A direct method of obtaining the Foci and Directrices from the general equation

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2 = 0.$$

By D. K. Picken, M.A.

The general equation of the second degree in two variables

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad - \quad - \quad (1)$$

can be brought by a direct process into the form

$$(x - \xi)^2 + (y - \eta)^2 = (lx + my + n)^2,$$

the determination of the constants ξ, η, l, m, n depending only on the solution of quadratic equations; so that the method is suitable for determining the foci, directrices, and eccentricities of conics with given numerical equations.

The equation (1) may be written

$$(\lambda - a)x^2 - 2hxy + (\lambda - b)y^2 = \lambda(x^2 + y^2) + 2gx + 2fy + c$$

and if λ is a root of the quadratic equation

$$(\lambda - a)(\lambda - b) = h^2 \text{ or } \phi(\lambda) \equiv \lambda^2 - (a + b)\lambda + ab - h^2 = 0, \quad (2)$$
$$[\text{Discriminant } \{(a - b)^2 + 4h^2\}]$$

the equation (1) becomes

$$(lx + my)^2 = \lambda(x^2 + y^2) + 2gx + 2fy + c$$

where $\qquad l^2 = \lambda - a, \quad m^2 = \lambda - b \quad$ and $\quad lm = -h,$

or $\qquad (lx + my + \nu)^2 = \lambda(x^2 + y^2) + 2(l\nu + g)x + 2(m\nu + f)y + \nu^2 + c$

$$= \lambda\left\{\left(x + \frac{l\nu + g}{\lambda}\right)^2 + \left(y + \frac{m\nu + f}{\lambda}\right)^2\right\} \quad - \quad (3)$$

if ν be so chosen that

$$(l\nu + g)^2 + (m\nu + f)^2 = \lambda(\nu^2 + c),$$

i.e., if ν be a root of the quadratic equation

$$(l^2 + m^2 - \lambda)\nu^2 + 2(gl + fm)\nu + g^2 + f^2 - \lambda c = 0 \quad - \quad (4)$$

https://doi.org/10.1017/S0013091500033150 Published online by Cambridge University Press

or $$\frac{ab-h^2}{\lambda}\nu^2 - 2(gl+fm)\nu + \lambda c - g^2 - f^2 = 0$$

$$\left(\text{since} \qquad \lambda - l^2 - m^2 = a + b - \lambda = \frac{ab-h^2}{\lambda} \text{ by } (2)\right),$$

of which the discriminant is

$$\left[g^2(\lambda - a) + f^2(\lambda - b) - 2fgh - c(ab - h^2) + (g^2 + f^2)\frac{ab-h^2}{\lambda}\right]$$

$$\equiv \left[(g^2+f^2)\left(\lambda + \frac{ab-h^2}{\lambda}\right) - ag^2 - bf^2 + ch^2 - 2fgh - abc\right]$$

$$\equiv (g^2+f^2)(a+b) - ag^2 - bf^2 + ch^2 - 2fgh - abc, \text{ by } (2)$$

$$\equiv -\Delta.$$

If we suppose a, b, c, f, g, h all real and a positive, the equation (1) is satisfied by *real* point-pairs unless $(ab-h^2)$ and Δ are both positive, *i.e.*, the equation (1) corresponds to a curve which can be drawn on the xy plane unless $(ab-h^2)$ and Δ are both positive.

The roots of the quadratic (2) are then real and they are separated by a or by b (*i.e.*, neither of them lies between a and b); hence, if their values are λ_1 and λ_2 we shall have $\lambda_1 - a$, $\lambda_1 - b$ both positive and $\lambda_2 - a$, $\lambda_2 - b$ both negative, and therefore l_1, m_1 are real numbers and l_2, m_2 are imaginary numbers.

Also $\quad l_1^2 l_2^2 = (a-\lambda_1)(a-\lambda_2) = \phi(a) = -h^2 = (b-\lambda_1)(b-\lambda_2) = m_1^2 m_2^2$

and $\therefore$ since $$l_1 m_1 l_2 m_2 = h^2,$$

we have $$l_1 l_2 + m_1 m_2 = 0,$$

i.e., the two straight lines given by the equations

$$l_1 x + m_1 y = 0, \quad l_2 \iota x + m_2 \iota y = 0$$

(in which the coefficients are real numbers) intersect at right angles.

Consider separately the cases in which the equation (1) represents

(i) an ellipse, (ii) an hyperbola.

(i) *For the Ellipse*: $ab > h^2$ and $\Delta < 0$;

therefore b is of the same sign as a, that is positive, and $(a+b)$ is positive; hence λ_1 and λ_2 are both positive.

Let ν_1, ν_1' be the roots of equation (4) when $\lambda = \lambda_1$

and let ν_2, ν_2' " " " " " " " $\lambda = \lambda$;

then ν_1, ν_1' are real since $\sqrt{-\Delta}$ and $(gl_1 + fm_1)$ are real

and ν_2, ν_2' are complex since $(gl_2 + fm_2)$ is imaginary.

https://doi.org/10.1017/S0013091500033150 Published online by Cambridge University Press

Hence the two equations

$$l_1x + m_1y + \nu_1 = 0, \quad l_1x + m_1y + \nu_1' = 0$$

give "real directrices"; the corresponding "real foci" being

$$\left(-\frac{l_1\nu_1 + g}{\lambda_1}, \ -\frac{m_1\nu_1 + f}{\lambda_1}\right) \text{ and } \left(-\frac{l_1\nu_1' + g}{\lambda_1}, \ -\frac{m_1\nu_1' + f}{\lambda_1}\right);$$

these foci clearly both lie on the line given by

$$l_2(\lambda_1x + g) + m_2(\lambda_1y + f) = 0, \ \textit{i.e.}, \text{ the major axis};$$

and, by a similar process, the "imaginary foci" lie on the perpendicular line given by

$$l_1(\lambda_2x + g) + m_1(\lambda_2y + f) = 0, \ \textit{i.e.}, \text{ the minor axis.}$$

The "eccentricity" e_1 corresponding to the real foci and directrices is given by

$$e_1^2 = \frac{l_1^2 + m_1^2}{\lambda_1} = \frac{+\sqrt{(l_1^2 - m_1^2)^2 + 4l_1^2m_1^2}}{\lambda_1} = \frac{+\sqrt{(a-b)^2 + 4h^2}}{\lambda_1}.$$

(ii) *For the Hyperbola*: $ab < h^2$ and therefore λ_1 is positive and λ_2 negative; Δ may be either negative or positive:

(A) *If* Δ *is negative*, the work is the same as for the case (i);

ν_1, ν_1' are real and the corresponding eccentricity, foci and directrices are real, while ν_2, ν_2' are complex and the corresponding eccentricity, foci and directrices are not real.

(B) *If* Δ *is positive*,

ν_1, ν_1' are complex and the corresponding foci and directrices are not real; the eccentricity is real;

ν_2, ν_2' are pure imaginary numbers, therefore the equations

$$l_2x + m_2y + \nu_2 = 0, \quad l_2x + m_2y + \nu_2' = 0$$

represent straight lines, the real directrices; and the corresponding foci are

$$\left(-\frac{l_2\nu_2 + g}{\lambda_2}, \ -\frac{m_2\nu_2 + f}{\lambda_2}\right) \text{ and } \left(-\frac{l_2\nu_2' + g}{\lambda_2}, \ -\frac{m_2\nu_2' + f}{\lambda_2}\right)$$

and the eccentricity is given by $e^2 = \dfrac{-\sqrt{(a-b)^2 + 4h^2}}{\lambda_2}$.

In this case (B), the introduction of imaginary numbers into the determination of the real foci and directrices may be avoided by writing the original equation

$$(a+\mu)x^2 + 2h + y + (b+\mu)y^2 = \mu(x^2+y^2) - 2gx - 2fy - c$$

and proceeding as above.

Example of Numerical Case.

C. Smith, p. 210.

$$x^2 - 6xy + y^2 - 2x - 2y + 5 = 0 \qquad (\Delta \ \textit{negative})$$

can be written

$$(\lambda - 1)x^2 + 6xy + (\lambda - 1)y^2 = \lambda(x^2+y^2) - 2x - 2y + 5.$$

Choose λ to satisfy

$$(\lambda - 1)^2 = 9, \text{ so that } \lambda_1 = 4, \ \lambda_2 = -2.$$

then λ_1 gives $\quad 3(x+y)^2 = 4(x^2+y^2) - 2x - 2y + 5$

$$i.e., \quad 3(x+y+\nu)^2 = 4\left\{\left(x + \frac{3\nu - 1}{4}\right)^2 + \left(y + \frac{3\nu - 1}{4}\right)^2\right\}$$

if ν be so chosen that $\quad (3\nu - 1)^2 = 2(3\nu^2 + 5)$

$$i.e., \quad 3\nu^2 - 6\nu - 9 = 0 \quad \text{or} \quad \nu^2 - 2\nu - 3 = 0$$

$$\therefore \quad \nu_1 = 3, \ \nu_1' = -1.$$

The directrices are $\quad x + y + 3 = 0, \ x + y = 1$;

the corresponding foci are $\quad (-2, -2)$ and $(1, 1)$

and the eccentricity is $\sqrt{\frac{3}{2}}$.

The Ratio of Incommensurables in Elementary Geometry.

By Professor A. Brown.

https://doi.org/10.1017/S0013091500033150 Published online by Cambridge University Press