u_3,\ \Lambda_2^{\frac{5}{4}} u_3,\ \Lambda_1^{\frac{5}{4}} \nabla u_3,\ \Lambda_2^{\frac{5}{4}} \nabla u_3 \in L^2(0,\infty;\ L^2),\\
&\Lambda_2^{\frac{5}{4}} w_1,\ \Lambda_3^{\frac{5}{4}} w_1,\ \Lambda_2^{\frac{5}{4}} \nabla w_1,\ \Lambda_3^{\frac{5}{4}} \nabla w_1 \in L^2(0,\infty;\ L^2),\\
&\Lambda_1^{\frac{5}{4}} w_2,\ \Lambda_3^{\frac{5}{4}} w_2,\ \Lambda_1^{\frac{5}{4}} \nabla w_2,\ \Lambda_3^{\frac{5}{4}} \nabla w_2 \in L^2(0,\infty;\ L^2),\\
&\Lambda_1^{\frac{5}{4}} w_3,\ \Lambda_2^{\frac{5}{4}} w_3,\ \Lambda_1^{\frac{5}{4}} \nabla w_3,\ \Lambda_2^{\frac{5}{4}} \nabla w_3 \in L^2(0,\infty;\ L^2).
\end{aligned}
\tag{1.6}
$$

*The bound of* $(\mathbf{u},\ \mathbf{w})$ *in* (1.6) *is uniform in time.*

***Remark 1.1*** Due to the symmetric, one can easily check the similar results as Theorem 1.1 holds for the cases that if $(u_1,\ \mathrm{w}_1)$ are only lack of the hyperdissipation in the $x_2$ direction, $(u_2,\ \mathrm{w}_2)$ are only lack of the hyperdissipation in the $x_3$ direction and $(u_3,\ \mathrm{w}_3)$ are only lack of the hyperdissipation in the $x_1$ direction or $(u_1,\ \mathrm{w}_1)$ are only lack of the hyperdissipation in the $x_3$ direction, $(u_2,\ \mathrm{w}_2)$ are only lack of the hyperdissipation in the $x_1$ direction and $(u_3,\ \mathrm{w}_3)$ are only lack of the hyperdissipation in the $x_2$ direction.

The rest of this paper is arranged as follows: Some notation and preliminaries will be given in Section 2. In Section 3, we will prove our main results. The proof of Theorem 1.1 will be divided into three stages. First, we will show the $L^2$-bound of $(\mathbf{u},\ \mathbf{w})$. Second, we will obtain the $L^2$-bound of $(\nabla\mathbf{u},\ \nabla\mathbf{w})$, and then we establish the global a priori bound for $(\mathbf{u},\ \mathbf{w})$ in $H^1$. This section is the main parts of the proof of Theorem 1.1. There are a lot of triple product terms bounded by using divergence-free condition, Sobolev's inequalities, Minkowski's inequality, and so forth. Finally, we will prove the uniqueness.

## 2 Notation and preliminaries

For simplicity, some notations will be introduced before we prove our main results, which are used throughout this paper. We denote

$$
\begin{aligned}
&\|f\|_{L^2(\mathbb{R}^3)} = \|f\|_2,\ \frac{\partial f}{\partial x_i} = \partial_i f,\\
&\|\Lambda_i^{\frac{5}{4}} f_k\|_{L^2(\mathbb{R}^3)} + \|\Lambda_j^{\frac{5}{4}} f_k\|_{L^2(\mathbb{R}^3)} \triangleq \|(\Lambda_i^{\frac{5}{4}},\ \Lambda_j^{\frac{5}{4}}) f_k\|_2\ \ (i,j,k) \in \{1,2,3\},\\
&\|\partial_i f_k\|_{L^2(\mathbb{R}^3)} + \|\partial_j f_k\|_{L^2(\mathbb{R}^3)} \triangleq \|(\partial_i,\ \partial_j) f_k\|_2\ \ (i,j,k) \in \{1,2,3\},\\
&\int f\,dxdydz = \iiint_{\mathbb{R}^3} f\,dxdydz,
\end{aligned}
$$

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and

$$\|f_1, f_2, \dots, f_n\|_{L^2(\mathbb{R}^3)}^2 = \|f_1\|_2^2 + \|f_2\|_2^2 + \cdots + \|f_n\|_2^2.$$

We denote the one-dimensional $L^2$-norm with respect to $x_i$ by $\|f\|_{L^2_{x_i}}$ $(i = 1, 2, 3)$ and $\|f\|_{L^2_{x_i x_j}}$ $(i, j \in \{1, 2, 3\})$ denote the two-dimensional $L^2$-norm with respect to $x_i$ and $x_j$. In addition, we denote

$$\|f\|_{L^s_{x_i} L^q_{x_j} L^p_{x_k}} \triangleq \big\| \big\| \|f\|_{L^p_{x_k}} \big\|_{L^q_{x_j}} \big\|_{L^s_{x_i}}.$$

The following lemma is Minkowski's inequality (see, e.g., [17]), which will be useful.

***Lemma 2.1*** *Assume that $f = f(x, y)$ with $(x, y) \in (\mathbb{R}^m \times \mathbb{R}^n)$ is a measurable function. Let $1 \le q \le p \le \infty$. Then,*

$$\big\| \|f\|_{L^q_y(\mathbb{R}^n)} \big\|_{L^p_x(\mathbb{R}^m)} \le \big\| \|f\|_{L^p_x(\mathbb{R}^m)} \big\|_{L^q_y(\mathbb{R}^n)}. \tag{2.1}$$

The next lemma is the Sobolev embedding inequality, which will be used frequently in this paper (see [32]).

***Lemma 2.2*** *Assume that $2 \le p \le \infty$ and $s > d\left(\frac{1}{2} - \frac{1}{p}\right)$. Then, there exists a constant $C = C(d, p, s)$ such that, for any d-dimensional functions $f \in H^s(\mathbb{R}^d)$,*

$$\|f\|_{L^p(\mathbb{R}^d)} \le C \|f\|_{L^2(\mathbb{R}^d)}^{1 - \frac{d}{s}(\frac{1}{2} - \frac{1}{p})} \|\Lambda^s f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{s}(\frac{1}{2} - \frac{1}{p})}. \tag{2.2}$$

*In particular, when $p \ne \infty$, (2.2) also holds for $s = d\left(\frac{1}{2} - \frac{1}{p}\right)$.*

The following is the Hölder-type inequality, which will be useful as well.

***Lemma 2.3*** *Assume that $f_1,\ f_2 \ge 0$ and $f_1,\ f_2 \in L^p$. Assume that $s_1,\ s_2 \in [0, 1]$ and $s_1 + s_2 = 1$. Assume that $1 \le p \le \infty$, then*

$$\|f_1^{s_1} f_2^{s_2}\|_{L^p} \le \|f_1\|_{L^p}^{s_1} \|f_2\|_{L^p}^{s_2}. \tag{2.3}$$

## 3 Global regularity for the strong solution to the 3D incompressible micropolar fluid flows

In this section, we will prove Theorem 1.1. Theorem 1.1 is proved through three stages. The first step is to establish the $L^2$-estimate of $(\mathbf{u}, \mathbf{w})$. Second, we will obtain the $H^1$-bound for $(\mathbf{u}, \mathbf{w})$. Finally, we will achieve the uniqueness of $(\mathbf{u}, \mathbf{w})$.

Step 1. Global $L^2$-bound.

***Proposition 3.1*** *Suppose that $(\mathbf{u}_0,\ \mathbf{w}_0) \in H^1(\mathbb{R}^3)$. Then, system (1.4) has a global solution $(\mathbf{u}, \mathbf{w})$ satisfying*

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$$\begin{aligned}&\|\mathbf{u},\ \mathbf{w}\|_2^2+4\kappa\|\mathbf{w}\|_2^2+\mu\|div\mathbf{w}\|_2^2+(\nu+\kappa)\\&\times\int_0^T\|(\Lambda_2^{\frac54},\ \Lambda_3^{\frac54})u_1,(\Lambda_1^{\frac54},\ \Lambda_3^{\frac54})u_2,(\Lambda_1^{\frac54},\ \Lambda_2^{\frac54})u_3\|_2^2\,dt\\&+\gamma\int_0^T\|(\Lambda_2^{\frac54},\ \Lambda_3^{\frac54})w_1,(\Lambda_1^{\frac54},\ \Lambda_3^{\frac54})w_2,(\Lambda_1^{\frac54},\ \Lambda_2^{\frac54})w_3\|_2^2\,dt\le C,\end{aligned}\tag{3.1}$$

*for any* $T>0$, *where* $C>0$ *is a constant, depending on* $\|\mathbf{u}_0,\mathbf{w}_0\|_2^2$.

**Proof** Multiplying equations $(1.4)_1$, $(1.4)_2$, $(1.4)_3$, $(1.4)_4$, $(1.4)_5$, and $(1.4)_6$ by $u_1$, $u_2$, $u_3$, $\mathrm{w}_1$, $\mathrm{w}_2$, and $\mathrm{w}_3$, respectively, and taking the $L^2$-inner product, integrating by parts, using the divergence-free condition $\nabla\cdot\mathbf{u}=0$ and adding them together, yield that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\mathbf{u},\ \mathbf{w}\|_2^2+4\kappa\|\mathbf{w}\|_2^2+\mu\|\mathrm{div}\mathbf{w}\|_2^2\\&\quad+(\nu+\kappa)\|(\Lambda_2^{\frac54},\ \Lambda_3^{\frac54})u_1,\ (\Lambda_1^{\frac54},\ \Lambda_3^{\frac54})u_2,\ (\Lambda_1^{\frac54},\ \Lambda_2^{\frac54})u_3\|_2^2\\&\quad+\gamma\|(\Lambda_2^{\frac54},\ \Lambda_3^{\frac54})\mathrm{w}_1,\ (\Lambda_1^{\frac54},\ \Lambda_3^{\frac54})\mathrm{w}_2,\ (\Lambda_1^{\frac54},\ \Lambda_2^{\frac54})\mathrm{w}_3\|_2^2=4\kappa\int\nabla\times\mathbf{w}\cdot\mathbf{u}dxdydz.\end{aligned}\tag{3.2}$$

The right-hand side of (3.2) can be estimated as

$$\begin{aligned}4\kappa\int\nabla\times\mathbf{w}\cdot\mathbf{u}dxdydz&=4\kappa\int(\varepsilon_{1jk}\partial_j\mathrm{w}_ku_1+\varepsilon_{2jk}\partial_j\mathrm{w}_ku_2+\varepsilon_{3jk}\partial_j\mathrm{w}_ku_3)dxdydz\\&=I_1+I_2+I_3.\end{aligned}$$

To begin with the term $I_1$, applying Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}I_1&=4\kappa\int(\partial_2\mathrm{w}_3-\partial_3\mathrm{w}_2)u_1\,dxdydz\\&\le C\|u_1\|_2(\|\partial_2\mathrm{w}_3\|_2+\|\partial_3\mathrm{w}_2)\|_2\\&\le C\|u_1\|_2(\|\mathrm{w}_3\|_2^{\frac15}\|\Lambda_2^{\frac54}\mathrm{w}_3\|_2^{\frac45}+\|\mathrm{w}_2\|_2^{\frac15}\|\Lambda_3^{\frac54}\mathrm{w}_2\|_2^{\frac45})\\&\le\varepsilon\|\Lambda_3^{\frac54}\mathrm{w}_2,\ \Lambda_2^{\frac54}\mathrm{w}_3\|_2^2+C_\varepsilon\|u_1\|_2^2.\end{aligned}\tag{3.3}$$

Similarly,

$$\begin{aligned}I_2&=4\kappa\int(\partial_3\mathrm{w}_1-\partial_1\mathrm{w}_3)u_2\,dxdydz\\&\le C\|u_2\|_2(\|\mathrm{w}_1\|_2^{\frac15}\|\Lambda_3^{\frac54}\mathrm{w}_1\|_2^{\frac45}+\|\mathrm{w}_3\|_2^{\frac15}\|\Lambda_1^{\frac54}\mathrm{w}_3\|_2^{\frac45})\\&\le\varepsilon\|\Lambda_3^{\frac54}\mathrm{w}_1,\ \Lambda_1^{\frac54}\mathrm{w}_3\|_2^2+C_\varepsilon\|u_2\|_2^2,\end{aligned}\tag{3.4}$$

and

$$I_3=4\kappa\int(\partial_1\mathrm{w}_2-\partial_2\mathrm{w}_1)u_3\,dxdydz\le\varepsilon\|\Lambda_1^{\frac54}\mathrm{w}_2,\ \Lambda_2^{\frac54}\mathrm{w}_1\|_2^2+C_\varepsilon\|u_3\|_2^2.\tag{3.5}$$

Inserting the above inequalities (3.3)–(3.5) into (3.2), choosing $\varepsilon$ small enough, and integrating from 0 to $T>0$ yield the desired estimate (3.1).

Step 2. Global $H^1$-bound.

The goal of this section is to establish the global $L^2$-estimate of $(\nabla \mathbf{u},\ \nabla \mathbf{w})$. The process of this section is more complex. ∎

***Proposition 3.2*** *Suppose that* $(\mathbf{u}_0,\ \mathbf{w}_0) \in H^1(\mathbb{R}^3)$. *Then, system* (1.4) *has a global solution* $(\mathbf{u}, \mathbf{w})$ *satisfying*

$$\begin{aligned} &\|\nabla \mathbf{u},\ \nabla \mathbf{w}\|_2^2 + 4\kappa \|\nabla \mathbf{w}\|_2^2 + \mu \|\nabla div \mathbf{w}\|_2^2 \\ &\quad + (\nu+\kappa)\int_0^T \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2\, dt \\ &\quad + \gamma \int_0^T \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla w_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla w_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla w_3\|_2^2\, dt \le C, \end{aligned} \tag{3.6}$$

*for any* $T > 0$, *where* $C > 0$ *is a constant, depending on* $\|\mathbf{u}_0, \mathbf{w}_0\|_{H^1(\mathbb{R}^3)}^2$.

**Proof** In order to obtain the global $H^1$-bound, we apply the operator $\nabla$ to system (1.4), and taking the inner product by the resulting equations with $\nabla \mathbf{u}$ and $\nabla \mathbf{w}$, respectively, integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u},\ \nabla \mathbf{w}\|_2^2 + 4\kappa \|\nabla \mathbf{w}\|_2^2 + \mu \|\nabla \mathrm{div} \mathbf{w}\|_2^2 \\ &\quad + (\nu+\kappa)\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2 \\ &\quad + \gamma \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla \mathrm{w}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla \mathrm{w}_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla \mathrm{w}_3\|_2^2 \\ &= -\int \nabla(\mathbf{u}\cdot\nabla \mathbf{u})\cdot \nabla \mathbf{u}\, dxdydz - \int \nabla(\mathbf{u}\cdot\nabla \mathbf{w})\cdot\nabla \mathbf{w}\, dxdydz \\ &\quad + 4\kappa \int \nabla\nabla\times \mathbf{u}\cdot\nabla \mathbf{w}\, dxdydz \\ &= J_1 + J_2 + J_3. \end{aligned} \tag{3.7}$$

Due to the divergence-free condition $\nabla\cdot \mathbf{u} = 0$, integrating by parts, we can rewrite $J_1$ and $J_2$ as follows:

$$J_1 = -\int \nabla(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{u}\, dxdydz = -\int \partial_i u_k \partial_k u_j \partial_i u_j\, dxdydz$$

and

$$J_2 = -\int \nabla(\mathbf{u}\cdot\nabla\mathbf{w})\cdot\nabla\mathbf{w}\, dxdydz = -\int \partial_i u_k \partial_k \mathrm{w}_j \partial_i \mathrm{w}_j\, dxdydz.$$

To start with the term $J_1$, similar to [32], we consider the first nine terms in $J_1$ denoted by $J_{11}$. The remaining terms in $J_1$ can be handled similar to $J_{11}$.

$$\begin{aligned} J_{11} = &-\int ((\partial_1 u_1)^3 + \partial_1 u_1 \partial_1 u_2 \partial_2 u_1 + \partial_1 u_1 \partial_1 u_3 \partial_3 u_1 \\ &+ (\partial_2 u_1)^2 \partial_1 u_1 + (\partial_2 u_1)^2 \partial_2 u_2 + \partial_2 u_1 \partial_2 u_3 \partial_3 u_1 \\ &+ (\partial_3 u_1)^2 \partial_1 u_1 + \partial_3 u_1 \partial_3 u_2 \partial_2 u_1 + (\partial_3 u_1)^2 \partial_3 u_3)\, dxdydz \\ = &\sum_{m=1}^{9} J_{11m}. \end{aligned} \tag{3.8}$$

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For the term $J_{111}$, integrating by parts and applying Lemmas 2.2 and 2.3, using the divergence-free condition $\nabla\cdot\mathbf{u}=0$, one has

$$\begin{aligned} J_{111} &= -\int((\partial_1 u_1)^3\,dxdydz = 2\int u_1\partial_{11}u_1\partial_1 u_1\,dxdydz\\ &\le C\|\partial_{11}u_1\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_1 u_1\|_{L^4_{x_1}L^\infty_{x_2}L^2_{x_3}}\|u_1\|_{L^2_{x_1x_2}L^\infty_{x_3}}\\ &\le \|\Lambda_1^{\frac54}\partial_1 u_1\|_2\|\Lambda_1^{\frac54}u_1\|_{L^2_{x_1x_3}L^\infty_{x_2}}\|u_1\|_2^{\frac35}\|\Lambda_3^{\frac54}u_1\|_2^{\frac25}\\ &\le \varepsilon\|\Lambda_1^{\frac54}\partial_1 u_1\|_2^2 + C_\varepsilon\|u_1\|_2^{\frac65}\|\Lambda_3^{\frac54}u_1\|_2^{\frac45}\|\Lambda_1^{\frac54}u_1\|_2\|\Lambda_1^{\frac54}\Lambda_2 u_1\|_2\\ &\le \varepsilon(\|\Lambda_1^{\frac54}\partial_1 u_1\|_2^2+\|\Lambda_1^{\frac54}\Lambda_2 u_1\|_2)+C_\varepsilon(\|u_1\|_2^2+\|\Lambda_3^{\frac54}u_1\|_2^2)^2(\|u_1\|_2^{\frac89}\|\Lambda_1^{\frac54}\partial_1 u_1\|_2^{\frac{10}{9}})\\ &\le 4\varepsilon\|\Lambda_1^{\frac54}\partial_2 u_2,\ \Lambda_1^{\frac54}\partial_3 u_3,\ \Lambda_2^{\frac54}\partial_1 u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2+\|\Lambda_3^{\frac54}u_1\|_2^2)^{\frac92}\|u_1\|_2^2. \end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned} J_{112} &= -\int \partial_1 u_1\partial_1 u_2\partial_2 u_1\,dxdydz = \int u_1\partial_{12}u_1\partial_1 u_2\,dxdydz + \int u_1\partial_{12}u_2\partial_1 u_1\,dxdydz\\ &= J_{1121}+J_{1122}. \end{aligned}$$

Using Lemmas 2.1 and 2.3, we can bound the term $J_{1121}$ as follows:

$$\begin{aligned} |J_{1121}| &\le C\|\partial_{12}u_1\|_{L^4_{x_2}L^2_{x_1x_3}}\|\partial_1 u_2\|_{L^4_{x_1}L^2_{x_2}L^\infty_{x_3}}\|u_1\|_{L^4_{x_1x_2}L^2_{x_3}}\\ &\le C\|\Lambda_2^{\frac54}\partial_1 u_1\|_2\|\Lambda_1^{\frac54}u_2\|_2^{\frac12}\|\Lambda_1^{\frac54}\Lambda_3 u_2\|_2^{\frac12}\|u_1\|_2^{\frac12}\|\partial_1 u_1,\ \partial_2 u_1\|_2^{\frac12}\\ &\le \varepsilon(\|\Lambda_2^{\frac54}\partial_1 u_1\|_2^2+\|\Lambda_1^{\frac54}\partial_3 u_2\|_2^2)+C_\varepsilon(\|u_1\|_2^2+\|\Lambda_1^{\frac54}u_2\|_2^2)^2\|\nabla\mathbf{u}\|_2^2 \end{aligned}$$

and

$$\begin{aligned} |J_{1122}| &\le C\|\partial_{12}u_2\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_1 u_1\|_{L^4_{x_1}L^2_{x_2}L^\infty_{x_3}}\|u_1\|_{L^2_{x_1x_3}L^\infty_{x_2}}\\ &\le C\|\Lambda_1^{\frac54}\partial_2 u_2\|_2\|\Lambda_1^{\frac54}u_1\|_2^{\frac12}\|\Lambda_1^{\frac54}\Lambda_3 u_1\|_2^{\frac12}\|u_1\|_2^{\frac35}\|\Lambda_2{}^{\frac54}u_1\|_2^{\frac25}\\ &\le \varepsilon\|\Lambda_1^{\frac54}\partial_2 u_2\|_2^2 + C_\varepsilon\|u_1\|_2^{\frac65}\|\Lambda_2{}^{\frac54}u_1\|_2^{\frac45}\|\Lambda_1^{\frac54}u_1\|_2\|\Lambda_1^{\frac54}\Lambda_3 u_1\|_2\\ &\le 4\varepsilon\|\Lambda_1^{\frac54}\partial_2 u_2,\ \Lambda_1^{\frac54}\partial_3 u_3,\ \Lambda_3^{\frac54}\partial_1 u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2+\|\Lambda_2^{\frac54}u_1\|_2^2)^{\frac92}\|u_1\|_2^2. \end{aligned}$$

Combining the above two estimates, we obtain

$$\begin{aligned} |J_{112}| &\le 4\varepsilon\|(\Lambda_2^{\frac54},\ \Lambda_3^{\frac54})\nabla u_1,\ (\Lambda_1^{\frac54},\ \Lambda_3^{\frac54})\nabla u_2,\ (\Lambda_1^{\frac54},\ \Lambda_2^{\frac54})\nabla u_3\|_2^2\\ &\quad + C_\varepsilon(\|u_1\|_2^2+\|\Lambda_2^{\frac54}u_1\|_2^2)^{\frac92}\|u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2+\|\Lambda_1^{\frac54}u_2\|_2^2)^2\|\nabla\mathbf{u}\|_2^2. \end{aligned} \tag{3.10}$$

Next, we will bound the term $J_{113}$, and integrating by parts, we find that

$$\begin{aligned} J_{113} &= -\int \partial_1 u_1\partial_1 u_3\partial_3 u_1\,dxdydz = \int u_1\partial_{13}u_1\partial_1 u_3\,dxdydz + \int u_1\partial_{13}u_3\partial_1 u_1\,dxdydz\\ &= J_{1131}+J_{1132}. \end{aligned}$$

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Similar to $J_{1121}$, one has

$$
\begin{aligned}
|J_{1131}| &\le C\|\partial_{13}u_1\|_{L^4_{x_3}L^2_{x_1x_2}}\|\partial_1 u_3\|_{L^4_{x_1}L^\infty_{x_2}L^2_{x_3}}\|u_1\|_{L^4_{x_1x_3}L^2_{x_3}}\\
&\le C\|\Lambda_3^{\frac{5}{4}}\partial_1 u_1\|_2\|\Lambda_1^{\frac{5}{4}}u_3\|_2^{\frac{1}{2}}\|\Lambda_1^{\frac{5}{4}}\Lambda_3 u_3\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{1}{2}}\|(\partial_1,\ \partial_3)u_1\|_2^{\frac{1}{2}}\\
&\le \varepsilon\|\Lambda_3^{\frac{5}{4}}\partial_1 u_1,\ \Lambda_1^{\frac{5}{4}}\partial_3 u_3\|_2^2 + C_\varepsilon\|u_1\|_2^2 + \|\Lambda_1^{\frac{5}{4}}u_3\|_2^2)^2\|\nabla u_1\|_2^2.
\end{aligned}
$$

Applying the similar method to $J_{1122}$, we have

$$
\begin{aligned}
|J_{1132}| &\le C\|\partial_{13}u_3\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_1 u_2\|_{L^4_{x_1}L^2_{x_2}L^\infty_{x_3}}\|u_1\|_{L^2_{x_1x_3}L^\infty_{x_2}}\\
&\le C\|\Lambda_1^{\frac{5}{4}}\partial_3 u_3\|_2\|\Lambda_1^{\frac{5}{4}}u_1\|_2^{\frac{1}{2}}\|\Lambda_1^{\frac{5}{4}}\Lambda_3 u_1\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{3}{5}}\|\Lambda_2^{\frac{5}{4}}u_1\|_2^{\frac{2}{5}}\\
&\le 4\varepsilon\|\Lambda_1^{\frac{5}{4}}\partial_2 u_2,\ \Lambda_1^{\frac{5}{4}}\partial_3 u_3,\ \Lambda_3^{\frac{5}{4}}\partial_1 u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_1\|_2^2)^{\frac{9}{2}}\|u_1\|_2^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|J_{113}| &\le 6\varepsilon\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2\\
&\quad + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_1\|_2^2)^{\frac{9}{2}}\|u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_1^{\frac{5}{4}}u_3\|_2^2)^2\|\nabla\mathbf{u}\|_2^2. \qquad (3.11)
\end{aligned}
$$

Integrating by parts and invoking the divergence-free condition $\nabla\cdot\mathbf{u} = 0$, one has

$$
\begin{aligned}
J_{114} + J_{115} &= -\int(\partial_2 u_1)^2(\partial_1 u_1 + \partial_2 u_2)\,dxdydz\\
&= \int(\partial_2 u_1)^2\partial_3 u_3\,dxdydz = -2\int u_3\partial_{23}u_1\partial_2 u_1\,dxdydz\\
&\le C\|\partial_{23}u_1\|_{L^4_{x_2}L^2_{x_1x_3}}\|\partial_2 u_1\|_{L^4_{x_2}L^2_{x_1}L^\infty_{x_3}}\|u_3\|_{L^2_{x_2x_3}L^\infty_{x_1}}\\
&\le C\|\Lambda_2^{\frac{5}{4}}\partial_3 u_1\|_2\|\Lambda_2^{\frac{5}{4}}u_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{5}{4}}\Lambda_3 u_1\|_2^{\frac{1}{2}}\|u_3\|_2^{\frac{1}{2}}\|\Lambda_1 u_3\|_2^{\frac{1}{2}}\\
&\le 2\varepsilon\|\Lambda_2^{\frac{5}{4}}\partial_3 u_1\|_2^2 + C_\varepsilon(\|u_3\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_1\|_2^2)^2\|\partial_1 u_3\|_2^2. \qquad (3.12)
\end{aligned}
$$

To bound the term $J_{116}$, integrating by parts, one yields that

$$
\begin{aligned}
J_{116} &= \int u_1\partial_{23}u_3\partial_2 u_1\,dxdydz + \int u_1\partial_{23}u_1\partial_2 u_3\,dxdydz\\
&= J_{1161} + J_{1162}.
\end{aligned}
$$

Applying the similar method to $J_{1131}$, one can easily find that

$$
\begin{aligned}
|J_{1161}| &\le C\|\partial_{23}u_3\|_{L^4_{x_2}L^2_{x_1x_3}}\|\partial_2 u_1\|_{L^4_{x_2}L^\infty_{x_3}L^2_{x_1}}\|u_1\|_{L^2_{x_2x_3}L^\infty_{x_1}}\\
&\le C\|\Lambda_2^{\frac{5}{4}}\partial_3 u_3\|_2\|\Lambda_2^{\frac{5}{4}}u_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{5}{4}}\Lambda_3 u_1\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{1}{2}}\|\Lambda_1 u_1\|_2^{\frac{1}{2}}\\
&\le \varepsilon\|\Lambda_2^{\frac{5}{4}}\partial_3 u_3,\ \Lambda_3^{\frac{5}{4}}\partial_3 u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_1\|_2^2)^2\|\nabla u_1\|_2^2
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|J_{1162}| &\le C\|\partial_{23}u_1\|_{L^4_{x_2}L^2_{x_1x_3}}\|\partial_2 u_3\|_{L^4_{x_2}L^\infty_{x_3}L^2_{x_1}}\|u_1\|_{L^2_{x_2x_3}L^\infty_{x_1}}\\
&\le \varepsilon\|\Lambda_2^{\frac{5}{4}}\partial_3 u_1,\ \Lambda_2^{\frac{5}{4}}\partial_3 u_3\|_2^2 + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_3\|_2^2)^2\|\nabla u_1\|_2^2.
\end{aligned}
$$

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Furthermore,

$$|J_{116}| \le 2\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\nabla u_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\nabla u_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2 + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_3\|_2^2)^2\|\nabla u_1\|_2^2. \tag{3.13}$$

Due to the divergence-free condition $\nabla \cdot \mathbf{u} = 0$, we will rewrite the last three terms as follows:

$$-\int (\partial_3 u_1 \partial_3 u_2 \partial_2 u_1 - (\partial_3 u_1)^2 \partial_2 u_2)\, dxdydz := J_{117} + J_{118}.$$

Integrating by parts, the term $J_{117}$ can be bounded as

$$\begin{aligned} J_{117} &= \int u_1 \partial_{23} u_1 \partial_3 u_2 \, dxdydz + \int u_1 \partial_{23} u_2 \partial_3 u_1 \, dxdydz \\ &\le C\|\partial_{23}u_1\|_{L^4_{x_3}L^2_{x_1x_2}} \|\partial_3 u_2\|_{L^4_{x_3}L^\infty_{x_1}L^2_{x_2}} \|u_1\|_{L^2_{x_1x_3}L^\infty_{x_2}} \\ &\quad + C\|\partial_{23}u_2\|_{L^4_{x_3}L^2_{x_1x_2}} \|\partial_3 u_1\|_{L^4_{x_3}L^\infty_{x_2}L^2_{x_1}} \|u_1\|_{L^2_{x_2x_3}L^\infty_{x_1}} \\ &\le C\|\Lambda_3^{\frac{5}{4}}\partial_2 u_1\|_2 \|\Lambda_3^{\frac{5}{4}}u_2\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{5}{4}}\Lambda_1 u_2\|_2^{\frac{1}{2}} \|u_1\|_2^{\frac{1}{2}} \|\Lambda_2 u_1\|_2^{\frac{1}{2}} \\ &\quad + C\|\Lambda_3^{\frac{5}{4}}\partial_2 u_2\|_2 \|\Lambda_3^{\frac{5}{4}}u_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{5}{4}}\Lambda_2 u_2\|_2^{\frac{1}{2}} \|u_1\|_2^{\frac{1}{2}} \|\Lambda_1 u_1\|_2^{\frac{1}{2}} \\ &\le 2\varepsilon\|\Lambda_3^{\frac{5}{4}}\nabla u_2, \Lambda_3^{\frac{5}{4}}\nabla u_1\|_2^2 + C_\varepsilon(\|u_1\|_2^2 + \|\Lambda_3^{\frac{5}{4}}u_1\|_2^2 + \|\Lambda_3^{\frac{5}{4}}u_2\|_2^2)^2\|\nabla u_1\|_2^2. \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} J_{118} &= -2\int u_2 \partial_{23} u_1 \partial_3 u_1 \, dxdydz \\ &\le C\|\partial_{23}u_1\|_{L^4_{x_3}L^2_{x_1x_2}} \|\partial_3 u_1\|_{L^4_{x_3}L^\infty_{x_2}L^2_{x_1}} \|u_2\|_{L^\infty_{x_1}L^2_{x_2x_3}} \\ &\le C\|\Lambda_3^{\frac{5}{4}}\partial_2 u_1\|_2 \|\Lambda_3^{\frac{5}{4}}u_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{5}{4}}\Lambda_2 u_1\|_2^{\frac{1}{2}} \|u_2\|_2^{\frac{1}{2}} \|\Lambda_1 u_2\|_2^{\frac{1}{2}} \\ &\le 2\varepsilon\|\Lambda_3^{\frac{5}{4}}\nabla u_1\|_2^2 + C_\varepsilon(\|u_2\|_2^2 + \|\Lambda_3^{\frac{5}{4}}u_1\|_2^2)^2\|\nabla u_2\|_2^2. \end{aligned} \tag{3.15}$$

Inserting the bounds (3.9)–(3.15) into equation (3.8), we obtain

$$\begin{aligned} |J_{11}| &\le 24\varepsilon\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\nabla u_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\nabla u_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2 \\ &\quad + C_\varepsilon(\|\mathbf{u}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_1\|_2^2)^{\frac{9}{2}}\|\mathbf{u}\|_2^2 \\ &\quad + C_\varepsilon(\|\mathbf{u}, (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})u_3\|_2^2)^2\|\nabla\mathbf{u}\|_2^2. \end{aligned} \tag{3.16}$$

Next, we will bound the second nine terms of $J_1$ denoted by $J_{12}$. We write it explicitly,

$$\begin{aligned} J_{12} &= -\int (\partial_1 u_1(\partial_1 u_2)^2 + (\partial_1 u_2)^2\partial_2 u_2 + \partial_1 u_3 \partial_3 u_2 \partial_1 u_2 \\ &\quad + \partial_2 u_1 \partial_1 u_2 \partial_2 u_2 + (\partial_2 u_2)^3 + \partial_2 u_3 \partial_3 u_2 \partial_2 u_2 \\ &\quad + \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 + \partial_3 u_2 \partial_2 u_2 \partial_3 u_2 + \partial_3 u_3(\partial_3 u_2)^2)\, dxdydz \\ &= \sum_{l=1}^{9} J_{12l}. \end{aligned} \tag{3.17}$$

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Most terms in (3.17) can be bounded similarly as $J_{11}$, we just estimate one of the most difficult terms, such as $J_{125}$, integrating by parts, yields

$$\begin{aligned} J_{125} &= -\int (\partial_2 u_2)^3 \, dxdydz = 2\int u_2 \partial_2 u_2 \partial_{22} u_2 \, dxdydz \\ &\le C\|\partial_{22}u_2\|_{L^4_{x_2}L^2_{x_1x_3}} \|\partial_2 u_2\|_{L^4_{x_2}L^\infty_{x_3}L^2_{x_1}} \|u_2\|_{L^\infty_{x_1}L^2_{x_2x_3}} \\ &\le C\|\Lambda_2^{\frac{5}{4}}\partial_2 u_2\|_2 \|\Lambda_2^{\frac{5}{4}} u_2\|_2^{\frac{1}{2}} \|\Lambda_2^{\frac{5}{4}}\Lambda_3 u_2\|_2^{\frac{1}{2}} \|u_2\|_2^{\frac{3}{5}} \|\Lambda_1^{\frac{5}{4}} u_2\|_2^{\frac{2}{5}} \\ &\le 4\varepsilon \|\Lambda_2^{\frac{5}{4}}\partial_1 u_1,\ \Lambda_2^{\frac{5}{4}}\partial_3 u_3,\ \Lambda_3^{\frac{5}{4}}\partial_2 u_2\|_2^2 + C_\varepsilon(\|u_2\|_2^2 + \|\Lambda_1^{\frac{5}{4}} u_2\|_2^2)^{\frac{9}{2}} \|u_2\|_2^2. \end{aligned} \tag{3.18}$$

Furthermore, one can easily check that

$$\begin{aligned} |J_{12}| &\le 24\varepsilon \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2 \\ &\quad + C_\varepsilon(\|\mathbf{u}\|_2^2 + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_2\|_2^2)^{\frac{9}{2}} \|\mathbf{u}\|_2^2 \\ &\quad + C_\varepsilon(\|\mathbf{u},\ (\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})u_3\|_2^2)^2 \|\nabla \mathbf{u}\|_2^2. \end{aligned} \tag{3.19}$$

Similarly,

$$\begin{aligned} |J_{13}| &\le 24\varepsilon \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2 \\ &\quad + C_\varepsilon(\|\mathbf{u}\|_2^2 + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})u_3\|_2^2)^{\frac{9}{2}} \|\mathbf{u}\|_2^2 \\ &\quad + C_\varepsilon(\|\mathbf{u},\ (\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})u_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})u_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})u_3\|_2^2)^2 \|\nabla \mathbf{u}\|_2^2. \end{aligned} \tag{3.20}$$

Next, we will bound the term $J_2$. Similar to $J_1$, we consider the first nine terms in $J_2$ denoted by $J_{21}$, firstly.

$$\begin{aligned} J_{21} &= -\int \big(\partial_1 u_1(\partial_1 \mathrm{w}_1)^2 + \partial_1 u_2 \partial_1 \mathrm{w}_1 \partial_2 \mathrm{w}_1 + \partial_1 u_3 \partial_3 \mathrm{w}_1 \partial_1 \mathrm{w}_1 \\ &\quad + \partial_2 u_1 \partial_1 \mathrm{w}_1 \partial_2 \mathrm{w}_1 + (\partial_2 \mathrm{w}_1)^2 \partial_2 u_2 + \partial_2 u_3 \partial_3 \mathrm{w}_1 \partial_2 \mathrm{w}_1 \\ &\quad + \partial_3 u_1 \partial_1 \mathrm{w}_1 \partial_3 \mathrm{w}_1 + \partial_3 u_2 \partial_2 \mathrm{w}_1 \partial_3 \mathrm{w}_1 + \partial_3 u_3 (\partial_3 \mathrm{w}_1)^2\big) \, dxdydz \\ &= \sum_{n=1}^{9} J_{21n}. \end{aligned} \tag{3.21}$$

To start with term $J_{211}$, we rewrite it as follows:

$$\begin{aligned} J_{211} &= -\int \partial_1 u_1 (\partial_1 \mathrm{w}_1)^2 \, dxdydz = -\int \partial_1 u_1 (\mathrm{div}\mathbf{w} - \partial_2 \mathrm{w}_2 - \partial_3 \mathrm{w}_3)\partial_1 \mathrm{w}_1 \, dxdydz \\ &\quad - \int (\partial_1 u_1 \mathrm{div}\mathbf{w} \partial_1 \mathrm{w}_1 - \partial_1 u_1 \partial_2 \mathrm{w}_2 \partial_1 \mathrm{w}_1 - \partial_1 u_1 \partial_3 \mathrm{w}_3 \partial_1 \mathrm{w}_1) \, dxdydz \\ &= J_{2111} + J_{2112} + J_{2113}. \end{aligned} \tag{3.22}$$

Applying Lemmas 2.2 and 2.3, we can bound the term $J_{2111}$ as follows:

$$\begin{aligned} J_{2111} &= -\int \partial_1 u_1 \mathrm{div}\mathbf{w} \partial_1 \mathrm{w}_1 \, dxdydz \\ &\le C\|\mathrm{div}\mathbf{w}\|_2 \|\partial_1 u_1\|_{L^4_{x_2}L^\infty_{x_1}L^2_{x_3}} \|\partial_1 \mathrm{w}_1\|_{L^2_{x_1}L^4_{x_2}L^\infty_{x_3}} \end{aligned}$$

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$$\leq C\|\mathrm{div}\mathbf{w}\|_2\|\Lambda_2^{\frac{1}{4}}\partial_1 u_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{1}{4}}\Lambda_1\partial_1 u_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{1}{4}}\partial_1 \mathrm{w}_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{1}{4}}\Lambda_3\partial_1 \mathrm{w}_1\|_2^{\frac{1}{2}}$$

$$\leq \varepsilon\|\Lambda_2^{\frac{5}{4}}\partial_1 u_1,\ \Lambda_1^{\frac{5}{4}}\partial_2 u_2,\ \Lambda_1^{\frac{5}{4}}\partial_3 u_3,\ (\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\partial_1 \mathrm{w}_1\|_2^2 + C_\varepsilon\|\mathrm{div}\mathbf{w}\|_2^{\frac{5}{2}}\|\partial_1 u_1,\ \partial_1 \mathrm{w}_1\|_2^2. \tag{3.23}$$

Integrating by parts, by Lemmas 2.1 and 2.2, $J_{2112}$ can be bounded as

$$\begin{aligned}
J_{2112} &= \int \partial_1 u_1\partial_2 \mathrm{w}_2\partial_1 \mathrm{w}_1\, dxdydz = -\int \mathrm{w}_1\partial_{11}u_1\partial_2\mathrm{w}_2\, dxdydz\\
&\quad - \int \mathrm{w}_1\partial_{12}\mathrm{w}_2\partial_1 u_1\, dxdydz\\
&\leq C\|\partial_{11}u_1\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_2\mathrm{w}_2\|_{L^4_{x_1}L^\infty_{x_3}L^2_{x_2}}\|\mathrm{w}_1\|_{L^\infty_{x_2}L^2_{x_1x_3}}\\
&\quad + C\|\partial_{12}\mathrm{w}_2\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_1 u_1\|_{L^4_{x_1}L^\infty_{x_3}L^2_{x_2}}\|\mathrm{w}_1\|_{L^\infty_{x_2}L^2_{x_1x_3}}\\
&\leq C\|\Lambda_1^{\frac{5}{4}}\partial_1 u_1\|_2\|\Lambda_1^{\frac{1}{4}}\partial_2\mathrm{w}_2\|_2^{\frac{1}{2}}\|\Lambda_1^{\frac{1}{4}}\Lambda_3\partial_2\mathrm{w}_2\|_2^{\frac{1}{2}}\|\mathrm{w}_1\|_2^{\frac{3}{5}}\|\Lambda_2^{\frac{5}{4}}\mathrm{w}_1\|_2^{\frac{2}{5}}\\
&\quad + C\|\Lambda_1^{\frac{5}{4}}\partial_2\mathrm{w}_2\|_2\|\Lambda_1^{\frac{1}{4}}\partial_1 u_1\|_2^{\frac{1}{2}}\|\Lambda_1^{\frac{1}{4}}\Lambda_3\partial_1 u_1\|_2^{\frac{1}{2}}\|\mathrm{w}_1\|_2^{\frac{3}{5}}\|\Lambda_2^{\frac{5}{4}}\mathrm{w}_1\|_2^{\frac{2}{5}}\\
&\leq 2\varepsilon\|\Lambda_1^{\frac{5}{4}}\partial_2 u_2,\ \Lambda_1^{\frac{5}{4}}\partial_3 u_3,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\partial_2\mathrm{w}_2\|_2^2\\
&\quad + C_\varepsilon(\|\mathrm{w}_1\|_2^2 + \|\Lambda_2^{\frac{5}{4}}\mathrm{w}_1\|_2^2)^{\frac{5}{2}}\|\partial_1 u_1,\ \partial_2\mathrm{w}_2\|_2^2.
\end{aligned} \tag{3.24}$$

Similarly, one can easily check that $J_{2113}$ satisfies

$$\begin{aligned}
J_{2113} &= \int \partial_1 u_1\partial_3 \mathrm{w}_3\partial_1 \mathrm{w}_1\, dxdydz = -\int \mathrm{w}_1\partial_{11}u_1\partial_3\mathrm{w}_3\, dxdydz\\
&\quad - \int \mathrm{w}_1\partial_{13}\mathrm{w}_3\partial_1 u_1\, dxdydz\\
&\leq C\|\partial_{11}u_1\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_3\mathrm{w}_3\|_{L^4_{x_1}L^\infty_{x_2}L^2_{x_3}}\|\mathrm{w}_1\|_{L^\infty_{x_3}L^2_{x_1x_2}}\\
&\quad + C\|\partial_{13}\mathrm{w}_3\|_{L^4_{x_1}L^2_{x_2x_3}}\|\partial_1 u_1\|_{L^4_{x_1}L^\infty_{x_2}L^2_{x_3}}\|\mathrm{w}_1\|_{L^\infty_{x_3}L^2_{x_1x_2}}\\
&\leq 2\varepsilon\|\Lambda_1^{\frac{5}{4}}\partial_2 u_2,\ \Lambda_1^{\frac{5}{4}}\partial_3 u_3,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\partial_3\mathrm{w}_3\|_2^2\\
&\quad + C_\varepsilon(\|\mathrm{w}_1\|_2^2 + \|\Lambda_3^{\frac{5}{4}}\mathrm{w}_1\|_2^2)^{\frac{5}{2}}\|\partial_1 u_1,\ \partial_3\mathrm{w}_3\|_2^2.
\end{aligned} \tag{3.25}$$

Inserting the estimates (3.23)–(3.25) into (3.22), we obtain

$$\begin{aligned}
|J_{211}| &\leq 6\varepsilon\|\Lambda_2^{\frac{5}{4}}\nabla u_1,\ \Lambda_1^{\frac{5}{4}}\partial_2 u_2,\ \Lambda_1^{\frac{5}{4}}\partial_3 u_3,\ (\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla\mathrm{w}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla\mathrm{w}_2,\\
&\qquad \times(\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla\mathrm{w}_3\|_2^2\\
&\quad + C_\varepsilon(\|\mathrm{div}\mathbf{w}\|_2^2 + \|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1\|_2^2)^{\frac{5}{2}}\|\nabla\mathbf{u},\ \nabla\mathbf{w}\|_2^2.
\end{aligned} \tag{3.26}$$

Next, we will bound $J_{212}$. Integrating by parts, using Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
J_{212} &= -\int \partial_1 u_2\partial_1\mathrm{w}_1\partial_2\mathrm{w}_1\, dxdydz\\
&= \int u_2\partial_{12}\mathrm{w}_1\partial_1\mathrm{w}_1\, dxdydz + \int u_2\partial_{11}\mathrm{w}_1\partial_2\mathrm{w}_1\, dxdydz\\
&= J_{2121} + J_{2122}.
\end{aligned} \tag{3.27}$$

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Employing Lemmas 2.1 and 2.2 yields

$$
\begin{aligned}
J_{2121} &= \int u_2 \partial_{12} w_1 \partial_1 w_1 \, dxdydz \\
&\le C \|\partial_{12} w_1\|_{L^4_{x_2} L^2_{x_1 x_3}} \|\partial_1 w_1\|_{L^4_{x_2} L^\infty_{x_3} L^2_{x_1}} \|u_2\|_{L^2_{x_2} L^\infty_{x_1} L^2_{x_3}} \\
&\le C \|\Lambda_2^{\frac{5}{4}} \partial_1 w_1\|_2 \|\Lambda_2^{\frac{1}{4}} \partial_1 w_1\|_2^{\frac{1}{2}} \|\Lambda_2^{\frac{1}{4}} \Lambda_3 \partial_1 w_1\|_2^{\frac{1}{2}} \|u_2\|_2^{\frac{3}{5}} \|\Lambda_1^{\frac{5}{4}} u_2\|_2^{\frac{2}{5}} \\
&\le 4\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_1 w_1\|_2^2 + C_\varepsilon (\|\mathbf{u}\|_2^2 + \|\Lambda_1^{\frac{5}{4}} u_2\|_2^2)^{\frac{5}{2}} \|\partial_1 w_1\|_2^2. \qquad (3.28)
\end{aligned}
$$

For the term $J_{2122}$, we rewrite it as follows:

$$
\begin{aligned}
J_{2122} &= \int u_2 \partial_{11} w_1 \partial_2 w_1 \, dxdydz = \int u_2 \partial_1 (\mathrm{div}\mathbf{w} - \partial_2 w_2 - \partial_3 w_3) \partial_2 w_1 \, dxdydz \\
&= \int (u_2 \partial_1 \mathrm{div}\mathbf{w} \partial_2 w_1 - u_2 \partial_{12} w_2 \partial_2 w_1 - u_2 \partial_{13} w_3 \partial_2 w_1) \, dxdydz \\
&= J_{21221} + J_{21222} + J_{21223}. \qquad (3.29)
\end{aligned}
$$

Now, we will estimate $J_{21221}$, and one can easily find that

$$
\begin{aligned}
J_{21221} &= \int u_2 \partial_1 \mathrm{div}\mathbf{w} \partial_2 w_1 \, dxdydz \\
&\le C \|\partial_1 \mathrm{div}\mathbf{w}\|_2 \|\partial_2 w_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|u_2\|_{L^\infty_{x_1} L^2_{x_2} L^4_{x_3}} \\
&\le C \|\partial_1 \mathrm{div}\mathbf{w}\|_2 \|\Lambda_3^{\frac{1}{4}} \partial_2 w_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \Lambda_2 \partial_1 w_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} u_2\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \Lambda_1 u_2\|_2^{\frac{1}{2}} \\
&\le 4\varepsilon \|\partial_1 \mathrm{div}\mathbf{w}, (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_2 w_1\|_2^2 + C_\varepsilon (\|\mathbf{u}\|_2^2 + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_2\|_2^2)^{\frac{5}{2}} \|\partial_2 w_1\|_2^2. \qquad (3.30)
\end{aligned}
$$

Similar to $J_{2121}$, one can estimate $J_{21222}$ as follows:

$$
\begin{aligned}
J_{21222} &= \int u_2 \partial_{12} w_2 \partial_2 w_1 \, dxdydz \\
&\le C \|\partial_{12} w_2\|_{L^4_{x_1} L^2_{x_2 x_3}} \|\partial_2 w_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|u_2\|_{L^2_{x_2} L^4_{x_1 x_3}} \\
&\le C \|\Lambda_1^{\frac{5}{4}} \partial_2 w_2\|_2 \|\Lambda_3^{\frac{1}{4}} \partial_2 w_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \Lambda_2 \partial_2 w_1\|_2^{\frac{1}{2}} \|u_2\|_2^{\frac{1}{2}} \|(\partial_1, \partial_3) u_2\|_2^{\frac{1}{2}} \\
&\le 4\varepsilon \|\Lambda_1^{\frac{5}{4}} \partial_2 w_2, (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_2 w_1\|_2^2 + C_\varepsilon (\|\mathbf{u}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{2}} \|(\partial_1, \partial_3) u_2\|_2^2. \qquad (3.31)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_{21223} &= \int u_2 \partial_{13} w_3 \partial_2 w_1 \, dxdydz \\
&\le C \|\partial_{13} w_3\|_{L^4_{x_1} L^2_{x_2 x_3}} \|\partial_2 w_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|u_2\|_{L^2_{x_2} L^4_{x_1 x_3}} \\
&\le 4\varepsilon \|\Lambda_1^{\frac{5}{4}} \partial_3 w_3, (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_2 w_1\|_2^2 + C_\varepsilon (\|\mathbf{u}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^2 \|(\partial_1, \partial_3) u_2\|_2^2. \qquad (3.32)
\end{aligned}
$$

Combining the inequalities (3.28) and (3.30)–(3.32) with (3.27), one has

$$
\begin{aligned}
|J_{212}| &\le 16\varepsilon \|\nabla \mathrm{div}\mathbf{w}, (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla w_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla w_3\|_2^2 \\
&\quad + C_\varepsilon (\|\mathbf{u}\|_2^2 + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_2\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{2}} \|\nabla \mathbf{u}, \nabla \mathbf{w}\|_2^2. \qquad (3.33)
\end{aligned}
$$

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For the term $J_{213}$, using the similar method to $J_{212}$, one can easily check that

$$|J_{213}| \leq 16\varepsilon\|\nabla\mathrm{div}\mathbf{w},\ (\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla \mathrm{w}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla \mathrm{w}_3\|_2^2$$
$$+ C_\varepsilon(\|\mathbf{u}\|_2^2 + \|(\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})u_3\|_2^2 + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1\|_2^2)^{\frac{5}{2}}\|\nabla\mathbf{u},\ \nabla\mathbf{w}\|_2^2. \tag{3.34}$$

Next, we will estimate the term $J_{214}$. Integrating by parts, one has

$$\begin{aligned} J_{214} &= -\int \partial_2 u_1 \partial_1 \mathrm{w}_1 \partial_2 \mathrm{w}_1\, dxdydz \\ &= \int \mathrm{w}_1 \partial_{22} u_1 \partial_1 \mathrm{w}_1\, dxdydz + \int \mathrm{w}_1 \partial_{12}\mathrm{w}_1 \partial_2 u_1\, dxdydz \\ &= J_{2141} + J_{2142}. \end{aligned} \tag{3.35}$$

First, we can bound the second term $J_{2142}$ as

$$\begin{aligned} J_{2142} &= \int \mathrm{w}_1 \partial_{12}\mathrm{w}_1 \partial_2 u_1\, dxdydz \\ &\leq C\|\partial_{12}\mathrm{w}_1\|_{L^4_{x_2}L^2_{x_1x_3}}\|\partial_2 u_1\|_{L^4_{x_2}L^\infty_{x_3}L^2_{x_1}}\|\mathrm{w}_1\|_{L^\infty_{x_1}L^2_{x_2x_3}} \\ &\leq C\|\Lambda_2^{\frac{5}{4}}\partial_1\mathrm{w}_1\|_2\|\Lambda_2^{\frac{5}{4}}u_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{5}{4}}\Lambda_3 u_1\|_2^{\frac{1}{2}}\|\mathrm{w}_1\|_2^{\frac{1}{2}}\|\Lambda_1\mathrm{w}_1\|_2^{\frac{1}{2}} \\ &\leq \varepsilon\|\Lambda_2^{\frac{5}{4}}\partial_3 u_1,\ \Lambda_2^{\frac{5}{4}}\partial_1\mathrm{w}_1\|_2^2 + C_\varepsilon(\|\mathbf{w}\|_2^2 + \|\Lambda_2^{\frac{5}{4}}u_1\|_2^2)^2\|\partial_1\mathrm{w}_1\|_2^2. \end{aligned} \tag{3.36}$$

Now, we return to estimate the term $J_{2141}$. Similar to $J_{2122}$, we rewrite it as

$$\begin{aligned} J_{2141} &= \int \mathrm{w}_1\partial_{22}u_1\partial_1\mathrm{w}_1\, dxdydz = \int \mathrm{w}_1\partial_{22}u_1(\mathrm{div}\mathbf{w} - \partial_2\mathrm{w}_2 - \partial_3\mathrm{w}_3)\, dxdydz \\ &= \int (\mathrm{w}_1\partial_{22}u_1\mathrm{div}\mathbf{w} - \mathrm{w}_1\partial_{22}u_1\partial_2\mathrm{w}_2 - \mathrm{w}_1\partial_{22}u_1\partial_3\mathrm{w}_3)\, dxdydz \\ &= J_{21411} + J_{21412} + J_{21413}. \end{aligned} \tag{3.37}$$

Applying Lemmas 2.1–2.3, we have

$$\begin{aligned} J_{21411} &= \int \mathrm{w}_1\partial_{22}u_1\mathrm{div}\mathbf{w}\, dxdydz \\ &\leq C\|\partial_{22}u_1\|_{L^4_{x_2}L^2_{x_1x_3}}\|\mathrm{div}\mathbf{w}\|_{L^\infty_{x_1}L^2_{x_2x_3}}\|\mathrm{w}_1\|_{L^\infty_{x_3}L^2_{x_1}L^4_{x_2}} \\ &\leq C\|\Lambda_2^{\frac{5}{4}}\partial_2 u_1\|_2\|\mathrm{div}\mathbf{w}\|_2^{\frac{1}{2}}\|\Lambda_1\mathrm{div}\mathbf{w}\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{1}{4}}\mathrm{w}_1\|_2^{\frac{1}{2}}\|\Lambda_2^{\frac{1}{4}}\Lambda_3\mathrm{w}_1\|_2^{\frac{1}{2}} \\ &\leq \varepsilon\|\partial_1\mathrm{div}\mathbf{w},\ \Lambda_2^{\frac{5}{4}}\partial_2 u_1\|_2^2 + C_\varepsilon(\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1\|_2^2)^{\frac{5}{2}}\|\partial_2\mathrm{w}_1\|_2^2 \\ &\quad + C_\varepsilon(\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1\|_2^2)^{\frac{5}{2}}\|\mathrm{w}_1\|_2^2. \end{aligned} \tag{3.38}$$

Similarly,

$$\begin{aligned} J_{21412} &= \int \mathrm{w}_1\partial_{22}u_1\partial_2\mathrm{w}_2\, dxdydz \\ &\leq C\|\partial_{22}u_1\|_{L^4_{x_2}L^2_{x_1x_3}}\|\partial_2\mathrm{w}_2\|_{L^2_{x_2x_3}L^\infty_{x_1}}\|\mathrm{w}_1\|_{L^2_{x_1}L^4_{x_2}L^\infty_{x_3}} \\ &\leq \varepsilon\|\Lambda_2^{\frac{5}{4}}\partial_2 u_1,\ \Lambda_1^{\frac{5}{4}}\partial_2\mathrm{w}_2\|_2^2 + C_\varepsilon(\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1\|_2^2)^{\frac{5}{3}}\|\partial_2\mathrm{w}_2\|_2^2 \end{aligned} \tag{3.39}$$

https://doi.org/10.4153/S0008414X22000414 Published online by Cambridge University Press

and

$$\begin{aligned} J_{21413} &= \int \mathrm{w}_1 \partial_{22} u_1 \partial_3 \mathrm{w}_3 \, dxdydz \\ &\le C \|\partial_{22} u_1\|_{L^4_{x_2} L^2_{x_1 x_3}} \|\partial_3 \mathrm{w}_3\|_{L^2_{x_2 x_3} L^\infty_{x_1}} \|\mathrm{w}_1\|_{L^2_{x_1} L^4_{x_2} L^\infty_{x_3}} \\ &\le \varepsilon \|\Lambda_2^{\frac{5}{4}} \partial_2 u_1, \ \Lambda_1^{\frac{5}{4}} \partial_3 \mathrm{w}_3\|_2^2 + C_\varepsilon (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2)^{\frac{5}{3}} \|\partial_3 \mathrm{w}_3\|_2^2. \end{aligned} \tag{3.40}$$

Furthermore, inserting the inequalities (3.38)–(3.40) into (3.37), we obtain

$$\begin{aligned} |J_{214}| &\le 16\varepsilon \|\nabla \mathrm{div}\mathbf{w}, \ (\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \partial_2 u_1, \ (\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \nabla \mathrm{w}_1, \ \Lambda_1^{\frac{5}{4}} \partial_2 \mathrm{w}_2, \ \Lambda_1^{\frac{5}{4}} \partial_3 \mathrm{w}_3\|_2^2 \\ &\quad + C_\varepsilon ((\|\mathbf{w}\|_2^2 + \|\Lambda_2^{\frac{5}{4}}) u_1\|_2^2)^2 + (\|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2 + \|\mathrm{div}\mathbf{w}\|_2^2)^{\frac{5}{2}} \\ &\quad + (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2)^{\frac{5}{3}}) \|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2 \\ &\quad + C_\varepsilon (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2)^{\frac{5}{2}} \|\mathrm{w}_1\|_2^2. \end{aligned} \tag{3.41}$$

Integrating by parts, the term $J_{215}$ can be estimated as

$$\begin{aligned} J_{215} &= -\int (\partial_2 \mathrm{w}_1)^2 \partial_2 u_2 \, dxdydz = 2 \int u_2 \partial_{22} \mathrm{w}_1 \partial_2 \mathrm{w}_1 \, dxdydz \\ &\le C \|\partial_{22} \mathrm{w}_1\|_{L^4_{x_2} L^2_{x_1 x_3}} \|\partial_2 \mathrm{w}_1\|_{L^4_{x_2} L^\infty_{x_1} L^2_{x_3}} \|u_2\|_{L^\infty_{x_3} L^2_{x_1 x_2}} \\ &\le C \|\Lambda_2^{\frac{5}{4}} \partial_2 \mathrm{w}_1\|_2 \|\Lambda_2^{\frac{5}{4}} \mathrm{w}_1\|_2^{\frac{1}{2}} \|\Lambda_2^{\frac{5}{4}} \Lambda_1 \mathrm{w}_1\|_2^{\frac{1}{2}} \|u_2\|_2^{\frac{1}{2}} \|\Lambda_3 u_2\|_2^{\frac{1}{2}} \\ &\le 2\varepsilon \|\Lambda_2^{\frac{5}{4}} \partial_2 \mathrm{w}_1\|_2^2 + C_\varepsilon (\|\mathbf{u}\|_2^2 + \|\Lambda_2^{\frac{5}{4}} \mathrm{w}_1\|_2^2)^2 \|\partial_3 u_2\|_2^2. \end{aligned} \tag{3.42}$$

Similarly,

$$\begin{aligned} J_{216} &= -\int \partial_2 u_3 \partial_3 \mathrm{w}_1 \partial_2 \mathrm{w}_1 \, dxdydz \\ &= \int \mathrm{w}_1 \partial_{23} u_3 \partial_2 \mathrm{w}_1 \, dxdydz + \int \mathrm{w}_1 \partial_{23} \mathrm{w}_1 \partial_2 u_3 \, dxdydz \\ &\le C \|\partial_{23} u_3\|_{L^4_{x_2} L^2_{x_1 x_3}} \|\partial_2 \mathrm{w}_1\|_{L^4_{x_2} L^\infty_{x_1} L^2_{x_3}} \|\mathrm{w}_1\|_{L^\infty_{x_3} L^2_{x_1 x_2}} \\ &\quad + C \|\partial_{23} \mathrm{w}_1\|_{L^4_{x_2} L^2_{x_1 x_3}} \|\partial_2 u_3\|_{L^4_{x_2} L^\infty_{x_1} L^2_{x_3}} \|\mathrm{w}_1\|_{L^\infty_{x_3} L^2_{x_1 x_2}} \\ &\le 2\varepsilon \|\Lambda_2^{\frac{5}{4}} \nabla u_3, \ \Lambda_2^{\frac{5}{4}} \nabla \mathrm{w}_1\|_2^2 + C_\varepsilon (\|\mathbf{w}\|_2^2 + \|\Lambda_2^{\frac{5}{4}} \mathrm{w}_1\|_2^2)^2 \|\partial_3 \mathrm{w}_1\|_2^2. \end{aligned} \tag{3.43}$$

Employing the similar method to $J_{214}$, we can bound the term $J_{217}$ as

$$\begin{aligned} J_{217} &= -\int \partial_3 u_1 \partial_1 \mathrm{w}_1 \partial_3 \mathrm{w}_1 \, dxdydz \\ &= \int \mathrm{w}_1 \partial_{33} u_1 \partial_1 \mathrm{w}_1 \, dxdydz + \int \mathrm{w}_1 \partial_{13} \mathrm{w}_1 \partial_3 u_1 \, dxdydz \\ &= J_{2171} + J_{2172}. \end{aligned} \tag{3.44}$$

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Furthermore, we will estimate the second term $J_{2172}$ first as follows:

$$\begin{aligned} J_{2172} &= \int w_1 \partial_{13} w_1 \partial_3 u_1 \, dxdydz \\ &\le C \|\partial_{13} w_1\|_{L^4_{x_3} L^2_{x_1 x_2}} \|\partial_3 u_1\|_{L^4_{x_3} L^\infty_{x_2} L^2_{x_1}} \|w_1\|_{L^\infty_{x_1} L^2_{x_2 x_3}} \\ &\le \varepsilon \|\Lambda_3^{\frac{5}{4}} \partial_2 u_1, \ \Lambda_3^{\frac{5}{4}} \partial_1 w_1\|_2^2 + C_\varepsilon (\|\mathbf{w}\|_2^2 + \|\Lambda_3^{\frac{5}{4}} u_1\|_2^2)^2 \|\partial_1 w_1\|_2^2. \end{aligned} \tag{3.45}$$

Now, we return to estimate the term $J_{2171}$. Similar to $J_{2122}$, we rewrite it as

$$\begin{aligned} J_{2171} &= \int w_1 \partial_{33} u_1 \partial_1 w_1 \, dxdydz = \int w_1 \partial_{33} u_1 (\mathrm{div}\mathbf{w} - \partial_2 w_2 - \partial_3 w_3) \, dxdydz \\ &= \int (w_1 \partial_{33} u_1 \mathrm{div}\mathbf{w} - w_1 \partial_{33} u_1 \partial_2 w_2 - w_1 \partial_{33} u_1 \partial_3 w_3) \, dxdydz \\ &= J_{21711} + J_{21712} + J_{21713}. \end{aligned} \tag{3.46}$$

Applying Lemmas 2.1–2.3, we have

$$\begin{aligned} J_{21711} &= \int w_1 \partial_{33} u_1 \mathrm{div}\mathbf{w} \, dxdydz \\ &\le C \|\partial_{33} u_1\|_{L^4_{x_3} L^2_{x_1 x_2}} \|\mathrm{div}\mathbf{w}\|_{L^\infty_{x_1} L^2_{x_2 x_3}} \|w_1\|_{L^\infty_{x_2} L^2_{x_1} L^4_{x_3}} \\ &\le \varepsilon \|\partial_1 \mathrm{div}\mathbf{w}, \ \Lambda_3^{\frac{5}{4}} \partial_3 u_1\|_2^2 + C_\varepsilon (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{2}} \|\partial_2 w_1\|_2^2 \\ &\quad + C_\varepsilon (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{2}} \|w_1\|_2^2. \end{aligned} \tag{3.47}$$

Similarly,

$$\begin{aligned} J_{21712} &= \int w_1 \partial_{33} u_1 \partial_2 w_2 \, dxdydz \\ &\le C \|\partial_{33} u_1\|_{L^4_{x_2} L^2_{x_1 x_3}} \|\partial_2 w_2\|_{L^2_{x_2 x_3} L^\infty_{x_1}} \|w_1\|_{L^2_{x_1} L^4_{x_2} L^\infty_{x_3}} \\ &\le \varepsilon \|\Lambda_3^{\frac{5}{4}} \partial_3 u_1, \ \Lambda_1^{\frac{5}{4}} \partial_2 w_2\|_2^2 + C_\varepsilon (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{3}} \|\partial_2 w_2\|_2^2 \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} J_{21713} &= \int w_1 \partial_{33} u_1 \partial_3 w_3 \, dxdydz \\ &\le C \|\partial_{33} u_1\|_{L^4_{x_3} L^2_{x_1 x_2}} \|\partial_3 w_3\|_{L^2_{x_2 x_3} L^\infty_{x_1}} \|w_1\|_{L^2_{x_1} L^4_{x_3} L^\infty_{x_2}} \\ &\le \varepsilon \|\Lambda_3^{\frac{5}{4}} \partial_3 u_1, \ \Lambda_1^{\frac{5}{4}} \partial_3 w_3\|_2^2 + C_\varepsilon (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{3}} \|\partial_3 w_3\|_2^2. \end{aligned} \tag{3.49}$$

Furthermore, inserting the inequalities (3.47)–(3.49) and (3.45) into (3.44), we obtain

$$\begin{aligned} |J_{217}| &\le 16\varepsilon \|\nabla \mathrm{div}\mathbf{w}, \ (\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \nabla u_1, \ (\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) \nabla w_1, \ \Lambda_1^{\frac{5}{4}} \partial_2 w_2, \ \Lambda_1^{\frac{5}{4}} \partial_3 w_3\|_2^2 \\ &\quad + C_\varepsilon \big( (\|\mathbf{w}\|_2^2 + \|\Lambda_3^{\frac{5}{4}} u_1\|_2^2)^2 + (\|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2 + \|\mathrm{div}\mathbf{w}\|_2^2)^{\frac{5}{2}} \\ &\quad + (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{3}} \big) \|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2 \\ &\quad + C_\varepsilon (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}}) w_1\|_2^2)^{\frac{5}{2}} \|w_1\|_2^2. \end{aligned} \tag{3.50}$$

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Integrating by parts, one can dominate $J_{218}$ as

$$\begin{aligned}
J_{218} &= -\int \partial_3 u_2 \partial_2 \mathrm{w}_1 \partial_3 \mathrm{w}_1 \, dxdydz \\
&= \int \mathrm{w}_1 \partial_{33} u_2 \partial_2 \mathrm{w}_1 \, dxdydz + \int \mathrm{w}_1 \partial_{23} \mathrm{w}_1 \partial_3 u_2 \, dxdydz \\
&\le C \|\partial_{33} u_2\|_{L^4_{x_3} L^2_{x_1x_2}} \|\partial_2 \mathrm{w}_1\|_{L^4_{x_2} L^\infty_{x_1} L^2_{x_3}} \|\mathrm{w}_1\|_{L^2_{x_1} L^4_{x_2x_3}} \\
&\quad + C \|\partial_{23} \mathrm{w}_1\|_{L^4_{x_2} L^2_{x_1x_3}} \|\partial_3 u_2\|_{L^4_{x_3} L^\infty_{x_1} L^2_{x_2}} \|\mathrm{w}_1\|_{L^2_{x_1} L^4_{x_2x_3}} \\
&\le 2\varepsilon \|\Lambda_3^{\frac{5}{4}} \nabla u_2, \, \Lambda_2^{\frac{5}{4}} \nabla \mathrm{w}_1\|_2^2 + C_\varepsilon (\|\mathbf{w}\|_2^2 + \|\Lambda_2^{\frac{5}{4}} \mathrm{w}_1\|_2^2 + \|\Lambda_3^{\frac{5}{4}} u_2\|_2^2)^2 \|\nabla \mathrm{w}_1\|_2^2. \qquad (3.51)
\end{aligned}$$

Finally, we will estimate the term $J_{219}$, and applying the divergence-free condition $\nabla \cdot \mathbf{u} = 0$ and integrating by parts, we obtain

$$\begin{aligned}
J_{219} &= -\int \partial_3 u_3 (\partial_3 \mathrm{w}_1)^2 \, dxdydz \\
&= \int \partial_1 u_1 (\partial_3 \mathrm{w}_1)^2 \, dxdydz + \int \partial_2 u_2 (\partial_3 \mathrm{w}_1)^2 \, dxdydz \\
&= -2 \int u_1 \partial_{13} \mathrm{w}_1 \partial_3 \mathrm{w}_1 \, dxdydz - 2 \int u_2 \partial_{23} \mathrm{w}_1 \partial_3 \mathrm{w}_1 \, dxdydz \\
&\le C \|\partial_{13} \mathrm{w}_1\|_{L^4_{x_3} L^2_{x_1x_2}} \|\partial_3 \mathrm{w}_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|u_1\|_{L^\infty_{x_1} L^2_{x_2x_3}} \\
&\quad + C \|\partial_{23} \mathrm{w}_1\|_{L^4_{x_3} L^2_{x_1x_2}} \|\partial_3 \mathrm{w}_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|u_1\|_{L^\infty_{x_1} L^2_{x_2x_3}} \\
&\le 4\varepsilon \|\Lambda_3^{\frac{5}{4}} \nabla \mathrm{w}_1\|_2^2 + C_\varepsilon (\|u_1\|_2^2 + \|\Lambda_3^{\frac{5}{4}} \mathrm{w}_1\|_2^2)^2 \|\partial_1 u_1\|_2^2. \qquad (3.52)
\end{aligned}$$

Combining the estimates (3.26), (3.33), (3.34), (3.41)–(3.43), and (3.50)–(3.52), we obtain

$$\begin{aligned}
|J_{21}| &\le 80\varepsilon (\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla u_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla u_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_3\|_2^2 \\
&\quad + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla \mathrm{w}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla \mathrm{w}_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla \mathrm{w}_3, \nabla \mathrm{div}\mathbf{w}\|_2^2) \\
&\quad + C_\varepsilon ((\|\mathbf{u}, \mathbf{w}, (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_3\|_2^2 \\
&\quad + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \mathrm{w}_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \mathrm{w}_3\|_2^2)^2 \\
&\quad + (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2)^{\frac{5}{2}} + (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2)^{\frac{5}{3}}) \|\nabla \mathbf{u}, \nabla \mathbf{w}\|_2^2 \\
&\quad + C_\varepsilon (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \mathrm{w}_1\|_2^2)^{\frac{5}{2}} \|\mathbf{w}\|_2^2. \qquad (3.53)
\end{aligned}$$

The remaining terms in $J_2$ can be handled similar to $J_{21}$. We omit it here. Next, we will estimate the term $J_3$:

$$\begin{aligned}
J_3 &= 4\kappa \int \nabla \nabla \times \mathbf{u} \cdot \nabla \mathbf{w} \, dxdydz = 4\kappa \int \nabla(\partial_2 u_3 - \partial_3 u_2) \nabla \mathrm{w}_1 \, dxdydz \\
&\quad + 4\kappa \int \nabla(\partial_3 u_1 - \partial_1 u_3) \nabla \mathrm{w}_2 \, dxdydz + 4\kappa \int \nabla(\partial_1 u_2 - \partial_2 u_1) \nabla \mathrm{w}_3 \, dxdydz \\
&= J_{31} + J_{32} + J_{33}. \qquad (3.54)
\end{aligned}$$

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To start with $J_{31}$, we write it explicitly:

$$\begin{aligned}J_{31} &= 4\kappa \int \nabla(\partial_2 u_3 - \partial_3 u_2)\nabla \mathrm{w}_1 \, dxdydz \\ &\le \|\nabla u_3\|_2^{\frac{1}{5}} \|\Lambda_2^{\frac{5}{4}} \nabla u_3\|_2^{\frac{4}{5}} \|\nabla \mathrm{w}_1\|_2 + \|\nabla u_2\|_2^{\frac{1}{5}} \|\Lambda_3^{\frac{5}{4}} \nabla u_2\|_2^{\frac{4}{5}} \|\nabla \mathrm{w}_1\|_2 \\ &\le \varepsilon \|\Lambda_2^{\frac{5}{4}} \nabla u_3, \ \Lambda_3^{\frac{5}{4}} \nabla u_2\|_2^2 + C_\varepsilon \|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2.\end{aligned}$$

Similarly, one can easily check that

$$J_{32} = 4\kappa \int \nabla(\partial_3 u_1 - \partial_1 u_3)\nabla \mathrm{w}_2 \, dxdydz \le \varepsilon \|\Lambda_3^{\frac{5}{4}} \nabla u_1, \ \Lambda_1^{\frac{5}{4}} \nabla u_3\|_2^2 + C_\varepsilon \|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2$$

Furthermore,

$$J_{33} = 4\kappa \int \nabla(\partial_1 u_2 - \partial_2 u_1)\nabla \mathrm{w}_3 \, dxdydz \le \varepsilon \|\Lambda_1^{\frac{5}{4}} \nabla u_2, \ \Lambda_2^{\frac{5}{4}} \nabla u_1\|_2^2 + C_\varepsilon \|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2.$$

Inserting the estimates of $J_{31}$, $J_{32}$, and $J_{33}$ into (3.31), we obtain

$$|J_3| \le \varepsilon \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\nabla u_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\nabla u_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\nabla u_1\|_2^2 + C_\varepsilon \|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2. \tag{3.55}$$

Combining the estimates (3.16), (3.19), (3.20), (3.53), and (3.55), choosing $\varepsilon$ small enough, we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2 + 4\kappa \|\nabla \mathbf{w}\|_2^2 + \mu \|\nabla \mathrm{div}\mathbf{w}\|_2^2 \\ &\quad + (\nu + \kappa)\|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\nabla u_1, (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\nabla u_2, (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\nabla u_3\|_2^2 \\ &\quad + \gamma \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\nabla \mathrm{w}_1, (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\nabla \mathrm{w}_2, (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\nabla \mathrm{w}_3\|_2^2 \\ &\quad \le C_\varepsilon((\|\mathbf{u}, \ \mathbf{w}, \ \mathrm{div}\mathbf{w}, \ (\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})u_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})u_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})u_3\|_2^2 \\ &\quad + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\mathrm{w}_3\|_2^2)^2 + (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1\|_2^2)^{\frac{5}{2}} \\ &\quad + (\|\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\mathrm{w}_3\|_2^2)^{\frac{5}{3}})\|\nabla \mathbf{u}, \ \nabla \mathbf{w}\|_2^2 \\ &\quad + ((\|\mathbf{u}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_1, \ (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_2, \ (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})u_3\|_2^2)^{\frac{9}{2}} \\ &\quad + (\|\mathrm{div}\mathbf{w}\|_2^2 + \|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\mathrm{w}_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\mathrm{w}_3\|_2^2)^{\frac{5}{2}})\|\mathbf{u}, \ \mathbf{w}\|_2^2.\end{aligned} \tag{3.56}$$

Applying Gronwall's inequality, integrating from 0 to $T$, we complete the proof of Proposition 3.2.

Step 3. Uniqueness

This section we will prove the uniqueness. ∎

***Proposition 3.3*** *Assume that the initial* $(\mathbf{u}_0, \mathbf{w}_0)$ *satisfies the conditions stated in Theorem* 1.1*. Suppose that* $(\mathbf{u}^{(1)}, \ \mathbf{w}^{(1)})$ *and* $(\mathbf{u}^{(2)}, \ \mathbf{w}^{(2)})$ *are two solutions to the system* (1.4)*, then* $(\mathbf{u}^{(1)}, \ \mathbf{w}^{(1)}) = (\mathbf{u}^{(2)}, \ \mathbf{w}^{(2)})$.

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**Proof** Denoting by $\pi^{(1)}$ and $\pi^{(2)}$ the associated pressures, then the differences

$$\tilde{\mathbf{u}} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \ \tilde{\pi} = \pi^{(1)} - \pi^{(2)}, \ \tilde{\mathbf{w}} = \mathbf{w}^{(1)} - \mathbf{w}^{(2)},$$

satisfy

(3.57)

$$\begin{cases} \partial_t \tilde{u}_1 + \mathbf{u}^{(1)} \cdot \nabla \tilde{u}_1 + \tilde{\mathbf{u}} \cdot \nabla u_1^{(2)} = -\partial_1 \tilde{\pi} - (\nu + \kappa) \Lambda_2^{\frac{5}{2}} \tilde{u}_1 - (\nu + \kappa) \Lambda_3^{\frac{5}{2}} \tilde{u}_1 + 2\kappa \varepsilon_{1jk} \partial_j \tilde{\mathrm{w}}_k, \\ \partial_t \tilde{u}_2 + \mathbf{u}^{(1)} \cdot \nabla \tilde{u}_2 + \tilde{\mathbf{u}} \cdot \nabla u_2^{(2)} = -\partial_2 \tilde{\pi} - (\nu + \kappa) \Lambda_1^{\frac{5}{2}} \tilde{u}_2 - (\nu + \kappa) \Lambda_3^{\frac{5}{2}} \tilde{u}_2 + 2\kappa \varepsilon_{2jk} \partial_j \tilde{\mathrm{w}}_k, \\ \partial_t \tilde{u}_3 + \mathbf{u}^{(1)} \cdot \nabla \tilde{u}_3 + \tilde{\mathbf{u}} \cdot \nabla u_3^{(2)} = -\partial_3 \tilde{\pi} - (\nu + \kappa) \Lambda_1^{\frac{5}{2}} \tilde{u}_3 - (\nu + \kappa) \Lambda_2^{\frac{5}{2}} \tilde{u}_3 + 2\kappa \varepsilon_{3jk} \partial_j \tilde{\mathrm{w}}_k, \\ \partial_t \tilde{\mathrm{w}}_1 + \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathrm{w}}_1 + \tilde{\mathbf{u}} \cdot \nabla \mathrm{w}_1^{(2)} + 4\kappa \tilde{\mathrm{w}}_1 = -\gamma \Lambda_2^{\frac{5}{2}} \tilde{\mathrm{w}}_1 - \gamma \Lambda_3^{\frac{5}{2}} \tilde{\mathrm{w}}_1 + \mu \partial_1 (\nabla \cdot \tilde{\mathbf{w}}) + 2\kappa \varepsilon_{1jk} \partial_j \tilde{u}_k, \\ \partial_t \tilde{\mathrm{w}}_2 + \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathrm{w}}_2 + \tilde{\mathbf{u}} \cdot \nabla \mathrm{w}_2^{(2)} + 4\kappa \tilde{\mathrm{w}}_2 = -\gamma \Lambda_1^{\frac{5}{2}} \tilde{\mathrm{w}}_2 - \gamma \Lambda_3^{\frac{5}{2}} \tilde{\mathrm{w}}_2 + \mu \partial_2 (\nabla \cdot \tilde{\mathbf{w}}) + 2\kappa \varepsilon_{2jk} \partial_j \tilde{u}_k, \\ \partial_t \tilde{\mathrm{w}}_3 + \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathrm{w}}_3 + \tilde{\mathbf{u}} \cdot \nabla \mathrm{w}_3^{(2)} + 4\kappa \tilde{\mathrm{w}}_3 = -\gamma \Lambda_1^{\frac{5}{2}} \tilde{\mathrm{w}}_3 - \gamma \Lambda_2^{\frac{5}{2}} \tilde{\mathrm{w}}_3 + \mu \partial_3 (\nabla \cdot \tilde{\mathbf{w}}) + 2\kappa \varepsilon_{3jk} \partial_j \tilde{u}_k, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ \tilde{\mathbf{u}}(x,0) = 0, \ \tilde{\mathbf{w}}(x,0) = 0. \end{cases}$$

Multiplying equations $(3.57)_1$, $(3.57)_2$, $(3.57)_3$, $(3.57)_4$, $(3.57)_5$, and $(3.57)_6$ with $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, $\tilde{\mathrm{w}}_1$, $\tilde{\mathrm{w}}_2$, and $\tilde{\mathrm{w}}_3$, respectively, integrating by parts, summing the results together, yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}, \ \tilde{\mathbf{w}}\|_2^2 + 4\kappa\|\tilde{\mathbf{w}}\|_2^2 + \mu\|\mathrm{div}\tilde{\mathbf{w}}\|_2^2 \\ &\qquad + (\nu+\kappa)\|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\tilde{u}_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\tilde{u}_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\tilde{u}_3\|_2^2 \\ &\qquad + \gamma\|(\Lambda_2^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_1, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_2, \ (\Lambda_1^{\frac{5}{4}}, \ \Lambda_2^{\frac{5}{4}})\tilde{\mathrm{w}}_3\|_2^2 \\ &\quad = -\int \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \tilde{\mathbf{u}} \, dxdydz - \int \tilde{\mathbf{u}} \cdot \nabla \mathbf{w}^{(2)} \tilde{\mathbf{w}} \, dxdydz + 4\kappa \int \nabla \times \tilde{\mathbf{w}} \cdot \tilde{\mathbf{u}} dxdydz \\ &\quad = K_1 + K_2 + K_3. \end{aligned}$$

(3.58)

To start with $K_1$, we write it explicitly:

$$\begin{aligned} K_1 = -\int \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \tilde{\mathbf{u}} \, dxdydz = -\int (&\tilde{u}_1 \partial_1 u_1^{(2)} \tilde{u}_1 + \tilde{u}_1 \partial_1 u_2^{(2)} \tilde{u}_2 + \tilde{u}_1 \partial_1 u_3^{(2)} \tilde{u}_3 \\ &+ \tilde{u}_2 \partial_2 u_1^{(2)} \tilde{u}_1 + \tilde{u}_2 \partial_2 u_2^{(2)} \tilde{u}_2 + \tilde{u}_2 \partial_2 u_3^{(2)} \tilde{u}_3 \\ &+ \tilde{u}_3 \partial_3 u_1^{(2)} \tilde{u}_1 + \tilde{u}_3 \partial_3 u_2^{(2)} \tilde{u}_2 + \tilde{u}_3 \partial_3 u_1^{(2)} \tilde{u}_3) \, dxdydz \\ = \sum_{i=1}^{9} K_{1i}. \qquad &(3.59) \end{aligned}$$

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Applying the divergence-free condition $\nabla \cdot \tilde{\mathbf{u}} = 0$, by Lemmas 2.1–2.3, we can bound $K_{11}$ as follows:

$$\begin{aligned} K_{11} &= -\int \tilde{u}_1 \partial_1 u_1^{(2)} \tilde{u}_1 \, dxdydz \\ &\le C \|\partial_1 u_1^{(2)}\|_{L^\infty_{x_1} L^2_{x_2 x_3}} \|\tilde{u}_1\|^2_{L^2_{x_1} L^4_{x_2 x_3}} \\ &\le C \|\partial_1 u_1^{(2)}\|_2^{\frac{3}{5}} \|\Lambda_1^{\frac{5}{4}} \partial_1 u_1^{(2)}\|_2^{\frac{2}{5}} \|\Lambda_2^{\frac{1}{4}} \tilde{u}_1\|_2^{\frac{3}{2}} \|\Lambda_2^{\frac{1}{4}} \Lambda_3 \tilde{u}_1\|_2^{\frac{1}{2}} \\ &\le \varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_1\|_2^2 + C_\varepsilon (\|\partial_1 u_1^{(2)}\|_2^2 + \|\Lambda_1^{\frac{5}{4}} \partial_2 u_2^{(2)}, \Lambda_1^{\frac{5}{4}} \partial_3 u_3^{(2)}\|_2^2)^{\frac{5}{6}} \|\tilde{u}_1\|_2^2. \end{aligned} \tag{3.60}$$

Using the similar method to $K_{11}$, the term $K_{12}$ can be estimated as

$$\begin{aligned} K_{12} &= -\int \tilde{u}_1 \partial_1 u_2^{(2)} \tilde{u}_2 \, dxdydz \\ &\le C \|\partial_1 u_2^{(2)}\|_2 \|\tilde{u}_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|\tilde{u}_2\|_{L^2_{x_2} L^\infty_{x_1} L^4_{x_3}} \\ &\le C \|\partial_1 u_2^{(2)}\|_2 \|\Lambda_3^{\frac{1}{4}} \tilde{u}_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \Lambda_2 \tilde{u}_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \tilde{u}_2\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \Lambda_1 \tilde{u}_2\|_2^{\frac{1}{2}} \\ &\le 2\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_2\|_2^2 + C_\varepsilon \|\partial_1 u_2^{(2)}\|_2^{\frac{5}{2}} \|\tilde{u}_1, \tilde{u}_2\|_2^2. \end{aligned} \tag{3.61}$$

Similarly,

$$\begin{aligned} K_{13} &= -\int \tilde{u}_1 \partial_1 u_3^{(2)} \tilde{u}_3 \, dxdydz \\ &\le C \|\partial_1 u_3^{(2)}\|_2 \|\tilde{u}_1\|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}} \|\tilde{u}_3\|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \\ &\le 2\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{u}_3\|_2^2 + C_\varepsilon \|\partial_1 u_3^{(2)}\|_2^{\frac{5}{2}} \|\tilde{u}_1, \tilde{u}_3\|_2^2. \end{aligned} \tag{3.62}$$

Furthermore,

$$\begin{aligned} K_{14} &= -\int \tilde{u}_2 \partial_2 u_1^{(2)} \tilde{u}_1 \, dxdydz \\ &\le C \|\partial_2 u_1^{(2)}\|_2 \|\tilde{u}_1\|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \|\tilde{u}_2\|_{L^2_{x_2} L^\infty_{x_1} L^4_{x_3}} \\ &\le 2\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_2\|_2^2 + C_\varepsilon \|\partial_2 u_1^{(2)}\|_2^{\frac{5}{2}} \|\tilde{u}_1, \tilde{u}_2\|_2^2. \end{aligned} \tag{3.63}$$

For the term $K_{15}$, applying the similar method to $K_{11}$, we find that

$$\begin{aligned} K_{15} &= -\int \tilde{u}_2 \partial_2 u_2^{(2)} \tilde{u}_2 \, dxdydz \\ &\le C \|\partial_2 u_2^{(2)}\|_{L^\infty_{x_2} L^2_{x_1 x_3}} \|\tilde{u}_2\|^2_{L^2_{x_2} L^4_{x_1 x_3}} \\ &\le \varepsilon \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{u}_2\|_2^2 + C_\varepsilon (\|\partial_2 u_2^{(2)}\|_2^2 + \|\Lambda_2^{\frac{5}{4}} \partial_1 u_1^{(2)}, \Lambda_2^{\frac{5}{4}} \partial_3 u_3^{(2)}\|_2^2)^{\frac{5}{6}} \|\tilde{u}_2\|_2^2. \end{aligned} \tag{3.64}$$

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Similar to $K_{12}$, we can easily bound the terms $K_{16}$, $K_{17}$, and $K_{18}$ as

$$\begin{aligned} K_{16} &= -\int \tilde{u}_2 \partial_2 u_3^{(2)} \tilde{u}_3 \, dxdydz \\ &\le C \|\partial_2 u_3^{(2)}\|_2 \|\tilde{u}_2\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|\tilde{u}_3\|_{L^2_{x_3} L^\infty_{x_2} L^4_{x_1}} \\ &\le 2\varepsilon \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\tilde{u}_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\tilde{u}_3\|_2^2 + C_\varepsilon \|\partial_2 u_3^{(2)}\|_2^{\frac{5}{2}} \|\tilde{u}_2, \tilde{u}_3\|_2^2, \end{aligned} \tag{3.65}$$

$$\begin{aligned} K_{17} &= -\int \tilde{u}_3 \partial_3 u_1^{(2)} \tilde{u}_1 \, dxdydz \\ &\le C \|\partial_3 u_1^{(2)}\|_2 \|\tilde{u}_1\|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}} \|\tilde{u}_3\|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \\ &\le 2\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\tilde{u}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\tilde{u}_3\|_2^2 + C_\varepsilon \|\partial_3 u_1^{(2)}\|_2^{\frac{5}{2}} \|\tilde{u}_1, \tilde{u}_3\|_2^2, \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} K_{18} &= -\int \tilde{u}_3 \partial_3 u_2^{(2)} \tilde{u}_2 \, dxdydz \\ &\le C \|\partial_3 u_2^{(2)}\|_2 \|\tilde{u}_2\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|\tilde{u}_3\|_{L^2_{x_3} L^\infty_{x_2} L^4_{x_1}} \\ &\le 2\varepsilon \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\tilde{u}_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\tilde{u}_3\|_2^2 + C_\varepsilon \|\partial_3 u_2^{(2)}\|_2^{\frac{5}{2}} \|\tilde{u}_2, \tilde{u}_3\|_2^2. \end{aligned} \tag{3.67}$$

Finally, we will estimate the term $K_{19}$. Using the similar method to $K_{11}$, we obtain

$$\begin{aligned} K_{19} &= -\int \tilde{u}_3 \partial_3 u_3^{(2)} \tilde{u}_3 \, dxdydz \\ &\le C \|\partial_3 u_3^{(2)}\|_{L^\infty_{x_3} L^2_{x_1 x_2}} \|\tilde{u}_3\|^2_{L^2_{x_3} L^4_{x_1 x_2}} \\ &\le \varepsilon \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\tilde{u}_3\|_2^2 + C_\varepsilon (\|\partial_3 u_3^{(2)}\|_2^2 + \|\Lambda_3^{\frac{5}{4}} \partial_1 u_1^{(2)}, \Lambda_3^{\frac{5}{4}} \partial_2 u_2^{(2)}\|_2^2)^{\frac{5}{6}} \|\tilde{u}_3\|_2^2. \end{aligned} \tag{3.68}$$

Inserting the above bounds (3.37)–(3.45) into equation (3.36) yields

$$\begin{aligned} |K_1| \le{}& 18\varepsilon \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\tilde{u}_1, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\tilde{u}_2, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\tilde{u}_3\|_2^2 + C_\varepsilon (\|\nabla \mathbf{u}^{(2)}\|_2^{\frac{5}{2}} \\ &+ (\|\partial_1 u_1^{(2)}, \partial_2 u_2^{(2)}, \partial_3 u_3^{(2)}\|_2^2 \\ &+ \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\partial_1 u_1^{(2)}, (\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\partial_2 u_2^{(2)}, (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\partial_3 u_3^{(2)}\|_2^2)^{\frac{5}{6}} \|\tilde{\mathbf{u}}\|_2^2. \end{aligned} \tag{3.69}$$

Next, we will estimate $K_2$, and we write it in terms of components:

$$K_2 = -\int \tilde{u}_i \partial_i \mathrm{w}_j^{(2)} \tilde{\mathrm{w}}_j \, dxdydz = \sum_{k=1}^{9} K_{2k}.$$

To start with $K_{21}$, we rewrite it as follows:

$$\begin{aligned} K_{21} &= -\int \tilde{u}_1 \partial_1 \mathrm{w}_1^{(2)} \tilde{\mathrm{w}}_1 \, dxdydz = -\int \tilde{u}_1 (\mathrm{div}\mathrm{w}^{(2)} - \partial_2 \mathrm{w}_2^{(2)} - \partial_3 \mathrm{w}_3^{(2)}) \tilde{\mathrm{w}}_1 \, dxdydz \\ &= K_{211} + K_{212} + K_{213}. \end{aligned}$$

Using Lemmas 2.1–2.3, we can estimate $K_{211}$ as

$$\begin{aligned}
K_{211} &= -\int \tilde{u}_1 \mathrm{div} \mathrm{w}^{(2)} \tilde{\mathrm{w}}_1 \, dxdydz \\
&\le C\|\mathrm{div}\mathrm{w}^{(2)}\|_{L^\infty_{x_1}L^2_{x_2x_3}} \|\tilde{u}_1\|_{L^\infty_{x_3}L^2_{x_1x_2}} \|\tilde{\mathrm{w}}_1\|_{L^\infty_{x_2}L^2_{x_1x_3}} \\
&\le C\|\mathrm{div}\mathrm{w}^{(2)}\|_2^{\frac{1}{2}} \|\Lambda_1 \mathrm{div}\mathrm{w}^{(2)}\|_2^{\frac{1}{2}} \|\tilde{u}_1\|_2^{\frac{3}{5}} \|\Lambda_3^{\frac{5}{4}}\tilde{u}_1\|_2^{\frac{2}{5}} \|\tilde{\mathrm{w}}_1\|_2^{\frac{3}{5}} \|\Lambda_2^{\frac{5}{4}}\tilde{\mathrm{w}}_1\|_2^{\frac{2}{5}} \\
&\le \varepsilon\|\Lambda_3^{\frac{5}{4}}\tilde{u}_1,\ \Lambda_2^{\frac{5}{4}}\tilde{\mathrm{w}}_1\|_2^2 + C_\varepsilon\big((\|\mathrm{div}\mathrm{w}^{(2)}\|_2^2 + \|\nabla \mathrm{div}\mathrm{w}^{(2)}\|_2^2)^{\frac{5}{3}} + 1\big)\|\tilde{u}_1,\ \tilde{\mathrm{w}}_1\|_2^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
K_{212} &= \int \tilde{u}_1 \partial_2 \mathrm{w}_2^{(2)} \tilde{\mathrm{w}}_1 \, dxdydz \\
&\le C\|\partial_2 \mathrm{w}_2^{(2)}\|_{L^\infty_{x_1}L^2_{x_2x_3}} \|\tilde{u}_1\|_{L^\infty_{x_3}L^2_{x_1x_2}} \|\tilde{\mathrm{w}}_1\|_{L^\infty_{x_2}L^2_{x_1x_3}} \\
&\le \varepsilon\|\Lambda_3^{\frac{5}{4}}\tilde{u}_1,\ \Lambda_2^{\frac{5}{4}}\tilde{\mathrm{w}}_1\|_2^2 + C_\varepsilon\big((\|\partial_2 \mathrm{w}_2^{(2)}\|_2^2 + \|\Lambda_1^{\frac{5}{4}}\partial_2 \mathrm{w}_2^{(2)}\|_2^2)^{\frac{5}{3}} + 1\big)\|\tilde{u}_1,\ \tilde{\mathrm{w}}_1\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
K_{213} &= \int \tilde{u}_1 \partial_3 \mathrm{w}_3^{(2)} \tilde{\mathrm{w}}_1 \, dxdydz \\
&\le C\|\partial_3 \mathrm{w}_3^{(2)}\|_{L^\infty_{x_1}L^2_{x_2x_3}} \|\tilde{u}_1\|_{L^\infty_{x_3}L^2_{x_1x_2}} \|\tilde{\mathrm{w}}_1\|_{L^\infty_{x_2}L^2_{x_1x_3}} \\
&\le \varepsilon\|\Lambda_3^{\frac{5}{4}}\tilde{u}_1,\ \Lambda_2^{\frac{5}{4}}\tilde{\mathrm{w}}_1\|_2^2 + C_\varepsilon\big((\|\partial_3 \mathrm{w}_3^{(2)}\|_2^2 + \|\Lambda_1^{\frac{5}{4}}\partial_3 \mathrm{w}_3^{(2)}\|_2^2)^{\frac{5}{3}} + 1\big)\|\tilde{u}_1,\ \tilde{\mathrm{w}}_1\|_2^2,
\end{aligned}$$

Combining the estimates of $K_{211}$–$K_{213}$, we obtain

$$\begin{aligned}
|K_{21}| &\le 3\varepsilon\|\Lambda_3^{\frac{5}{4}}\tilde{u}_1,\ \Lambda_2^{\frac{5}{4}}\tilde{\mathrm{w}}_1\|_2^2 + C_\varepsilon\big((\|\mathrm{div}\mathrm{w}^{(2)}\|_2^2 + \|\nabla \mathrm{div}\mathrm{w}^{(2)}\|_2^2)^{\frac{5}{3}} \\
&\quad + (\|\partial_2 \mathrm{w}_2^{(2)}\|_2^2 + \|\Lambda_1^{\frac{5}{4}}\partial_2 \mathrm{w}_2^{(2)}\|_2^2)^{\frac{5}{3}} \\
&\quad + (\|\partial_3 \mathrm{w}_3^{(2)}\|_2^2 + \|\Lambda_1^{\frac{5}{4}}\partial_3 \mathrm{w}_3^{(2)}\|_2^2)^{\frac{5}{3}} + 1\big)\|\tilde{u}_1,\ \tilde{\mathrm{w}}_1\|_2^2.
\end{aligned} \tag{3.70}$$

Similar to $K_{12}$, $K_{22}$ can be bounded as

$$\begin{aligned}
K_{22} &= -\int \tilde{u}_1 \partial_1 \mathrm{w}_2^{(2)} \tilde{\mathrm{w}}_2 \, dxdydz \\
&\le C\|\partial_1 \mathrm{w}_2^{(2)}\|_2 \|\tilde{u}_1\|_{L^\infty_{x_2}L^2_{x_1}L^4_{x_3}} \|\tilde{\mathrm{w}}_2\|_{L^\infty_{x_1}L^2_{x_2}L^4_{x_3}} \\
&\le C\|\partial_1 \mathrm{w}_2^{(2)}\|_2 \|\Lambda_3^{\frac{1}{4}}\tilde{u}_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}}\Lambda_2\tilde{u}_1\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}}\tilde{\mathrm{w}}_2\|_2^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}}\Lambda_1\tilde{\mathrm{w}}_2\|_2^{\frac{1}{2}} \\
&\le 2\varepsilon\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{u}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_2\|_2^2 + C_\varepsilon\|\partial_1 \mathrm{w}_2^{(2)}\|_2^{\frac{5}{2}}\|\tilde{u}_1,\ \tilde{\mathrm{w}}_2\|_2^2.
\end{aligned} \tag{3.71}$$

Furthermore,

$$\begin{aligned}
K_{23} &= -\int \tilde{u}_1 \partial_1 \mathrm{w}_3^{(2)} \tilde{\mathrm{w}}_3 \, dxdydz \\
&\le C\|\partial_1 \mathrm{w}_3^{(2)}\|_2 \|\tilde{u}_1\|_{L^\infty_{x_3}L^2_{x_1}L^4_{x_2}} \|\tilde{\mathrm{w}}_3\|_{L^\infty_{x_1}L^2_{x_3}L^4_{x_2}} \\
&\le \varepsilon\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{u}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{\mathrm{w}}_3\|_2^2 + C_\varepsilon\|\partial_1 \mathrm{w}_2^{(2)}\|_2^{\frac{5}{2}}\|\tilde{u}_1,\ \tilde{\mathrm{w}}_3\|_2^2.
\end{aligned} \tag{3.72}$$

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The other terms in $K_2$ can be bounded as $K_{21}$–$K_{23}$. We omit it here. Furthermore, one can easily check that

$$
\begin{aligned}
|K_2| \le\ & 18\varepsilon(\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{u}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{u}_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{u}_3,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{\mathbf{w}}\|_2^2 \\
& + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{\mathrm{w}}_3\|_2^2) \\
& + C_\varepsilon((\|\nabla \mathbf{w}^{(2)}\|_2^2 + \|\mathrm{div}\mathbf{w}^{(2)}\|_2^2 + \|\nabla \mathrm{div}\mathbf{w}^{(2)}\|_2^2 \\
& + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla\tilde{\mathrm{w}}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla\tilde{\mathrm{w}}_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla\tilde{\mathrm{w}}_3\|_2^2)^{\frac{5}{3}} \\
& + \|\nabla \mathbf{w}^{(2)}\|_2^{\frac{5}{2}} + 1)\|\tilde{\mathbf{u}},\ \tilde{\mathbf{w}}\|_2^2. \qquad (3.73)
\end{aligned}
$$

Finally, we will bound the term $K_3$. Applying the similar methods to (3.3)–(3.5), one can easily check that

$$
|K_3| \le 4\varepsilon\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{\mathrm{w}}_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{\mathrm{w}}_3\|_2^2 + C_\varepsilon\|\tilde{\mathbf{u}},\ \tilde{\mathbf{w}}\|_2^2. \qquad (3.74)
$$

Inserting the estimates of (3.69), (3.73), and (3.74) into (3.58), and choosing $\varepsilon$ small enough, we have

$$
\begin{aligned}
& \frac{d}{dt}\|\tilde{\mathbf{u}},\ \tilde{\mathbf{w}}\|_2^2 + 4\kappa\|\tilde{\mathbf{w}}\|_2^2 + \mu\|\mathrm{div}\tilde{\mathbf{w}}\|_2^2 \\
& \quad + (\nu+\kappa)\|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{u}_1, (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\tilde{u}_2, (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{u}_3\|_2^2 + \gamma\|(\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\tilde{\mathbf{w}}\|_2^2 \\
& \le C_\varepsilon(\|\nabla \mathbf{u}^{(2)}\|_2^{\frac{5}{2}} + (\|\partial_1 u_1^{(2)},\ \partial_2 u_2^{(2)},\ \partial_3 u_3^{(2)}\|_2^2 \\
& \quad + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\partial_1 u_1^{(2)},\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\partial_2 u_2^{(2)},\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\partial_3 u_3^{(2)}\|_2^2)^{\frac{5}{6}} \\
& \quad + (\|\nabla \mathbf{w}^{(2)}\|_2^2 + \|\mathrm{div}\mathbf{w}^{(2)}\|_2^2 + \|\nabla \mathrm{div}\mathbf{w}^{(2)}\|_2^2 \\
& \quad + \|(\Lambda_2^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla\tilde{\mathrm{w}}_1,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_3^{\frac{5}{4}})\nabla\tilde{\mathrm{w}}_2,\ (\Lambda_1^{\frac{5}{4}},\ \Lambda_2^{\frac{5}{4}})\nabla\tilde{\mathrm{w}}_3\|_2^2)^{\frac{5}{3}} \\
& \quad + \|\nabla \mathbf{w}^{(2)}\|_2^{\frac{5}{2}} + 1)\|\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}\|_2^2. \qquad (3.75)
\end{aligned}
$$

Applying Gronwall's inequality and the previous estimates, we obtain $(\tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) \equiv 0$. Then, we complete the proof of Proposition 3.3. Furthermore, combining Propositions 3.1–3.3, we complete the proof of Theorem 1.1. ■

**Acknowledgment** The author would like to thank the editors for the excellent handling of our manuscript and to express our thanks to the anonymous reviewers for the constructive and valuable suggestions to update our manuscript.

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