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HYPOELLIPTICITY FOR A CLASS OF THE SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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§1. Introduction.

In this paper, we shall investigate the hypoellipticity for a class of degenerate equations of the second order with complex coefficients as a direct extension of the results obtained in [8]. As is well known, the satisfactory general results about hypoellipticity of real operators of the second order have been obtained in [3] and [9], where the assumption that the operators are real plays a crucial role and our aim of this paper is to study the operators with complex coefficients. Our method may be considered as a generalization of the usual variational method replacing the Gårding inequality by the estimate (2.15), (cf. [3], [5]).

Let $\boldsymbol{R}^N$ be N-dimensional Euclidean space regarded as a direct product of three Euclidean spaces $\boldsymbol{R}_x^m, \boldsymbol{R}_y^n$ and $\boldsymbol{R}_t^1$ $(m+n+1=N)$ and generic point of $\boldsymbol{R}^N$ will be denoted by $(x, y, t)=(x_1, \cdots, x_m, y_1, \cdots, y_n, t)$. We shall mainly consider a partial differential equation of the form

$$\begin{aligned} L(x, y, t, D) = D_t u &- \sum_{k,j=1}^{m} D_{x_j}(a^{kj} D_{x_k} u) - \sum_{k,j=1}^{m} D_{y_j}(a_1^{kj} D_{y_k} u) \\ &- 2\sum_{k=1}^{m}\sum_{j=1}^{n} D_{y_j}(g^{kj} D_{x_k} u) + \sum_{k=1}^{m} b^k D_{x_k} u + \sum_{j=1}^{n} b_1^j D_{y_j} u \\ &+ cu = f \qquad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $D_{x_j}=\partial/\partial x_j$ and $a^{kj}, a_1^{kj}, g^{kj}, b^k, b_1^j, c$ and f are complex valued C^∞ functions defined in a domain $\Omega \subset \boldsymbol{R}^N$ which is supposed to contain the origin $\{0\}$ of $\boldsymbol{R}^N$.

The following notations are convenient for the later discussions:

$$A = (a^{kj}(x, y, t))_{1 \leq k,j \leq m}\,, \qquad \mathscr{R}e\, A = (\mathscr{R}e\, a^{kj})_{1 \leq k,j \leq m}\,,$$
$$\operatorname{Im} A = (\operatorname{Im} a^{kj})_{1 \leq k,j \leq m}\,,$$

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$$A_{x_\sigma} = (a^{kj}_{x_\sigma}(x, y, t))_{1\leqq k,j\leqq m}\,,$$
$$Q_0 = Q_0(x, y, t, D) = D_t + \sum_{k=1}^{m} \mathscr{R}e\, b^k D_{x_k}\,,$$
$$Q_j = \sum_{k=1}^{m} \mathscr{R}e\, a^{kj} D_{x_k},\ j = 1, \cdots, m\,, \qquad Q_{m+j} = D_{y_j},\ j = 1, \cdots, n\,.$$

Now we set the hypotheses on L:

(1.2) $$\mathscr{R}e \sum_{k,j=1}^{m} a^{kj}\xi_k\xi_j \geqq 0 \quad \text{in } \Omega \text{ for all } \xi \in \boldsymbol{R}^m\,,$$

(1.3) $$a^{kk}(0) = 0,\ k = 1, \cdots, m\,, \quad a^{kj} = a^{jk}\,, \quad 1 \leqq j,\ k \leqq m\,,$$

(1.4) $$\mathscr{R}e \sum_{k,j=1}^{n} a_1^{kj}\eta_k\eta_j \geqq \alpha\,|\eta|^2 \quad \text{in } \Omega,\ \eta \in \boldsymbol{R}^n\ (\alpha > 0)\,.$$

(1.5) $$\sum_{\sigma=1}^{m} [\mathscr{R}e\, A_{x_\sigma}]^2 + [\mathscr{R}e\, A_t]^2 \leqq C^{1)}\, \mathscr{R}e\, A \quad \text{in } \Omega\,,$$

(1.6) $$\left|\operatorname{Im} a^{kj} \sum_{k,j=1}^{m} \xi_k\xi_j\right| \leqq C\, \mathscr{R}e \sum_{k,j=1}^{m} a^{kj}\xi_k\xi_j \quad \text{in } \Omega,\ \xi \in \boldsymbol{R}^m\,,$$

(1.7) every vector field D_{x_j}, $j = 1, \cdots, m$, (on Ω) can be expressed as a linear combination (with C^∞ coefficients) of $Q_0, Q_1, \cdots, Q_{m+n}, \cdots, [Q_k, Q_j], \cdots, [Q_i, [Q_k, Q_j]], \cdots, [Q_{j_1}, [Q_{j_2}, \cdots, Q_{j_p}], \cdots,], \cdots,$

(1.8) denoting by $A_1 = (a_1^{kj})_{1\leqq k,j\leqq n}$ and $G = (g^{kj})_{1\leqq k\leqq m, 1\leqq j\leqq n}$, we have
$$\mathscr{R}e \begin{pmatrix} \mu A & G \\ {}^tG & \mu A_1 \end{pmatrix} \geqq 0 \quad \text{in } \Omega$$
for some positive constant μ, $0 < \mu < 1$,

(1.9) $$\sum_{j=1}^{n} \left|\sum_{k=1}^{m} g^{kj}\xi_k\right|^2 \leqq C\, \mathscr{R}e\, a^{kj}\xi_k\xi_j \quad \text{in } \Omega,\ \xi \in \boldsymbol{R}^m\,,$$

(1.10) $$\left|\sum_{k=1}^{m} \operatorname{Im} b^k\xi_k\right|^2 \leqq C\, \mathscr{R}e\, a^{kj}\xi_k\xi_j \quad \text{in } \Omega,\ \xi \in \boldsymbol{R}^m\,.$$

We remark that there is no restriction on $\mathscr{R}e\, b^k$, $k = 1, \cdots, m$.

Our main result is to prove the following theorem.

THEOREM 1.1. *Suppose that the operator L given in* (1.1) *satisfies the condition* (1.2)$\sim$(1.10). *Then any distribution $u \in \mathscr{D}'(\Omega)$ satisfying* (1.1) *with $f \in C^\infty(\Omega)$ must be a C^∞ function in Ω.*

EXAMPLES. The following operators satisfy the above conditions:

1) We use the symbols C, $C^1, \cdots$ to express the different positive constants throughout this paper.

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1) $L = D_t - at^2(t^2 + y^2)D_y^2$ in $\boldsymbol{R}^2$, $\mathscr{Re}\, a > 0$,

2) $L = D_t - y^2(x^2 + y^2)D_x^2 - D_y^2 + (1 + ia)xyD_xD_y + bD_x$ in $\boldsymbol{R}^3_{x,y,t}$, a, b real,

3) degenerate elliptic operator treated in [8] considered as a stationary case in the variable t in (1.1).

The inequality (2.10) plays an essential role in the proof of Theorem 1.1 and the hypothese (1.7) is a sufficient condition so that (2.10) is valid. We can get the following result by the same manner as in Theorem 1.1:

Let $\boldsymbol{R}^N$ be N-dimensional Euclidean space regarded as a direct product of three Euclidean spaces $\boldsymbol{R}^m, \boldsymbol{R}^n$ and $\boldsymbol{R}^p$ $(m + n + p = N)$ and generic point of $\boldsymbol{R}^N$ will be denoted by $(x, y, t) = (x_1, \cdots, x_m, y_1, \cdots, y_n, t_1, \cdots, t_p)$.

We consider a partial differential equation of the form

$$
\begin{aligned}
L(x, y, t, D)u = &- \sum_{k,j=1}^{m} D_{x_j}(a^{kj}D_{x_k}u) - \sum_{k,j=1}^{m} D_{y_j}(a_1^{kj}D_{y_k}u) \\
&- 2\sum_{k=1}^{m}\sum_{j=1}^{n} D_{y_j}(g^{kj}D_{x_k}u) + \sum_{k=1}^{m} b^k D_{x_k}u \\
&+ \sum_{j=1}^{n} b_1^j D_{y_j}u + \sum_{\ell=1}^{p} d^\ell D_{t_\ell}u + cu = f \qquad \text{in } \Omega\,,
\end{aligned}
\tag{1.1)'}
$$

where $a^{kj}, a_1^{kj}, g^{kj}, b^k, b_1^j, c$ and f are complex valued C^∞ functions in $\Omega \subset \boldsymbol{R}^N$ as in (1.1) and we remark that only the coefficients $d^\ell = d^\ell(x, y, t)$ $(\ell = 1, \cdots, p)$ are supposed to be real valued C^∞ functions in Ω.

THEOREM 1.2. *For the operator L defined by* (1.1)′ *we suppose that the hypotheses* (1.2)$\sim$(1.6), (1.8)$\sim$(1.10) *and the estimate* (2.10) *are valid, where we take*

$$
Q_0 = \sum_{k=1}^{m} \mathscr{Re}\, b^k \frac{\partial}{\partial x_k} + \sum_{\ell=1}^{p} d^\ell \frac{\partial}{\partial t_\ell}
$$

and

$$
\sum_{1\sigma=}^{m} [\mathscr{Re}\, A_{x_\sigma}]^2 + \sum_{\ell=1}^{p} [\mathscr{Re}\, A_{t_\ell}]^2 \leqq C\, \mathscr{Re}\, A \qquad \text{in } \Omega\,. \tag{1.5}
$$

Then any distribution $u \in \mathfrak{D}'(\Omega)$ satisfying (1.1)′ *with $f \in C^\infty(\Omega)$ must be a C^∞ function in Ω.*

EXAMPLE. The following operator satisfies the condition of Theorem 1.2:

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4) $L = a_1 D_y^2 + dy^k D_t$,
$\mathscr{R}e\, a_1 > 0$, k integer $\geqq 0$,
d real, $d \neq 0$: Fokker-Plank type.

Finally, we remark that we can prove the hypoellipticity of the first boundary value problem for the equation (1.1) by the similar way as in [8].

§ 2. Preliminaries for the proof of Theorem 1.1.

The proof will be obtained by the same steps as in the proof of Theorem 1.1 of [8]. Suggested by [8], we introduce the norm $|||\cdot|||$ and its dual norm $|||\cdot|||'$ by

$$|||u|||^2 = \sum_{k,j=1}^{m} \int_\Omega \mathscr{R}e\, a^{kj} u_{x_j} \bar{u}_{x_j} dxdydt + \sum_{j=1}^{n} \|u_{y_j}\|^2 + \|u\|^2 ,$$

$$|||v|||' = \sup_{w \in C_0^\infty(\Omega)} \frac{|\langle v, w\rangle|}{|||w|||} ,$$

where $\|\cdot\|$ is the usual L^2-norm on Ω and $\langle v, w\rangle$ is the value of $v \in \mathscr{D}'(\Omega)$ evaluated at w.

LEMMA 2.1. *Let L be the operator given in* (1.1). *We have the following estimate with some positive constant C*:

(2.1) $$|||v||| + |||Q_0 v|||' \leqq C(\|v\| + |||Lv|||') , \qquad v \in C_0^\infty(\Omega) .$$

Proof. Obviously we have

(2.2) $$|\langle Lv, \bar{v}\rangle| \leqq |||Lv|||'\, |||v||| , \qquad v \in C_0^\infty(\Omega) .$$

Next, integrating by parts, we have

$$\begin{aligned}\mathscr{R}e \langle Lv, \bar{v}\rangle = &\sum_{k,j=1}^{m} \int_\Omega (\mathscr{R}e\, a^{kj}) v_{x_k} \bar{v}_{x_j} dV + \sum_{k,j=1}^{n} \int_\Omega (\mathscr{R}e\, a_1^{kj}) v_{y_k} \bar{v}_{y_j} dV \\ &+ 2 \sum_{k=1}^{m} \sum_{j=1}^{n} \int_\Omega (\mathscr{R}e\, g^{kj}) v_{x_k} \bar{v}_{y_j} dV \\ &+ \mathscr{R}e \int_\Omega Q_0 v \cdot \bar{v} dV - \operatorname{Im} \sum_{k=1}^{m} \int (\operatorname{Im} b^k) v_{x_k} \bar{v} dV \\ &+ \mathscr{R}e \sum_{j=1}^{n} \int_\Omega b_1^j v_{y_j} \bar{v} dV + \mathscr{R}e \int_\Omega C\, |v|^2\, dV ,\end{aligned}$$

where $dV = dxdydt = dx_1 \cdots dx_m dy_1 \cdots dy_n dt$. By virtue of the hypotheses (1.4), (1.8) and (1.9) we have easily

(2.3) $$\mathscr{R}e \langle Lv, \bar{v}\rangle \geqq C_1 |||v|||^2 - C_2 \|v\|^2 , \qquad v \in C_0^\infty(\Omega)$$

for some positive constants C_1 and C_2. With a new constant C we obtain from (2.2) and (2.3)

$$|||v||| \leqq C(\|v\| + |||Lv|||') , \qquad v \in C_0^\infty(\Omega) . \tag{2.4}$$

Noting that

$$\begin{aligned} Q_0 v = Lv &+ \sum_{k,j=1}^{m} (a^{kj} v_{x_k})_{x_j} + \sum_{k,j=1}^{n} (a_1^{kj} u_{y_k})_{y_j} \\ &+ 2 \sum_{k=1}^{m} \sum_{j=1}^{n} (g^{kj} v_{x_k})_{y_j} - i \sum_{k=1}^{m} \operatorname{Im} b^k \cdot v_{x_k} \\ &- \sum_{j=1}^{n} b_1^j v_{y_j} - cv , \end{aligned}$$

and using the conditions (1.4), (1.6), (1.9) and (1.10) we easily obtain for some constant C

$$|||Q_0 v|||' \leqq C(|||Lv|||' + |||v|||) , \qquad v \in C_0^\infty(\Omega) . \tag{2.5}$$

Indeed, for example, we have by (1.4)

$$\begin{aligned} \left| \left\langle \sum_{k,j=1}^{m} (a^{kj} v_{x_k})_{x_j}, w \right\rangle \right| &= \left| \sum_{k,j=1}^{m} \langle a^{kj} v_{x_k}, w_{x_j} \rangle \right| \\ &\leqq C\, |||v||| \cdot |||w||| \qquad w \in C_0^\infty(\Omega) , \end{aligned}$$

Thus we have

$$\left|\left|\left| \sum_{k,j=1}^{m} (a^{kj} v_{x_k})_{x_j} \right|\right|\right|' \leqq C\, |||v||| , \qquad v \in C_0^\infty(\Omega) .$$

Other terms can be treated similarly. We get the estimate (2.1) combining (2.4) with (2.5).

LEMMA 2.2. *Let L be as above, then we have*

$$\sum_{j=1}^{m} \left|\left|\left| \sum_{k=1}^{m} a_{x_\sigma}^{kj} v_{x_k} \right|\right|\right|' \leqq C\, \|v\| , \qquad \sigma = 1, \cdots, m , \tag{2.6}$$

$$\sum_{j=1}^{m} \left|\left|\left| \sum_{k=1}^{m} a_t^{kj} v_{x_k} \right|\right|\right|' \leqq C\, \|v\| , \tag{2.6'}$$

$$\sum_{j=1}^{m} \left|\left|\left| \sum_{k=1}^{m} a^{kj} v_{x_k} \right|\right|\right|' \leqq C\, \|v\| , \tag{2.7}$$

$$\left|\left|\left| \sum_{k=1}^{m} \operatorname{Im} b^k \cdot v_{x_k} \right|\right|\right|' \leqq C\, \|v\| \tag{2.8}$$

$$\sum_{j=1}^{n} |||v_{y_j}|||' \leqq C\, \|v\| \tag{2.9}$$

for all $v \in C_0^\infty(\Omega)$ *with some positive constant* C.

Proof. For any $w \in C_0^\infty(\Omega)$, we have

$$\left\langle \sum_{k,j=1}^m a_{x_\sigma}^{kj} v_{x_k}, w \right\rangle = -\sum_{j=1}^m \left\langle v, \sum_{k=1}^m a_{x_\sigma}^{kj} w_{x_k} \right\rangle - \sum_{j=1}^m \left\langle v, \sum_{k=1}^m a_{x_\sigma x_k}^{kj} w \right\rangle .$$

Taking account of the assumption (1.5), we have (2.6) by applying Schwartz inequality for the right hand side of the above equality. By the similar way we have (2.6′) and (2.7). Finally for $w \in C_0^\infty(\Omega)$ we have

$$\left\langle \sum_{k=1}^m \mathrm{Im}\, b^k \cdot v_{x_k}, w \right\rangle = -\left\langle v, \sum_{k=1}^m \mathrm{Im}\, b^k \cdot w_{x_k} \right\rangle - \left\langle v, \sum_{k=1}^m Im\, b_{x_k}^k \cdot w \right\rangle ,$$

which gives the estimate (2.8) by assumption (1.10). (2.9) is trivially obtained.

Now we introduce the norm $\|\cdot\|_{(s,r)}$, with s any real number and r nonnegative integer (cf. [2], §2.6), defined by

$$\|v\|_{(s,r)}^2 = (2\pi)^{-(m+1)} \int_{R_y^n} \int_{R_{\xi,\tau}^{m+1}} |\hat{V}(\xi, y, \tau)|^2 (1 + |\xi|^2 + |\tau|^2)^s d\xi dy d\tau$$
$$+ \sum_{\substack{|\alpha| \leqq r \\ \alpha = (\alpha_1, \cdots, \alpha_n)}} \|D_y^\alpha v\|_{L^2(R^{m+n+1})}^2 ,$$
$$\xi = (\xi_1, \cdots, \xi_m) \in R_\xi^m , \qquad \tau \in R_\tau^1 ,$$
$$\hat{V}(\xi, y, \tau) = \iint e^{-i(\langle x,\xi\rangle + \langle t,\tau\rangle)} v(x, y, t) dx dt , \qquad v \in C_0^\infty(R_{x,y,t}^{m+n+1}) .$$

We denote by $H_{(s,r)}(R_{x,y,t}^{m+n+1})$ the completion of $C_0^\infty(R^{m+n+1})$ in the norm $\|\cdot\|_{(s,r)}$.

LEMMA 2.3. *For any compact subset* K *of* Ω, *there exist positive constants* ε *and* $(0 < \varepsilon \leqq 1)$ *and* C *such that*

$$\|v\|_{(\varepsilon,1)} \leqq C(|||v||| + |||Q_0 v|||') , \qquad v \in C_0^\infty(K) . \tag{2.10}$$

Proof. We shall use tentatively the following notation:

$$\|v\|_{(s)}^2 = \int |\hat{v}(\xi, \eta, \tau)|^2 (1 + |\xi|^2 + |\eta|^2)^s d\xi d\eta d\tau , \qquad s \text{ real}$$
$$\|v\|_s^2 = \int |\hat{v}(\xi, \eta, \tau)|^2 (1 + |\xi|^2 + |\eta|^2 + |\tau|^2)^s d\xi d\eta d\tau .$$

Then by the assumption (1.7), we can apply the results of [3], §§4, 5 or the idea of [5] to get the following estimate for some number ε', $0 < \varepsilon' \leqq 1$:

(2.11) $$\|v\|_{(\varepsilon')} \leqq \sum_{j=1}^{m+n} \|Q_j v\| , \qquad v \in C_0^\infty(K) .$$

On the other hand, clearly we have

(2.12) $$\sum_{j=1}^{m+n} \|Q_j v\| \leqq C\,|||v||| , \qquad v \in C_0^\infty(K) .$$

Since $Q_0 v = v_t + \sum_{k=1}^m \mathscr{R}e\, b^k v_{x_k}$, it follows that

$$\|D_t v\|_{(-1)} \leqq C(|||Q_0 v|||' + \|v\|) .$$

Hence we have

(2.13) $$\|v\|_{(\varepsilon')} + \|D_t v\|_{(-1)} \leqq C(|||Q_0 v|||' + |||v|||) , \qquad v \in C_0^\infty(K) ,$$

from which we have

(2.14) $$\|v\|_{\varepsilon'/(1+\varepsilon')} \leqq C(|||Q_0 v|||' + |||v|||) , \qquad v \in C_0^\infty(K) .$$

Thus by the definition of norm $|||\cdot|||$ and $Q_{m+j} = D_{y_j}$, $j = 1, \cdots, n$, the estimate (2.10) follows with $\varepsilon = \varepsilon'/(\varepsilon' + 1)$. Q.E.D.

Combining (2.10) with (2.1) we now come to the main estimate:

(2.15) $$\|v\|_{(\varepsilon,1)} \leqq C(\|v\| + |||Lv|||') , \qquad v \in C_0^\infty(K) .$$

In § 3, we shall prove that it follows from (2.15) that L is hypoelliptic in Ω.

§ 3. Proof of Theorem 1.1.

The main step of the proof is to prove the following lemma which corresponds to Proposition 3.1 of [3].

LEMMA 3.1. *Every* $v \in H_{(0,2)}(\boldsymbol{R}^{m+n+1}) \cap \mathscr{E}'(\Omega)$ *such that* $|||Lv|||' < \infty$ *belongs to* $H_{(\varepsilon,2)}(\boldsymbol{R}^{m+n+1}) \cap \mathscr{E}'(\Omega)$ *with a positive number* ε, *where* Ω *is shrinked if necessary.*

Proof. We can easily see that (2.11) is valid for all $v \in H_{(2.2)}(\boldsymbol{R}^{m+n+1}) \cap \mathscr{E}'(\Omega)$ as in the proof of Lemma (2.6) of [8]. Next if v satisfies the required conditions, we choose $\chi \in C_0^\infty(\Omega)$ so that $0 \leqq \chi \leqq 1$ and $\chi = 1$ in a neighbourhood ω of supp. v and we set

$$v_\delta = \chi(1 - \delta^2 \Delta)^{-1} v , \quad \delta > 0 , \quad \Delta = \sum_{k=1}^m \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial t^2} .$$

Here $(1 - \delta^2 \Delta)^{-1} v$ is defined as the Fourier transform of

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$$(1 + \delta^2(|\xi|^2 + |\tau|^2))^{-1}\hat{v}(\xi, y, \tau)\,, \qquad \xi = (\xi_1, \cdots, \xi_m)\,,$$

$$v_\delta = \chi\cdot(2\pi)^{-m-1}\int_{R^{m+1}_{\xi,\tau}} e^{i(\langle x,\xi\rangle + \langle t,\tau\rangle)}(1 + \delta^2(|\xi|^2 + |\tau|^2))^{-1}\hat{v}(\xi, y, \tau)d\xi d\tau\,.$$

It is clear that v_δ is then in $H_{(2)}(\boldsymbol{R}^{m+n+1}) \cap \mathscr{E}'(\Omega)$, and that $v_\delta \to v$ in L^2 norm as $\delta \to 0$. Hence we may apply (2.15) to v_δ to conclude that $\|v\|_{(\varepsilon,1)} < \infty$, and hence $\|v\|_{(\varepsilon,2)} < \infty$ provided that we can show that $|||Lv_\delta|||'$ remains bounded as $\delta \to 0$. To prove the last assertion we prepare some remarks.

1°. We have

$$K_1(x) = \frac{1}{2}e^{-|x|} = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{ix\cdot\xi}\frac{d\xi}{1+\xi^2}\,, \qquad -\infty < x < \infty\,,$$

$$K_2(x) = C_2[1 + \sqrt{|x|}]^{-1}[1 + \log(1 + |x|^{-1})]e^{-|x|}$$

$$= (2\pi)^{-2}\iint_{-\infty}^{\infty} e^{i(x_1\xi_1 + x_2\xi_2)}\frac{d\xi_1 + d\xi_2}{1 + \xi_1^2 + \xi_2^2}\,, \qquad x \in \boldsymbol{R}^2\backslash\{0\}\,,$$

$$K_\ell(x) = C_\ell|x|^{2-\ell}e^{-|x|} = (2\pi)^{-\ell}\int_{-\infty}^{\infty}\cdots\int e^{i\langle x,\xi\rangle}\frac{d\xi_1\cdots d\xi_\ell}{1+|\xi|^2}\,,$$

$$\ell \geqq 3\,, \qquad x \in \boldsymbol{R}^\ell\backslash\{0\}\,.$$

2°. If Q is a differential operator of order $j \leqq 2$ in $\partial/\partial x$ and $\partial/\partial t$ with coefficients in $C^\infty(\bar{\Omega})$, it follows that

$$\|(1 - \delta^2\Delta)^{-1}Qu\| \leqq C\|u\|\,, \qquad u \in L^2(\Omega) \cap \mathscr{E}'(\Omega)\,. \tag{3.1}$$

3°. When $\chi \in C_0^\infty(\Omega)$ we have

$$|||\chi(1 - \delta^2\Delta)^{-1}w||| \leqq C\,|||w|||\,, \qquad w \in C_0^\infty(\Omega)\,. \tag{3.2}$$

Indeed we have for $w \in C_0^\infty(\Omega)$

$$\Big|\sum_{k,j=1}^{m}\langle \mathscr{R}e\,a^{kj}\cdot(\chi(1-\delta^2\Delta)^{-1}w)_{x_k},\ (\chi(1-\delta^2\Delta)^{-1}w)_{x_j}\rangle\Big|$$

$$\leqq C(\|w\|^2 + \|\chi\sqrt{\mathscr{R}e\,A}\cdot(1-\delta^2\Delta)^{-1}D_x w\|^2)\,.$$

Here we denoted by $D_x w = (w_{x_1}, \cdots, w_{x_m})$ and by $\|F\|^2 = \|F_1\|^2 + \cdots + \|F_m\|^2$ for a m-vector $F = (F_1, \cdots, F_m)$. If we denote by $\mathfrak{A} = \chi\sqrt{\mathscr{R}e\,A}$, $\mathfrak{A}$ is an $m \times m$ matrix which is uniformly Lipschitz continuous and compactly supported in Ω by the results of [1]. Now for the second term of the right hand side, we have

$$\|\mathfrak{A}(1-\delta^2\Delta)^{-1}D_x w\|^2$$

$$= \|(1-\delta^2\Delta)^{-1}\mathfrak{A}D_x w + [\mathfrak{A}D_x,\ (1-\delta^2\Delta)^{-1}I]w\|^2 \tag{3.3}$$

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$$\leqq 2\|\mathfrak{A}D_x w\|^2 + 2\|[\mathfrak{A}D_x,\ (1-\delta^2\Delta)^{-1}I]w\|^2$$
$$\leqq C|||w|||^2 + 2\|[\mathfrak{A}D_x,\ (1-\delta^2\Delta)^{-1}I]w\|^2 .$$

Partial integration proves

$$[\mathfrak{A}D_x,\ (1-\delta^2\Delta)^{-1}I]w(x,y,t)$$
$$= \delta^{-(m+1)}\int_{R^{m+1}_{\xi,\tau}} K_{m+1}\Big(\frac{x-\xi}{\delta},\frac{t-\tau}{\delta}\Big)[\mathfrak{A}(x,y,t) - \mathfrak{A}(\xi,y,\tau)D_\xi w(\xi,y,\tau)d\xi d\tau$$
$$= \delta^{-(m+1)}\int_{R^{m+1}} K_{m+1}\Big(\frac{x-\xi}{\delta},\frac{t-\tau}{\delta}\Big)w(\xi,y,\tau)D_\xi\mathfrak{A}(\xi,y,\tau)d\xi d\tau$$
$$= \delta^{-(m+2)}\int_{R^{m+1}} w(\xi,y,\tau)\cdot[\mathfrak{A}(x,y,t) - \mathfrak{A}(\xi,y,\tau)]D_\xi K_{m+1}\Big(\frac{x-\xi}{\delta},\frac{t-\tau}{\delta}\Big)d\xi d\tau .$$

By account of the explicit expression of K_{m+1} in 1° and by the uniform Lipschitz continuity of $\mathfrak{A}$, the L^2 norm of last two terms is bounded above by $\|w\|^2$. This estimate combined with (3.3) gives (3.2).

Completion of the proof of Lemma 3.1: We recall that it remains to prove that $|||Lv_\delta|||'$ is bounded as $\delta\to 0$. In the neighbourhood ω of supp. v we have $(1-\delta^2\Delta)v_\delta = v$ and

$$(1-\delta^2\Delta)Lv_\delta = Lv - \delta^2[L,\Delta]v_\delta$$
$$= Lv + 2\delta^2\sum_{\sigma=1}^{m}\sum_{k,j=1}^{m}(a^{kj}_{x_\sigma}v_{\delta x_k x_\sigma})_{x_j} + 2\delta^2\sum_{k,j=1}^{m}(a^{kj}_t v_{\delta x_k t})_{x_j} + \delta^2 Bv_\delta ,$$

where B is a differential operator of the form

$$B = \sum_{\substack{|\alpha|+\alpha'\leqq 2\\ |\alpha|\leqq 2,\ \alpha'\leqq 1\\ |\beta|\leqq 2}} B^{\alpha,\alpha',\beta}(x,y,t)D_x^\alpha D_y^\beta D_t^{\alpha'} ,$$

which may be considered with compact support. It follows that we have everywhere

$$(1-\delta^2\Delta)Lv_\delta = Lv + 2\delta^2\sum_{\sigma=1}^{m}\sum_{k,j=1}^{m}(a^{kj}_{x_\sigma}v_{\delta x_k x_\sigma})_{x_j} + 2\delta^2\sum_{k,j=1}^{m}(a^{kj}_t v_{\delta x_k t})_{x_j} + \delta^2 Bv_\sigma + h_\delta ,$$

where h_δ is a function such that it vanishes in ω, supp. $h_\delta\subset$ supp. χ and $\|h_\delta\|\to 0$ as $\delta\to 0$ in view of 1°. Hence

$$Lv_\delta = \chi_1\{(1-\delta^2\Delta)^{-1}Lv + 2\delta^2 B_1 v_\delta + 2\delta^2 B_2 v_\delta + (1-\delta^2\Delta)^{-1}\delta^2 Bv_\delta + (1-\delta^2\Delta)^{-1}h_\delta\} , \tag{3.4}$$

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where χ_1 is a function in $C_0^\infty(\Omega)$ which is equal to 1 in supp. χ. We remark that from 3° we have

$$(3.5)\qquad |||\chi_1(1-\delta^2\Delta)^{-1}f|||' \leqq C\,|||f|||'\,, \qquad f \in \mathscr{D}'(\Omega) \cap \mathscr{E}'(\Omega)\,.$$

Therefore, it follows that

$$|||\chi_1(1-\delta^2\Delta)^{-1}Lv|||' \leqq C\,|||Lv|||'\,.$$

The last two terms are bounded in L^2 norm in view of 2° and by the assumption that $v \in H_{(0,2)}(\Omega) \cap \mathscr{E}'(\Omega)$. For the second and third terms, we use (2.6), 2° and (3.5) then we have

$$|||\delta^2 B_1 v_\delta|||' + |||\delta^2 B_2 v_\delta|||' \leqq C\,\|v_\delta\| \leqq C'\,\|v\|\,.$$

Thus we have $|||Lv_\delta|||' < \infty$ as $\delta \to 0$. This completes the proof of Lemma 3.1.

Proof of Theorem 1.1. The following process is almost the same as in § 3 of [8].

Given a function $\psi \in C_0^\infty(\Omega)$ and an integer $r \geqq 2$, we may assume, by the partial hypoellipticity of L in the direction y (cf. [2], § 4.3), that $\psi u \in H_{(s,1)}(\boldsymbol{R}^{m+n+1}) \cap \mathscr{E}'(\Omega)$ for some real number s. For the proof of Theorem 1.1 it suffices to show that s can be replaced by $s+\varepsilon$. Indeed, it follows that $v \in H_{(s,r)}^{\mathrm{loc}}(\Omega)$ for any s and r, which means that $u \in C^\infty(\Omega)$ by the Sobolev lemma.

Let E be a pseudo-differential operator with symbol $e(\xi,\tau) = (1+|\xi|^2+|\tau|^2)^{s/2}$ (cf. [4]), and set $v = \chi E\psi u$ where $\chi \in C_0^\infty(\Omega)$. If we can show that $v \in H_{(\varepsilon,r)}$ for every χ and ψ we will have $E\psi u \in H_{(\varepsilon,2)}^{\mathrm{loc}}(\Omega)$, hence $u \in H_{(s+\varepsilon,2)}^{\mathrm{loc}}$ since E is elliptic. As $r \geqq 2$, it is clear that $v \in H_{(0,2)}(\Omega) \cap \mathscr{E}'(\Omega)$, so in view of Lemma 3.1 it remains only to show that $|||Lv|||' < \infty$. We note that $E' = \chi E\psi$ is considered as a compactly supported pseudo-differential operator of order s (in (x,x')) with parameter y (cf. [4]) and $Lv = LE'u$. Taking account of $E'Lu = E'f$ and $Lv = E'f + LE'u - E'f$, it now suffices to show that $|||LE'u - E'f|||' < \infty$ to prove $|||Lv|||' < \infty$. We have

$$LE'u - E'f = \sum_{k,j=1}^{m} [a^{kj}D_{x_k}D_{x_j}, E']u - \sum_{k,j=1}^{m} [a_{x_j}^{kj}D_{x_k}, E']u + \sum_{j=1}^{n} E_0^j u_{y_j} + E_1 u\,,$$

where $\sum_{k,j=1}^{m}[a_{x_j}^{kj}D_{x_k}, E']$, E_0^j and E_1 are compactly supported pseudo-differential operators of order $\leqq s$ in the direction (x, t) and we have

$$\sum_{k,j=1}^{m}\|[a_{x_j}^{kj}D_{x_k}, E']u\| + \sum_{j=1}^{n}\|E_0^j u_{y_j}\| + \|E_1 u\| < \infty$$

by assumption, so we have only to analyse the first summation in the right hand side. We have

$$\begin{aligned}[a^{kj}D_{x_k}D_{x_j}, E']u &= [a^{kj}D_{x_k}, E']D_{x_j}u + a^{kj}D_{x_k}[D_{x_j}, E']u \\ &= D_{x_j}[a^{kj}D_{x_k}, E']u + [D_{x_j}, [a^{kj}D_{x_k}, E']]u \\ &\quad + a^{kj}D_{x_k}[D_{x_j}, E']u\,.\end{aligned}$$

On the other hand, if we denote by $\sigma(E)$ a symbol of a pseudo-differential operator E, a simple calculation (cf. [4]) proves the equality

$$\sigma([a^{kj}D_{x_k}, E']) = a^{kj}\sigma(E_1^k) + \sum_{\nu=1}^{m} a_{x_\nu}^{kj}\sigma(E_2^\nu) + a_t^{kj}\sigma(E_3) + \sigma(E_4)$$

where E_1^k $(k = 1, \cdots, m)$, E_2^ν $(\nu = 1, \cdots, m)$, E_3 and E_4 are pseudo-differential operators with parameter y of order $\leqq s$ and $\leqq s-1$ respectively (in the direction (x, t)). This equality leads us, by using (2.6), (2.6′) and (2.7)

$$\left|\left|\left|\sum_{k,j=1}^{m} D_{x_j}[a^{kj}D_{x_k}, E']u\right|\right|\right|' < \infty\,.$$

Obviously we have

$$|||[D_{x_j}, [a^{kj}D_{x_k}, E']]u|||' < \infty$$

since the order of $[D_{x_j}, [a^{kj}D_{x_k}, E']]$ is less than or equal to s. Finally for any $w \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}&\left|\sum_{k,j=1}^{m}\langle a^{kj}D_{x_k}[D_{x_j}, E']u, w\rangle\right| \\ &\quad = \left|\sum_{j=1}^{m}\left\langle [D_{x_j}, E']u, \sum_{k=1}^{m} a^{kj}w_{x_k}\right\rangle\right| \\ &\quad \leqq C\,|||w|||\,,\end{aligned}$$

hence we have

$$\left|\left|\left|\sum_{k,j=1}^{m} a^{kj}D_{x_k}[D_{x_j}, E']u\right|\right|\right|' < \infty\,.$$

The above investigation implies that

$$|||LE'u - E'f|||' < \infty \ .$$

Thus we have $|||Lv|||' < \infty$ and this completes the proof of Theorem 1.1.

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