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ON ELLIPTICALLY EMBEDDED SUBGROUPS OF SOLUBLE GROUPS

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1. Introduction. We call a subset X of a group an *elliptic set* if there is an integer n such that each element of the group generated by X can be written as a product of at most n elements of $X \cup X^{-1}$. The terminology is due to Philip Hall, who investigated elliptic sets in lectures given in Cambridge in the 1960's. Hall was chiefly interested in sets X which are unions of conjugacy classes, but among other things he proved that if H, K are subgroups of a finitely generated nilpotent group then their union $H \cup K$ is elliptic. We shall say that a subgroup H of an arbitrary group G is *elliptically embedded* in G, and we write H ee G, if $H \cup K$ is an elliptic set for each subgroup K of G. Thus H ee G if for each subgroup K there is an integer n (depending on K) such that

$$\langle H, K \rangle = HK \dots HK$$

where the product has $2n$ factors.

The concept of elliptic embedding has no significance for finite groups and our principal results concern groups which are close to being torsion-free. Every quasinormal subgroup H of a group G is elliptically embedded, for to say that H is quasinormal is just to say that $\langle H, K \rangle = HK$ for each subgroup K. Further instances of elliptically embedded subgroups are given in Section 2. From the result of Hall mentioned above it follows easily that every subgroup of a finitely generated finite by nilpotent group is elliptically embedded (see Proposition 1 in Section 2). Our first main result is a partial converse to this:

THEOREM 1. *Let $G = \langle g_1, \dots, g_s \rangle$ be soluble and suppose that $\langle g_i \rangle$ is elliptically embedded in G for $i = 1, \dots, s$. Then G is finite by nilpotent.*

This has the immediate

COROLLARY 1. *Let G be a locally soluble group having no non-trivial normal torsion subgroup. If $\langle g \rangle$* ee *G then the normal closure of $\langle g \rangle$ in G is locally nilpotent.*

This follows since each subgroup generated by finitely many conjugates of g satisfies the hypothesis of Theorem 1. Our second main result is a stronger result for the case where G is soluble:

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THEOREM 2. *Let G be a soluble group having no non-trivial normal torsion subgroup. If $\langle g \rangle$ is elliptically embedded in G then $\langle g \rangle$ is subnormal in G.*

As a consequence of the above theorem and Proposition 3 we have the following result:

COROLLARY 2. *Let G be a torsion-free group and $\langle g \rangle$ a cyclic subgroup whose normal closure in G is soluble and either finitely generated or minimax. Then $\langle g \rangle$* ee *G if and only if $\langle g \rangle$ is a subnormal subgroup of G.*

Let N be the normal closure of $\langle g \rangle$ in G. If $\langle g \rangle$ is subnormal in G then N is locally nilpotent ([**3**], Section 2.3, p. 61), and so nilpotent (see for example [**3**], Theorem 6.36), and we shall prove in Proposition 3 that any subgroup of G whose normal closure is a nilpotent minimax group is elliptically embedded in G. The other implication of the corollary follows from Theorem 2.

We have already explained why torsion presents an obstacle in the study of elliptically embedded subgroups. A more serious restriction is the restriction to cyclic subgroups. Our treatment of cyclic subgroups rests on some delicate calculations with complex numbers of bounded modulus (Lemma 1 and Lemma 2). It is likely that results like Corollary 1 and Theorem 2 also hold for elliptically embedded free abelian subgroups of finite rank; but this appears to be a difficult problem. They certainly do not hold for free abelian subgroups of countably infinite rank. Let G be the group of matrices

$$\begin{pmatrix} u & a \\ 0 & 1 \end{pmatrix} \quad (u, a \in \mathbf{Q} \text{ and } u > 0),$$

and let H be the subgroup of diagonal matrices in G. It is straightforward, if a little tedious, to verify that H is elliptically embedded in G. On the other hand G is torsion-free and not locally nilpotent, and yet the normal closure H^G of H in G equals G so that H is not subnormal in G.

2. Some sufficient conditions for elliptic embedding.

PROPOSITION 1. (a) *If H is a subnormal subgroup of G and H^G satisfies the maximal condition for subnormal subgroups then H* ee *G.*

(b) *Let F be a finite normal subgroup of G. If $H \leqq G$ and HF/F* ee *G/F then H* ee *G.*

Proof. (a) Let $K \leqq G$ and define $G_1 = \langle H, K \rangle$. We show by induction on the defect of H in G_1 (that is, the least d for which there is a series

$$H = H_d \lhd \ldots \lhd H_1 \lhd H_0 = G_1)$$

that $G_1 = (HK)^n$ for some n. This is clear if $d = 1$, so we assume $d > 1$. Write

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$$L = H^{H_{d-2}}.$$

For each $g \in H_{d-2}$ we have $H^g \lhd H_{d-1}$, and since H^G satisfies the maximal condition for subnormal subgroups there is a finite set $\{g_1, \ldots, g_r\}$ of elements of H_{d-2} such that

$$L = H^{g_1} \ldots H^{g_r}.$$

Since $G_1 = \langle H, K \rangle$, some set $(HK)^s$ contains all the elements g_i and their inverses. Thus

$$H^{g_i} \leqq (HK)^s H (HK)^s = (HK)^{2s}$$

for each i, and $L \leqq (HK)^{2rs}$. On the other hand L is subnormal of defect at most $d - 1$ and therefore $G_1 = (LK)^m$ for some m. It follows that $G_1 = (HK)^{2rsm}$ as required.

(b) We have $\langle H, K \rangle \leqq (HK)^n F$ for some integer n, so that

$$\langle H, K \rangle = (HK)^n (F \cap \langle H, K \rangle).$$

There is an integer s such that the finite set $(F \cap \langle H, K \rangle)$ lies in $(HK)^s$, and so

$$\langle H, K \rangle = (HK)^{n+s}.$$

Proposition 1 shows in particular that all subgroups of finitely generated finite by nilpotent groups are elliptically embedded. However a direct approach yields a little more. Let C be nilpotent of class c and let $C = \langle A, B \rangle$ where

$$A = \langle a_1, \ldots, a_r \rangle \quad \text{and} \quad B = \langle b_1, \ldots, b_r \rangle.$$

If

$$\gamma_k(C) \leqq (AB)^f \text{ modulo } \gamma_{k+1}(C)$$

for some k, then modulo $\gamma_{k+2}(C)$ we have

$$\begin{aligned} \gamma_{k+1}(C) &\equiv [A, \gamma_k(C)][\gamma_k(C), B] \\ &\equiv \prod_{i=1}^{r} [a_i, \gamma_k(C)] \prod_{j=1}^{r} [\gamma_k(C), b_j] \\ &\leqq (A(AB)^f A(AB)^f)^r ((AB)^f B(AB)^f B)^r \\ &\equiv (AB)^{4fr}. \end{aligned}$$

It follows that $C = (AB)^t$ where $t = (4r)^c$. This leads to the first assertion in the next result:

PROPOSITION 2. (a) *Let G be nilpotent of finite (Prüfer) rank. Then every subgroup of G is elliptically embedded; indeed for all $H, K \leqq G$ one has $\langle H, K \rangle = (HK)^t$ where $t = (4r)^c$ and r, c are respectively the rank and* *class of G.*

(b) *If G is nilpotent of class* 2 *then every finitely generated subgroup H of G is elliptically embedded.*

Proof. (a) Let $g \in \langle H, K \rangle$. There are finitely generated subgroups A, B, of H, K respectively such that $g \in \langle A, B \rangle$, and each of A, B can be generated by r elements. Thus from above $g \in (AB)^t \leqq (HK)^t$.

(b) Let $H = \langle h_1, \ldots, h_n \rangle \leqq G$ and $K \leqq G$. Then each element of the subgroup $[H, K]$ may be written in the form $[h_1, k_1] \ldots [h_n, k_n]$ for suitable $k_1, \ldots, k_n$ in K. Thus $[H, K] \leqq (HK)^{2n}$ and

$$\langle H, K \rangle = H[H, K]K \leqq H(HK)^{2n}K = (HK)^{2n}.$$

It is not true in general that a subgroup of a nilpotent group of class 2 is elliptically embedded. For let A and B be abelian groups of exponent an odd prime p with bases $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ respectively, and let

$$G = \langle A, B; [a_i, b_j] = c_{ij}, [c_{ij}, G] = 1 \text{ for all } i, j \rangle.$$

Then $(AB)^k \neq G$ for each integer k. To see this, pick $n > 2k$ and let

$$A_n = \langle a_{n+1}, \ldots \rangle \ B_n = \langle b_{n+1}, \ldots \rangle \text{ and } \bar{G} = G/\langle A_n^G, B_n^G \rangle.$$

Then

$$|\bar{G}| = p^n \cdot p^n \cdot p^{n^2} = p^{2n+n^2}$$

while

$$(\bar{A}\bar{B})^k = p^{2nk} < p^{n^2}.$$

A similar example shows that cyclic subgroups of a group of class 3 are not in general elliptically embedded.

PROPOSITION 3. *If H is a subgroup of a group G such that H^G is a nilpotent minimax group, then H ee G.*

Proof. Let $K \leqq G$ and write $J = \langle H, K \rangle$. We need to show that $J \leqq (HK)^n$ for some integer n. The proof is by induction on the nilpotency class c of H^J. Let L be the cth term of the lower central series of H^J. Since H^J is generated by

$$X = \{h^k; h \in H, k \in K\},$$

the subgroup L is generated by

$$Y = \{ [x_1, \ldots, x_c]; x_i \in X, i, = 1, 2, \ldots, c\}.$$

Let N be a finitely generated subgroup of L such that L/N is periodic. By Lemma 11 of [**2**] there is an integer f, depending only on the Prüfer rank and number of non-trivial Sylow subgroups of L/N, such that every finitely generated subgroup of L/N is generated by the images in L/N of

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f suitably chosen elements of Y. It is easy to see that each element of Y, and indeed each power of an element of Y, lies in $K(HK)^t$ where

$$t = (3/2)(c^2 - c) + 1,$$

and so we have $L \leqq NK(HK)^t$. However N lies in a join of finitely many conjugates H^k of H under elements of K, and since each H^k is elliptic in H^J by Proposition 2, this join is a product of finitely many subgroups H^k and so lies in $(HK)^s$ for some s. It follows that

$$L \leqq (HK)^{s+t}.$$

If $c = 1$ we therefore have $H^J \leqq (HK)^{s+t}$ and so

$$\langle H, K \rangle \leqq H^J K \leqq (HK)^{s+t},$$

as required. If $c > 1$ then by induction we have $J = L(HK)^n$ for some integer n, so that

$$J \leqq (HK)^{s+t+n}$$

and again the result follows.

In Section 3 we shall need to study groups which are split extensions $A\langle g \rangle$ of an abelian normal subgroup A by a cyclic group $\langle g \rangle$ which is elliptically embedded in $A\langle g \rangle$. The following result is therefore of some interest.

PROPOSITION 4. *Let G be a split extension of an abelian normal subgroup A by a cyclic subgroup $\langle g \rangle$. If G is nilpotent then $\langle g \rangle$ ee $\langle G \rangle$.*

Proof. Let $K \leqq G$. If $K \leqq A$ then

$$[K, \langle g \rangle] = \{ [k_1, g] \dots [k_c, g]; k_i \in K\}$$

where c is the nilpotency class of G, and so

$$\langle K, \langle g \rangle \rangle = K[K, \langle g \rangle]\langle g \rangle \leqq K(K\langle g \rangle)^{2c}\langle g \rangle = (K\langle g \rangle)^{2c}.$$

If $K \nleqq A$, then $(K \cap A)^{\langle g \rangle}$ is normal and lies in $(K\langle g \rangle)^{2c}$. Passing to the quotient group $G/((K \cap A)^{\langle g \rangle})$, we may therefore assume $K \cap A = 1$. However K is then cyclic and $\langle K, g \rangle$ is a finitely generated nilpotent group, so that

$$\langle K, g \rangle = (K\langle g \rangle)^m \quad \text{for some integer } m.$$

3. Proofs of the theorems. We approach Theorem 1 through a series of lemmas. The first two lemmas which deal with complex numbers of bounded modulus are crucial for the proof of Theorem 1.

LEMMA 1. *Let k, t be positive integers and let $\lambda \in \mathbf{C}$ with $|\lambda| < 1$. There is a number $\omega = \omega(k, t, \lambda) > 0$ such that $|\theta| \geqq \omega$ for each non-zero expression*

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$$\theta = \sum_{i=1}^{k} a_i \lambda^{r_i}$$

with

$$\sum_{i=1}^{l} a_i \lambda^{r_i} \neq 0 \quad \textit{for } 1 \leqq l \leqq k,$$

with a_i an integer satisfying $|a_i| \leqq t$ for each i, and with $0 = r_1 < r_2 < \ldots r_k$.

Proof. We prove the result by induction on k. Clearly we may take

$$\omega(1, t, \lambda) = 1.$$

Suppose that $\omega(k - 1, t, \lambda) = \omega'$ is defined, and let m be the least integer with $|\lambda^m| < \omega'/2t$. Thus if θ is an expression

$$\sum_{i=1}^{k} a_i \lambda^{r_i}$$

of the sort under consideration and if $r_k \geqq m$ then we have

$$|\theta| \geqq \left| \sum_{i=1}^{k-1} a_i \lambda^{r_i} \right| - |a_k \lambda^{r_k}| \geqq \omega' - t(\omega'/2t) = \omega'/2.$$

Since there are only finitely many expressions

$$\sum_{i=1}^{k} a_i \lambda^{r_i}$$

with $r_k < m$ the result follows.

LEMMA 2. *Let α, θ be complex numbers with $|\alpha| > 1$ and $\theta \neq 0$, and let l be a positive integer. Then there exists a positive integer n such that $n\theta$ is not of the form*

$$\sum_{j=1}^{l} \epsilon_j \alpha^{m_j}$$

with each $\epsilon_j \in \{0, 1, -1\}$.

Proof. Let N be a positive integer. We shall estimate the number of integers n with $|n| \leqq N$ for which $n\theta$ can have the required form. Each sum

$$\sum_{j=1}^{l} \epsilon_j \alpha^{m_j}$$

can be written in the form

(*) $s = a_1\alpha^{m_1} + \ldots + a_k\alpha^{m_k}$,

with $k \leqq l$, $|a_i| \leqq l$, $m_1 > m_2 > \ldots > m_k$, and no partial sum equal to zero. Fix k and consider the s with $|s| \leqq N|\theta|$. We have

$$\alpha^{-m_1}s = \sum_{i=1}^{k} a_i\alpha^{m_i - m_1} = \sum_{i=1}^{k} a_i\lambda^{m_1 - m_i}$$

where $\lambda = 1/\alpha$, and so

$$|\alpha^{-m_1}s| \geqq \omega_k,$$

where $\omega_k = \omega(k, l, \lambda)$ is as defined in Lemma 1. Thus

$$\omega_k|\alpha|^{m_1} \leqq s \leqq N|\theta|$$

so that

$$m_1 + \log \omega_k < \log N + \log|\theta|$$

where logarithms are to base $|\alpha|$. Let $d_k = d_k(k, l, \alpha, \theta)$ be the least integer such that

$$l|\alpha|^{-d_k} < |\theta|/2k.$$

Thus the sum of the terms in (*) with exponent $m_i \leqq -d_k$ has absolute value bounded by

$$\sum_{i=1}^{k} (\max|a_i|)|\alpha|^{-d_k} < |\theta|/2.$$

The number of possibilities for the sum of the remaining terms is at most

$$[2l(\log N + \log|\theta| - \log \omega_k + d_k + 1)]^k = [2l(f_k + \log N)]^k$$

say, where $f_k = f_k(k, l, \alpha, \theta)$. Thus the number of $n\theta$ with $|n| \leqq N$ of form (*) is at most $[2l(f_k + \log N)]^k$ and the number of $n\theta$ with $|n| \leqq N$ of the form

$$\sum_{j=1}^{l} \epsilon_j\alpha^{m_j}$$

is at most $l[2l(f + \log N)]^l$ where

$$f = \max\{f_1, \ldots, f_l\}.$$

If the result is false we therefore have, for large N,

$$N \leqq l[2l(f + \log N)]^l \leqq l[2l \cdot 2 \log N]^l$$

$$\leqq l(\log N)^{2l} \leqq (\log N)^{2l+1}$$

which is clearly a contradiction.

LEMMA 3. *Let $\langle g \rangle$ be a cyclic group and A a torsion-free abelian group of finite rank on which $\langle g \rangle$ acts rationally irreducibly. If $\langle g \rangle$ is elliptically embedded in the split extension G of A by $\langle g \rangle$, then $\langle g \rangle$ acts trivially on A.*

Proof. Suppose otherwise, and choose $a \in A \backslash 1$. If $\langle g, g^a \rangle = \langle g \rangle$ then

$$[a, g] \in A \cap \langle g \rangle = 1,$$

so that $C_A(g)$ is a non-trivial $\langle g \rangle$-invariant subgroup of A and a contradiction ensues. Thus

$$\langle g \rangle < \langle g, g^a \rangle = \langle g \rangle A \cap \langle g, g^a \rangle = \langle g \rangle (A \cap \langle g, g^a \rangle),$$

and so

$$B = A \cap \langle g, g^a \rangle$$

is a non-trivial subgroup of A. Since $\langle g \rangle$ ee G there is an integer n such that each element of $\langle g, g^a \rangle$ is a product of n terms of the form

$$g^k (g^a)^l = g^{k+l} a a^{-g^l}.$$

Collecting the powers of g in such a product on the left, we see that each element of $\langle g, g^a \rangle$ is a product of a power of g and $2n$ conjugates of $a^{\pm 1}$ under elements of $\langle g \rangle$. Thus, in additive notation, each element of B is of the form

$$a \sum_{i=1}^{2n} \pm g^{u_i}$$

with each u_i in Z.

Now $V = A \times_{\mathbf{Z}} \mathbf{Q}$ is an irreducible $\mathbf{Q}\langle g \rangle$-module, and by Schur's Lemma the centralizer ring

$$\Gamma = \mathrm{End}_{\mathbf{Q}\langle g \rangle} V$$

is a division ring finite dimensional over $\mathbf{Q}$. The image of $\langle g \rangle$ in $\mathrm{End}_{\mathbf{Q}} V$ clearly lies in and spans Γ so that Γ is an algebraic number field. Further, regarded as a Γ-vector space, V must be one dimensional. Let α be the image of g in Γ and choose $b \in B \backslash 0$, so that $b = a\varphi$ for some φ in Γ. Thus for each integer m we can write $ma\varphi$ in the form

$$a \sum_{i=1}^{2n} \pm \alpha^{u_i},$$

so that each $m\varphi$ has the form

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$$\sum_{i=1}^{2n} \pm \alpha^{u_i}.$$

If α is not a root of 1 then Γ can be embedded in **C** so that $|\alpha| > 1$ (see for instance [**1**], p. 122), and we have contradiction to Lemma 2. If α is a root of 1 then in any embedding of Γ in **C** we have

$$\left| \sum_{i=1}^{2n} \pm \alpha^{u_i} \right| \leqq \sum_{i=1}^{2n} |\alpha^{u_i}| = 2n$$

so that $|m\varphi| \leqq 2n$ for each m. This too yields a contradiction and the lemma follows.

LEMMA 4. *Let $\langle g \rangle$ be a cyclic group and A a finitely generated $\mathbf{Z}\langle g \rangle$-module. If $\langle g \rangle$ is elliptically embedded in the split extension G of A by $\langle g \rangle$, then A is finitely generated as an abelian group.*

Proof. We suppose the result false. Since A is a noetherian module it has a maximal submodule L with respect to A/L not being finitely generated as an abelian group. Of course $\langle g \rangle$ will be elliptically embedded in the split extension of A/L by $\langle g \rangle$, and so we may replace A by A/L.

Let $a \in A$ and consider the group $\langle a, g \rangle$. Since $\langle g \rangle$ ee G, there is an integer n such that each element of $\langle a, g \rangle$ is a product of n terms $g^k a^l$ with k, l in **Z**. Collecting powers of g in such a product on the left, we can see that each element of $\langle a, g \rangle$ is a product of a power of g and n conjugates of powers of a under elements of $\langle g \rangle$. Thus, in additive notation, each element of the $\mathbf{Z}\langle g \rangle$-module A_0 generated by a has the form

$$a \sum_{i=1}^{n} l_i g^{u_i} \quad \text{with } l_i,\ u_i \text{ in } \mathbf{Z} \text{ for each } i.$$

Suppose that A is not **Z**-torsion-free and choose a to have prime order p. Then A_0 can be regarded as an $\mathbf{F}_p\langle g \rangle$-module. The map $\theta: r \mapsto ar$ from $\mathbf{F}_p\langle g \rangle$ to A_0 is surjective, but cannot be injective since each element of A_0 has the form

$$a \sum_{i=1}^{n} l_i g^{u_i}$$

with n fixed. The kernel of θ is an ideal $I \neq 0$ of $\mathbf{F}_p\langle g \rangle$, and so both $\mathbf{F}_p\langle g \rangle / I$ and A_0 are finite. However A/A_0 is finitely generated as an abelian group; so therefore is A, and this is a contradiction.

It follows that A is torsion-free. We choose $a \neq 0$ and consider the $\mathbf{Q}\langle g \rangle$-module $A_0 \times_{\mathbf{Z}} \mathbf{Q}$, each of whose elements has the form

$$a \sum_{i=1}^{n} l_i g^{u_i} \quad \text{with } l_i \in \mathbf{Q} \text{ and } u_i \in \mathbf{Z} \text{ for each } i.$$

Exactly the same argument as in the above paragraph shows that $A_0 \times_{\mathbf{Z}} \mathbf{Q}$ has finite dimension, so that A_0 has finite torsion-free rank. Let A_1 be a non-zero cyclic submodule of A_0 of least possible rank, generated by an element b, say. Then $\langle g \rangle$ acts rationally irreducibly on A_1, and so acts trivially on A_1 by Lemma 3. Thus A_1 is just the cyclic group generated by b. Since A/A_1 is finitely generated as a group so also must be A_1, and with this contradiction proof of the lemma is complete.

Proof of Theorem 1. We must prove that if $G = \langle g_1, \dots, g_s \rangle$ is soluble and $\langle g_i \rangle$ is elliptically embedded in G for $i = 1, \dots, s$, then G is finite by nilpotent.

Arguing by induction on the derived length of G, we may suppose that G has an abelian normal subgroup A such that G/A is finite by nilpotent. Since $\langle g_i \rangle$ ee G for each i, we have

$$G = \langle g_{i_1} \rangle \dots \langle g_{i_n} \rangle$$

for some n and some choice of $i_1, \dots, i_n$. Because G is abelian by polycyclic and finitely generated we have $A = B^G$ for some finitely generated subgroup B. Write

$$B_0 = B \quad \text{and} \quad B_j = B_{j-1}^{\langle g_{i_j} \rangle} \quad \text{for } j = 1, \dots, n,$$

so that $B_n = A$. If B_{j-1} is a finitely generated group then so is B_j by Lemma 4. We conclude by induction that A is a finitely generated group. Its torsion subgroup T is finite, and since we want to prove that G is a finite by nilpotent, there is no harm in assuming that $T = 1$.

We claim that each $\langle g_i \rangle$ acts nilpotently on A. If $A \cap \langle g_i \rangle = 1$ this follows by applying Lemma 3 to each factor in a maximal $\langle g_i \rangle$-invariant series for A with torsion-free factors. If instead $g_i^m \in A$ for some m then g_i centralizes both $\langle g_i^m \rangle$ and its isolator J in A since A is torsion-free. Thus $\langle A, g_i \rangle / J$ is the split extension of A/J by $\langle g_i J \rangle$, and the result follows from Lemma 3.

Let H/A be a nilpotent normal subgroup of G/A of finite index l, say, and let L/M be a factor in a maximal G-invariant series for A with torsion free factors. Fix i with $i \leqq s$. Since

$$\langle A, g_i^l \rangle \leqq H \lhd G,$$

the subgroup $\langle A, g_i^l \rangle$ is subnormal in G. From above it is also nilpotent, so it lies in the Fitting subgroup F of G. Thus g_i^l acts trivially on L/M. It follows that the minimal polynomial $f(t)$ of the action of g_i on L/M divides $t^l - 1$. Moreover g_i acts nilpotently on A, so $f(t)$ also divides $(t - 1)^k$ for some integer k. Therefore we must have $f(t) = t - 1$, and g_i centralizes L/M. Since this holds for each i, L/M is a central factor of G and it follows that A is in the hypercentre of G. Since G/A is finite by nilpotent and since a group is finite by nilpotent if and only if a finite term of its upper central series has finite index ([**3**], Theorem 4.25), the proof of the theorem is complete.

The next lemma provides the key to Theorem 2.

LEMMA 5. *Let $\langle g \rangle$ be an infinite cyclic group and A a $\mathbf{Z}\langle g \rangle$-module which is $\mathbf{Z}$-torsion free. If $\langle g \rangle$ is elliptically embedded in the split extension G of A by $\langle g \rangle$, then G is nilpotent.*

Proof. If B is a finitely generated submodule of A then B is a finitely generated abelian group by Lemma 4 and so it has a finite series whose factors are $\mathbf{Z}$-torsion-free and rationally irreducible. It follows from Lemma 3 that $\langle g \rangle$ acts nilpotently and that $B \leqq \zeta_n(G)$ for some n. Thus

$$A = \bigcup_{n=1}^{\infty} (\zeta_n(G) \cap A)$$

and G is hypercentral. If

$$\zeta_i(G) \cap A = \zeta_{i-1}(G) \cap A \quad \text{for some } i,$$

then G is nilpotent. Suppose then that

$$\zeta_i(G) \cap A > \zeta_{i-1}(G) \cap A \quad \text{for each } i.$$

For each k choose

$$f_k \in (\zeta_{k^2}(G) \backslash \zeta_{k^2-1}(G)) \cap A,$$

and for each i find the integer k with $(k - 1)^2 < i \leqq k^2$ and define

$$e_i = [f_k, \underbrace{g, \ldots, g}_{k^2 - i}].$$

Thus $e_i \in \zeta_i(G) \backslash \zeta_{i-1}(G)$ for each i.

Since the terms of the upper central series are isolated, the elements e_i freely generate a free abelian group V. Define U to be the group generated by the elements f_k. Clearly $U\mathbf{Z}\langle g \rangle = V$, or in multiplicative notation,

$$U^{\langle g \rangle} = V.$$

Consider the group $\langle U, g \rangle = \langle V, g \rangle$. Since $\langle g \rangle$ ee G there is an integer n such that each element of $\langle V, g \rangle$ is a product of n elements of the form $g^i u$ with $i \in \mathbf{Z}$ and $u \in U$. Collecting powers of g on the left we see that each element of $\langle V, g \rangle$ is a product of a power of g and n conjugates of elements of U under elements of $\langle g \rangle$. Thus, writing V additively again, we conclude that each element of V is a sum of n elements ug^i with $u \in U$ and $i \in \mathbf{Z}$. We consider the element e_{n^2+n} of V; say

$$e_{n^2+n} = \sum_{i=1}^{n} u_i g^{\gamma_i}.$$

Collecting terms with the same γ_i together and deleting zero terms, we may assume

$$e_{n^2+n} = \sum_{i=1}^{m} u_i g^{\gamma_i}$$

where $m \leqq n$, the γ_i are distinct, and $u_i \neq 0$ for each i. Let k be the greatest integer such that $u_i \notin \langle f_j; j < k \rangle$ for some i. Clearly $k > n$. Renumbering the u_i, if necessary, we may assume that

$$u_1, \dots, u_s \notin \langle f_j; j < k \rangle$$

and that

$$u_{s+1}, \dots, u_m \in \langle f_j; j < k \rangle.$$

Thus

$$u_i - l_i f_k \in \langle f_j; j < k \rangle$$

for $i = 1, \dots, s$ and some non-zero integers $l_1, \dots, l_s$. Thus modulo $W = \zeta_{(k-1)^2}(G)$, we have

$$e_{n^2+n} \equiv \sum_{i=1}^{m} u_i g^{\gamma_i} \equiv \sum_{i=1}^{s} l_i f_k g^{\gamma_i}.$$

Now $(g - 1)$ induces a nilpotent map on $V + W/W$, and so

$$f_k g^{\gamma} = f_k(1 + (g - 1))^{\gamma} \equiv \sum_{j=0}^{k^2} \binom{\gamma}{j} e_{k^2-j} \text{ modulo } W,$$

for each $\gamma \in \mathbf{Z}$. Thus

$$e_{n^2+n} \equiv \sum_{i=1}^{s} l_i \sum_{j=0}^{2k-2} \binom{\gamma_i}{j} e_{k^2-j} \text{ modulo } W.$$

Since $k > n$, we have

$$k^2 > (n + 1)^2 - 1 = n^2 + 2n$$

and hence $k^2 - n > n^2 + n$. Now $s \leqq n$; hence

$$\sum_{i=1}^{s} l_i \binom{\gamma_i}{l} = 0 \quad \text{for } j = 0, 1, \dots, s.$$

From these equations we deduce successively that

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$$\sum_{i=1}^{s} l_i \gamma_i^j = 0$$

for $j = 0, 1, \ldots, s$. Since the γ_i are distinct, the Vandermonde determinant

$$\begin{vmatrix} 1 & \ldots & 1 \\ \gamma_1 & \ldots & \gamma_s \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \gamma_1^{s-1} & \ldots & \gamma_s^{s-1} \end{vmatrix}$$

is non-zero. Thus $l_i = 0$ for each i. However, this is a contradiction, and the lemma follows.

We mention an immediate consequence of Lemma 5, which should be compared with Proposition 4.

PROPOSITION 4. *Let G be a split extension of a torsion-free nilpotent group N by infinite cyclic group $\langle g \rangle$. If $\langle g \rangle$ is elliptically embedded in G then G is nilpotent.*

Proof. This follows from Lemma 5 by induction on the nilpotency class of N.

Proof of Theorem 2. We know from Theorem 1 that $N = \langle g \rangle^G$ is locally finite-by-nilpotent. Thus the torsion elements of N form a normal subgroup of G, and we conclude that N is torsion-free and locally nilpotent. It suffices to show that $\langle g \rangle$ is subnormal in N. We argue by induction on the derived length of N. Let A be the isolator in N of the last non-trivial term of the derived series for N. Thus N/A is torsion-free and $\langle gA \rangle$ ee N/A, so that $\langle A, g \rangle$ is subnormal in N by induction. If $g \in A$ then clearly $\langle g \rangle$ is subnormal in N since A is abelian. Otherwise the extension of A by $\langle g \rangle$ is split and Lemma 5 applies; it shows that $\langle A, g \rangle$ is nilpotent, hence $\langle g \rangle$ is subnormal in $\langle A, g \rangle$. This concludes the proof of Theorem 2.

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