FUNCTION SPACES

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1. Introduction. This paper is the first in a series dealing with Banach spaces L whose elements are functions on a measure space S. If W is a family of non-negative weight functions w_α, we sometimes write $L_W{}^p$ when the norm is given as

$$|f| = \sup_\alpha \left(\int |f(P)|^p \, w_\alpha(P) \, d\gamma(P) \right)^{1/p} \qquad 1 \leqslant p < \infty,$$

$$= \sup_\alpha \left(w_\alpha \sup |f(P)| \right) \qquad p = \infty,$$

(here w_α-sup means: *supremum*, neglecting sets on which $w_\alpha(P) = 0$ for almost all P). We sometimes write $L_{(w)}{}^p$ when W consists of all the functions equimeasurable with a single $w(P)$ ($w(P)$ is required to satisfy a weak condition, see §2). When $w(P)$ is identically 1, $L_{(w)}{}^p$ reduces to classical L^p space.

In §§3 and 4, a Hölder type inequality, of some interest in itself, is proved (in more general form than actually required elsewhere in this paper). In §§5 and 6, using this inequality, we determine explicitly the conjugate spaces L^*, L^{**} when L is of type $L_{(w)}{}^p$. It turns out that the case: *S has infinite measure but w has finite integral on S*, is pathological. Excluding this case we show: (i) if $1 \leqslant p < \infty$ then L^* is a new generalization of classical L^p space; (ii) if $1 < p < \infty$ then $L_{(w)}{}^p$ is reflexive. In the pathological case, $L_{(w)}{}^p$ fails to be reflexive for every p.

The main result of the present paper solves for a special case a very deep problem indicated in [**1**, p. 182]: if a linear vector space carries a family of norms and a new norm is defined as the supremum of the given norms, what is the nature of the conjugate space to the new space? An extension to vector valued functions will be given by H. W. Ellis and the author in [**2**].

Since references to the present paper occur in the literature, it is remarked that the results of this paper were found by the author and embodied in a manuscript in the summer of 1950 at the Research Institute of the Canadian Mathematical Congress. Function spaces which could be considered as special cases of the $L_{(w)}{}^p$ had been defined previously by G. G. Lorentz [**3**] but discussed by him only for the case $p = 1$.

2. Terminology. Throughout the papers in this series we suppose $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, with $p^{-1} + q^{-1} = 1$, interpreting ∞^{-1} as 0. We let S denote a space of points P with a non-negative, countably additive set function ν defined for a non-empty family of ν-sets which includes relative complements

Received October 7, 1952; in revised form April 2, 1953.

https://doi.org/10.4153/CJM-1953-031-3 Published online by Cambridge University Press

and countable unions of its members and is such that $\nu(S_1) = 0$ implies that every subset of S_1 is a ν-set.

If ES_1 is a ν-set for every ν-set S_1 with $\nu(S_1)$ finite, we define $\gamma(E)$ to be the supremum of such $\nu(ES_1)$; measure, measurability, integral and ess. sup will refer to γ. The change from ν to γ enables us to disregard sets S_1 which are purely infinite (i.e. $\nu(S_1) = \infty$, but $0 < \nu(S_2) < \infty$ is false for all $S_2 \subset S_1$) without actually deleting them from S. Both for abstract S and for Euclidean space, where m in place of γ denotes Lebesgue measure, the letter E will be used for arbitrary measurable sets; *e will always denote a set of finite measure.* $\int$ and ess. sup refer to the entire space when no subset is indicated. We sometimes write γ for $\gamma(S)$.

$\{\phi(P)\} = \{P; \phi(P)\}$ means the set of P for which $\phi(P)$ holds.

B will denote a real or complex Banach space, B^* its conjugate (B may consist of the real or complex numbers). If $c \in B, v \in B^*$ then $cv = vc$ is the value of v at c; $c(P)$ is the function with value c for all P. For point or set functions $f(P)$, $F(e)$, we define f_E, F_E to coincide with f, F respectively on E and to vanish outside E; $f_N(P)$ shall equal $f(P)$ if $|f(P)| \leqslant N$ and $N|f(P)|^{-1}f(P)$ otherwise; $c_{e,i}$ is an abbreviation for c_e with $c = c_i$, $e = e_i$; f is finitely (countably) valued if $f = \sum_i c_{e,i}$ with finite (countable) disjoint e_i; f is *Bochner measurable* if for every e and every n there is a finitely valued f_1 such that the subset of e for which $|f(P) - f_1(P)| \geqslant 1/n$ has measure less than $1/n$.

Two functions will be identified without comment if they differ only on a set of measure zero; *all numerical valued functions considered in these papers will be measurable.*

Non-negative functions $f_1(P)$, $f_2(P)$ will be called *equimeasurable* if $\gamma\{f_1(P) > k\} = \gamma\{f_2(P) > k\}$ for all $k \geqslant 0$. This need not imply the stronger relation with $\geqslant$ inside the braces.

We allow ∞ as a value for a non-negative function with the usual conventions $0\,\infty = 0$, $k/0 = \infty$ if $k > 0$, $k/\infty = 0$ if $0 \leqslant k < \infty$; we adopt the convention that $k \leqslant 0/0$ is valid for all $k \geqslant 0$.

In §§3 and 4, u, v, and a fixed w, called the *weight* function, are non-negative functions on (a, b), $-\infty \leqslant a < b \leqslant \infty$ and we define $v^\circ(x)$, the *level function of v with respect to the fixed w*, in Definition 3.2.

$u(a_1, b_1)$ will denote the integral of u on (a_1, b_1); (a_1, b_1) is called u-null if $u(a_1, b_1) = 0$ and u-null maximal if, in addition, it is not contained in any other u-null interval. We shall suppose that $0 \leqslant w(a, x) < \infty$ for all $a < x < b$; $\tilde{a}$ means a if $w(a, x) > 0$ for all $a < x < b$, otherwise $\tilde{a} = \sup_x \{x; w(a, x) = 0\}$. We shall suppose that $0 < w(a, b) \leqslant \infty$. For given v we let $R(a_1, b_1)$ denote $v(a_1, b_1)/w(a_1, b_1)$ if $w(a_1, b_1) < \infty$ and lim sup $v(a_1, t)/w(a_1, t)$ as $t \to b_1$ if $w(a_1, b_1) = \infty$; R° refers to v° in place of v.

$|u| = |u|_p$ shall mean:

$$\left(\int u(x)^p w(x)\,dx\right)^{1/p}$$

if $1 \leqslant p < \infty$ and w-sup $u(x)$ if $p = \infty$. If $v(a, \tilde{a}) > 0$ then $[v] = [v]_q$ shall mean ∞; if $v(a, \tilde{a}) = 0$ then $[v] = [v]_q$ shall mean

$$\left(\int (v^\circ(x)/w(x))^q w(x)\, dx\right)^{1/q}$$

if $1 \leqslant q < \infty$ and w-sup $(v^\circ(x)/w(x))$ if $q = \infty$.

v will be said to have the Δ-*property* if $w(a_1, b_1) > 0$, $w(a_2, b_2) > 0$, $a_1 \leqslant a_2$, $b_1 \leqslant b_2 < b$ always imply $\infty > R(a_1, b_1) \geqslant R(a_2, b_2)$; v will be called *non-increasing relative to w* if $v(x) = D(x)w(x)$ for $\tilde{a} < x < b$ with $D(x)$ finite, non-negative, non-increasing. v will be called *w-infinite* if $v(x) = \infty$ whenever $w(x) > 0$. $v \prec v_1$ will mean: $v(a, x) \leqslant v_1(a, x)$ for all $a < x < b$; $v \ll v_1$ will mean the stronger relation: $v(x) = v_1(x)$ for $a < x \leqslant \tilde{a}$ and $v(\tilde{a}, x) \leqslant v_1(\tilde{a}, x)$ for all $\tilde{a} < x < b$.

In §§5, 6 we consider arbitrary $f(P)$, $g(P)$ and a fixed non-negative $w(P)$ on S (there should be no confusion between $w(P)$ and the $w(x)$ of §§3, 4). The left-continuous non-increasing rearrangement of $|f(P)|$ is defined to be a function $f^*(x)$ on $0 \leqslant x < \gamma$ as follows: $f^*(0) = \text{ess. sup}\, |f(P)|$ and for $x > 0$, $f^*(x) = \sup$ k with $\gamma\{|f(P)| \geqslant k\} \geqslant x$. In these sections, $|f| = |f|_p$ shall mean:

$$\sup_\alpha \left(\int |f(P)|^p w_\alpha(P)\, d\gamma(P)\right)^{1/p}$$

if $1 \leqslant p < \infty$, and $\sup_\alpha$ (w_α-sup $|f(P)|$) if $p = \infty$, where w_α varies over all functions equimeasurable with w.

Omitting some trivial cases we shall suppose that for every $k < \infty$, the supremum of the integral of w over sets of measure $\leqslant k$ is finite and that the integral of w over S is greater than zero. We shall distinguish the three possibilities: Case (C_1) with $\gamma < \infty$; Case (C_2) with $\gamma = \infty$ and integral of w over S infinite; Case (C_3) with $\gamma = \infty$ and integral of w over S finite. We shall suppose that w is restricted by the condition: $|f|_p$ defined above agrees with $|f^*|_p$ as defined for §§3, 4 with $w^*(x)$ as the weight function on $(0, \gamma)$ in place of $w(x)$ on (a, b). It is easy to verify that this condition is satisfied if $w(P)$ is constant on S, more generally if $w^*(x)$ is constant on $(0, \gamma)$; for any other w this condition is equivalent to the requirement that either S has no atomic sets e (i.e. $\gamma(e) > 0$ and $e_1 \subset e$ implies $\gamma(e_1) = 0$ or $\gamma(e - e_1) = 0$) or every measurable subset of S of finite measure is a union of atomic sets of equal measure.

With this $w(P)$ we define $[g] = [g]_q$ to agree with $[g^*]_q$ as defined for §3, 4 where the weight function to be used shall be w^* on $(0,\gamma)$.

$L = L_{(w)}{}^p$ and $M = M_{(w)}{}^q$ will denote the spaces whose elements are the numerical valued f, g with finite norm $|f|_p$, $[g]_q$ respectively. $L_{(w)}{}^p(B)$, $M_{(w)}{}^q(B)$ shall denote the corresponding spaces when f, g are valued in B and are Bochner measurable. $L_{(w)}{}^p(B)$, as well as the more general $L_W{}^p(B)$, are obviously linear, normed spaces and for $M_{(w)}{}^q(B)$ this will be shown in §5; the remainder of the proof that all $L_W{}^p(B)$ and all $M_{(w)}{}^q(B)$ are Banach spaces (i.e. the proof of completeness) will be omitted in this paper since a more general result will be given in [**2**, Theorem 3.1].

S is said to have property (R) if there is a family, not necessarily countable, of *disjoint* e_α such that an arbitrary S_1 is measurable and $\gamma(S_1) = 0$ whenever $S_1 e_\alpha$ is measurable with measure 0 for every α.

3. Level intervals and level functions. The constructions and results of this section are required to solve the Hölder inequality problem of the next section. We refer to §2 for terminology.

DEFINITION 3.1. (a_1, b_1), with $a \leqslant a_1 < b_1 \leqslant b$, is called a *level interval* (of v with respect to w), abbreviation l.i., *if for all* $a_1 < x < b_1$, $w(a_1, x) > 0$ and $R(a_1, x) \leqslant R(a_1, b_1)$. If the l.i. is not contained in a larger l.i. it is called a *maximal level interval*, abbreviation m.l.i.

We note that if $w(a_1, x) > 0$ for all $a_1 < x < b_1$ and $R(a_1, b_1) = \infty$ then (a_1, b_1) is a l.i., $R(\tilde{a}, b) = \infty$ and $(\tilde{a}, b)$ is a m.l.i.

THEOREM 3.1.

(i) *Every* l.i. *is contained in a* m.l.i.

(ii) *If* (a_1, b_1), (a_2, b_2) *are* l.i.'s *with* $a_1 < a_2 < b_1 < b_2$ *then* (a_1, b_2) *is a* l.i.

(iii) *The* m.l.i.'s *are non-overlapping and denumerable.*

Proof of (i). Suppose $a_1 \geqslant a_2 \geqslant \ldots$, $b_1 \leqslant b_2 \leqslant \ldots$, $a_0 = \inf a_n$, $b_0 = \sup b_n$. If each (a_n, b_n) is a l.i. it is easily verified that (a_0, b_0) is a l.i. Now for arbitrary l.i. (a_1, b_1) the a_n, b_n can clearly be chosen so that (a_0, b_0) will be a m.l.i.

Proof of (ii). $w(a_1, x) > 0$ for $a_1 < x < b_2$ since (a_1, b_1) is a l.i. We may suppose $R(a_1, b_2) < \infty$. Then $R(a_1, a_2) \leqslant R(a_1, b_1) \leqslant R(a_2, b_1) \leqslant R(a_2, b_2) \leqslant R(b_1, b_2)$, implying $R(a_1, b_1) \leqslant R(a_1, b_2) \leqslant R(a_2, b_2)$. It follows that (a_1, b_2) is a l.i. for if $a_1 < x \leqslant b_1$ then $R(a_1, x) \leqslant R(a_1, b_1) \leqslant R(a_1, b_2)$; and if $b_1 < x < b_2$ then $R(a_1, b_2) \leqslant R(a_2, b_2) \leqslant R(x, b_2)$ which implies $R(a_1, x) \leqslant R(a_1, b_2)$.

Proof of (iii). This follows at once from (ii).

Remark 1. A w-null, v-null interval, like a single point, may or may not be part of a l.i. But Definition 3.1 implies that it can not be at the beginning of a l.i., and either all or none of it is part of a m.l.i.

Remark 2. If (a_1, b_1) is w-null but not v-null and $b_1 > \tilde{a}$ then (a_1, b_1) is part of a l.i. Indeed $a < a_1$ and we may clearly suppose that $R(\tilde{a}, b)$ is finite and (a_1, b_1) is w-null maximal so that $w(x, b_1) > 0$ for all $a \leqslant x < a_1$. Then $R(x, b_1)$ is finite and continuous for $a \leqslant x_1 < a_1$ and diverges to ∞ as $x \rightarrow a_1$; hence it assumes its minimum value and we let a_2 denote the maximum x for which this mimimum is attained. Then $a \leqslant a_2 < a_1$ and for $a_2 < x < a_1$, $R(a_2, b_1) < R(x, b_1)$. This implies: $w(a_2, x) > 0$ for all $a_2 < x < b$ and (a_2, b_1) is a l.i. containing (a_1, b_1). (If $a_2 = a = -\infty$, the argument is still valid.)

Remark 3. If $a < \tilde{a}$ then no part of $(a, \tilde{a})$ can be part of a l.i.

DEFINITION 3.2. $v^\circ(x)$, the level function of v with respect to w is defined by:

(i) $v^\circ(x) = R(a_1, b_1)w(x)$ if x is interior to a m.l.i. (a_1, b_1),

(ii) $v^\circ(x) = v(x)$ for all other x.

If $v^\circ = v$ then v is called a *level function.*

Remark 1. If (a_1, b_1) is w-null and $b_1 > \tilde{a}$ then Remarks 1 and 2 following Theorem 3.1 imply that $v^\circ(x) = 0$ for $a_1 < x < b_1$.

Remark 2. If $a \neq \tilde{a}$ then $v^\circ(x)$ could be obtained by the equivalent definition: on $(a, \tilde{a})$ define $v^\circ(x)$ to be $v(x)$ and on $(\tilde{a}, b)$ use Definition 3.2 but with w, v considered as functions on $(\tilde{a}, b)$ in place of (a, b).

Remark 3. If (b_1, b) is v-null then $v^\circ(x) = 0$ for $b_1 < x < b$. If (b_1, b) is a l.i. of v and $R(b_1, b) = 0$ then (b_1, b) is v-null.

Remark 4. If $v_1(x) = v(x)$ except on a l.i. (a_1, b_1) of v and $v_1(x) = R(a_1, b_1) w(x)$ for $a_1 < x < b_1$ then $v_1{}^\circ = v^\circ$.

Remark 5. If $R(\tilde{a}, b) = \infty$ then v° is w-infinite.

THEOREM 3.2.

(i) $v(a_1, x) \leqslant v^\circ(a_1, x)$ *if* a_1 *is not interior to a* l.i. *of* v *and equality holds if neither* a_1 *nor* x *is interior to a* l.i.

(ii) $v \ll v^\circ$ *and* $v^\circ(a, b) = v(a, b)$.

(iii) $R(\tilde{a}, b) = R^\circ(\tilde{a}, b)$.

Proof of (i) *and* (ii). If (a_1, b_1) is a m.l.i. and $a_1 < x < b_1$ then Definition 3.2 implies that $v(a_1, x) \leqslant v^\circ(a_1, x)$ with equality if $x = b_1$. Since $v^\circ(x) = v(x)$ outside the m.l.i.'s, this gives (i) and (ii).

Proof of (iii). We may suppose $w(a, b) = \infty$ and $R(\tilde{a}, b) < \infty$. If now there is a m.l.i. (a_1, b) then (iii) holds; if there is no such m.l.i. then (iii) follows from the relations: $R(\tilde{a}, x) = R^\circ(\tilde{a}, x)$ when $x < b$ and x is not interior to a m.l.i., and $R(\tilde{a}, x) \leqslant R^\circ(\tilde{a}, x) \leqslant \max\ (R(\tilde{a}, a_1), R(\tilde{a}, b_1))$ when x is interior to a m.l.i. (a_1, b_1) with $a_1 > \tilde{a}$.

THEOREM 3.3.

(i) *Every* l.i. *of* v *is a* l.i. *of* v°.

(ii) *Every* m.l.i. *of* v° *is a* l.i. *of* v.

(iii) v *and* v° *have the same* m.l.i.'s.

(iv) *On each of its* l.i.'s $v^\circ(x) = kw(x)$ *with* k *constant on the* l.i.

(v) $v^{\circ\circ} = v^\circ$.

Proof of (i). On each l.i. (a_1, b_1) of v, $v^\circ(x) = kw(x)$ so that $R^\circ(a_1, x)$ is constant for $a_1 < x < b_1$. This implies (i).

Proof of (ii), (iii), (iv), *and* (v). If (a_1, b_1) is a m.l.i. of v° then by (i), neither a_1 nor b_1 is interior to a l.i. of v and hence for $a_1 < x < b_1$,

$$R(a_1, x) \leqslant R^\circ(a_1, x) \leqslant R^\circ(a_1, b_1) = R(a_1, b_1).$$

This proves (ii). Now (iii), (iv), and (v) follow from (i) and (ii).

THEOREM 3.4 *If* $a_1 < b_1 < b_2$ *and* $0 < w(b_1, b_2) < \infty$ *then*

$$R^\circ(b_1, b_2) \leqslant R^\circ(a_1, b_2).$$

Proof. We may suppose $R^\circ(a_1, b_2) < \infty$. Then $R^\circ(x, b_2)$ is finite and continuous for $a_1 \leqslant x \leqslant b_1$. Suppose, contrary to the statement of the theorem, that $R^\circ(a_1, b_2) < R^\circ(b_1, b_2)$ and let x_0 be the maximum x at which $R^\circ(x, b_2)$ assumes its minimum value. Then $x_0 < b$, $R^\circ(x_0, b_2) < R^\circ(x, b_2)$ for $x_0 < x \leqslant b_1$ and $w(x_0, x) > 0$ for $x_0 < x < b$. Now let x_1 be an x at which $R^\circ(x_0, x)$ on $b_1 \leqslant x \leqslant b_2$ assumes its maximum value. Necessarily $b_1 < x_1$. If $x_0 < x \leqslant b_1$ then $R^\circ(x_0, x) < R^\circ(x_0, b_2) \leqslant R^\circ(x_0, x_1)$; if $b_1 \leqslant x < x_1$, $R^\circ(x_0, x) \leqslant R^\circ(x_0, x_1)$. This implies that (x_0, x_1) is a l.i. of v°, hence that $R^\circ(x_0, x)$ is constant for $x_0 < x \leqslant x_1$, contradicting the previous inequality $R^\circ(x_0, b_1) < R^\circ(x_0, x_1)$.

THEOREM 3.5. *If* $a_1 < b_1 \leqslant b_2$, $0 < w(a_1, b_1)$ *and* $w(a_1, b_2) < \infty$ *then* $R^\circ(a_1, b_2) \leqslant R^\circ(a_1, b_1)$.

Proof. If $w(b_1, b_2) = 0$, the Remark 1 following Definition 3.2 implies $v^\circ(x) = 0$ for $b_1 < x < b_2$ and hence the theorem. If $w(b_1, b_2) > 0$, the theorem is a corollary of Theorem 3.4.

THEOREM 3.6. *For a function v the following are equivalent: to be a level function but not w-infinite; to be non-increasing relative to w; to have the* Δ*-property.*

Proof. In view of Theorems 3.4 and 3.5 we need only prove that the Δ-property implies that v is non-increasing relative to w. Let E denote the set union of the denumerable family of closed w-null maximal intervals and E' the set of x with $a < x < b$ and x not in E. For x in E' and $t > 0$ set $H(x, t) = R(x, t_1)$ with $t_1 = \min\,(x + t, \frac{1}{2}(x + b))$. For fixed x, $H(x, t)$ is non-increasing as t decreases to zero and for fixed t, $H(x, \mathrm{t})$ is non-increasing in x. Hence $D(x) = \lim H(x, t)$ as $t \to 0$ exists for all x in E' and is finite, non-negative and non-increasing for these x. Since

$$v(x, t_1)/(t_1 - x) = H(x, t)\, w(x, t_1)/(t_1 - x),$$

the fundamental theorem of the (Lebesgue) calculus shows that $v(x) = D(x)w(x)$ for almost all x in E'. But for almost all x in E with $x > \tilde{a}$, we have $v(x) = w(x) = 0$ since the Δ-property implies $v(a_1, b_1) = 0$ whenever (a_1, b_1) is w-null with $b_1 > \tilde{a}$. It follows that $D(x)$ can be so defined for the x which are $> \tilde{a}$ and in the closed intervals which constitute E that $v(x) = D(x)w(x)$ will hold for all $x > \tilde{a}$.

THEOREM 3.7.

(i) $v \prec v_1$ *implies* $v^\circ \prec v_1^\circ$ *and* $v \ll v_1$ *implies* $v^\circ \ll v_1^\circ$.

(ii) *If* v_1 *is a level function,* $v \prec v_1$ *implies* $v^\circ \prec v_1$ *and* $v \ll v_1$ *implies* $v^\circ \ll v_1$.

(iii) v° *can be characterized among the level functions (equivalently, the functions which are w-infinite or non-increasing relative to w)* v_1 *with* $v \ll v_1$ *as the one for which* $v_1(\tilde{a}, x)$ *attains the minimum value for every* $\tilde{a} < x < b$.

https://doi.org/10.4153/CJM-1953-031-3 Published online by Cambridge University Press

Proof. We shall show that $v \prec v_1$ implies $v^\circ(a, x) \leqslant v_1^\circ(a, x)$ for $\tilde{a} < x < b$. We may clearly suppose $v(a, x) < \infty$ for all $a < x < b$ and $R(a, b) < \infty$.

If x is not interior to a m.l.i. of v then

$$v^\circ(a, x) = v(a, x) \leqslant v_1(a, x) \leqslant v_1^\circ(a, x).$$

If x is interior to a m.l.i. (a_1, b_1) of v with $w(a_1, b_1) < \infty$, then

$$\begin{aligned} v^\circ(a, x) &= v(a, a_1) + \frac{w(a_1, x)}{w(a_1, b_1)} v(a_1, b_1) \\ &= v(a, a_1)\left(1 - \frac{w(a_1, x)}{w(a_1, b_1)}\right) + v(a, b_1)\,\frac{w(a_1, x)}{w(a_1, b_1)} \\ &\leqslant v_1^\circ(a, a_1)\left(1 - \frac{w(a_1, x)}{w(a_1, b_1)}\right) + v_1^\circ(a, b_1)\,\frac{w(a_1, x)}{w(a_1, b_1)} \\ &= v_1(a, a_1) + \frac{w(a_1, x)}{w(a_1, b_1)} v_1^\circ(a_1, b_1) \leqslant v_1^\circ(a, x), \end{aligned}$$

since v_1° is either w-infinite or non-increasing relative to w.

Finally, if x is interior to a m.l.i. (a_1, b_1) of v with $w(a_1, b_1) = \infty$, then $b_1 = b$ and

$$\begin{aligned} v^\circ(a, x) &= v(a, a_1) + R(a_1, b)w(a_1, x) \\ &\leqslant v_1^\circ(a, a_1) + \frac{v_1^\circ(a_1, x)}{w(a_1, x)} w(a_1, x) = v_1^\circ(a, x), \end{aligned}$$

since $v_1^\circ(a_1, t)/w(a_1, t)$ is non-increasing and has limit $\geqslant R(a, b)$ when $t \to b$.

4. *D*-type Hölder inequalities. We refer to §2 for terminology. u, v, w are non-negative throughout this section.

THEOREM 4.1. *If $u(x)$ is non-increasing on (a, b) then $v \prec v_1$ implies that for all $a < x < b$,*

$$\int_a^x v(t)u(t)\,dt \leqslant \int_a^x v_1(t)u(t)\,dt.$$

Proof. It is sufficient to prove the theorem for u of the form: $u(x) = k_i$ on (a_i, a_{i+1}) with $a = a_1 < a_2 < \ldots < a_{m+1} = b$ and $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_m \geqslant 0$; using Abel's rearrangement, it is sufficient to prove the theorem for $u(x) = k$ on (a, a_i) and 0 elsewhere, for arbitrary $k \geqslant 0$; and this follows directly from $v \prec v_1$.

THEOREM 4.2. *If u varies over all non-increasing functions with $|u|_p \leqslant 1$ then*

$$\sup \int_a^b u(x)v(x)\,dx = [v]_q, \tag{4.1}$$

https://doi.org/10.4153/CJM-1953-031-3 Published online by Cambridge University Press

and for arbitrary v and arbitrary non-increasing u, the D-type (i.e., decreasing type) Hölder inequality holds:

$$\int_a^b u(x)v(x)\,dx \leqslant |u|_p[v]_q.$$

Proof. Both sides of (4.1) are infinite if *any* of the following hold: (i) $v(a, \tilde{a}) > 0$, (ii) $R(\tilde{a}, b) = \infty$, or (iii) $w(a, b) = \infty$, $p < \infty$, and $R(\tilde{a}, b) > 0$. To verify that the right-hand side of (4.1) is infinite: if (i) holds, choose $u(x) = k$ on $(a, \tilde{a})$, $= 0$ elsewhere and let $k \to \infty$; if (ii) or (iii) holds, let t be fixed and choose $u(x) = w(a, t)^{-1/p}$ on (a, t), $= 0$ elsewhere and let $t \to b$. We may therefore suppose none of (i), (ii), or (iii) holds; then $v^\circ(x) = 0$ whenever $w(x) = 0$ and $v^\circ(x) = D(x)w(x)$ for all x for some $D(x)$, non-negative, non-increasing, and finite for $x > \tilde{a}$.

Now by Theorem 4.1 and the ordinary Hölder inequality,

$$\int_a^b u(x)v(x)\,dx \leqslant \int_a^b u(x)v^\circ(x)\,dx = \int_a^b u(x)w(x)^{1/p}v^\circ(x)w(x)^{-1/p}dx \leqslant [v]_q.$$

Thus $\leqslant$ holds in (4.1).

If $p = \infty$, then $\geqslant$ holds in (4.1) as can be verified by choosing $u(x)$ to be identically 1.

If $p < \infty$, consider any t not interior to a m.l.i. of v for which $w(a, t) < \infty$ and set $u_1(x) = (D_N(x))^{q-1}$ on (a, t), $= 0$ elsewhere. Let $u(x) = |u_1|_p^{-1}u_1(x)$. Then $|u|_p = 1$, $u(x)$ is non-negative, non-increasing, and constant on each m.l.i. of v. With this u,

$$\int_a^b u(x)v(x)\,dx \geqslant \left(\int_a^t (D_N(x))^q w(x)\,dx\right)^{1/q}.$$

Since N is arbitrary, this proves that the left-hand side of (4.1) is greater than or equal to

$$\left(\int_a^t D(x)^q w(x)\,dx\right)^{1/q} = \left(\int_a^t \left(\frac{v^\circ(x)}{w(x)}\right)^q w(x)\,dx\right)^{1/q}$$

for every such t. This obviously implies $\geqslant$ in (4.1) except when $w(a, b) = \infty$ and v has a m.l.i. (b_1, b); but for this case, $R(\tilde{a}, b) = 0$ (since $p < \infty$), hence $v^\circ(x) = 0$ on (b_1, b), and $\geqslant$ in (4.1) is obtained when $t = b_1$.

This completes the proof of the theorem.

COROLLARY. *$v \prec v_1$, in particular $v(x) \leqslant v_1(x)$ for all x, implies $[v]_q \leqslant [v_1]_q$; $v_n(x) \leqslant v(x)$ for all x together with $v_n(x) \to v(x)$ as $n \to \infty$ for each x, implies $[v_n]_q \to [v]_q$; $[v_N]_q \leqslant [v]_q$ and $[v_N]_q \to [v]_q$ as $N \to \infty$; $[v_1 + v_2]_q \leqslant [v_1]_q + [v_2]_q$.*

Remark 1. Theorem 4.2 implies the more general theorem: with fixed non-negative $u_0(x)$ let $u(x)$ be restricted to functions of the form $u_1(x)u_0(x)$ with $u_1(x)$ non-negative, non-increasing on (a, b) and $|u_1|_p \leqslant 1$; then

$$\sup \int_a^b u(x)v(x)\,dx = \sup \int_a^b u_1(x)u_0(x)v(x)\,dx = [vu_0]_q.$$

It follows that if u is fixed, not necessarily non-increasing, and v varies over *all* level functions on (a, b) with $[v]_q \leqslant 1$, then

$$\sup \int_a^b u(x)v(x)\,dx = [uw]_p \tag{4.2}$$

and clearly the right-hand side of (4.2) is equal to $|u|_p$ if u is non-increasing.

Remark 2. Standard arguments now show that the supremum in (4.1) is actually attained except when $p = 1$, $[v] < \infty$, $D(\tilde{a} + 0) > R(\tilde{a}, x)$ for all $\tilde{a} < x < \mathrm{b}$ all hold. Consequently, if $u(x)v(x)$ has a *finite* integral whenever $u(x)$ is non-increasing with $|u|_p \leqslant 1$ then $[v]_q < \infty$.

Remark 3. The supremum in (4.1) will not be changed if on each of an arbitrary family of non-overlapping level intervals (a_i, b_i) of v, $v(x)$ is replaced by $R(a_i, b_i)w(x)$. The proof of Theorem 4.2 also shows that the supremum in (4.1) will not be changed if $u(x)$ is further restricted to be *constant* on each of the level intervals (a_i, b_i) for which $w(a_i, b_i) < \infty$.

It follows that if on each of a family of *finite* non-overlapping intervals, $v(x)$ is *constant* and $w(x)$ is *non-increasing*, then the supremum in (4.1) will not be changed if $u(x)$ is further restricted to be constant on each of these intervals.

Remark 4. The preceding results of sections 3, 4 apply if all functions u, v as well as the weight function w, are required to be constant on each of a fixed but arbitrary family of non-overlapping sub-intervals of (a, b). If (a, b) is entirely subdivided into such intervals *of equal length*, there results the corresponding theory for finite, infinite, or doubly infinite sequences with integrals replaced by sums. In this case the supremum in (4.1) is attained for *all* cases.

Remark 5. Theorem 4.2 and the Remarks above remain valid if $u(x)$ is further restricted to have *finite* value for every x, with the following exception: if $v(a, \tilde{a}) > 0$ then the supremum in (4.1), namely ∞, may not be attained for such u in the general situation described in Remark 4. Thus, for *sequences*, if $a < \tilde{a}$, the finiteness of $\Sigma\, u_n v_n$ for all non-increasing u_n with $|u|_p \leqslant 1$ and $u_n < \infty$ for every n, need *not* imply that $[v]_q < \infty$.

5. The spaces $L_{(w)}{}^p$ and $M_{(w)}{}^q$. We refer to §2 for terminology.

The definition of f^* implies: $f^*(x) \leqslant f_1^*(x)$ for all x if $|f(P)| \leqslant |f_1(P)|$ for almost all P; $f_e^*(x) = 0$ for all $x > \gamma(e)$; $f_N^*(x) = (f^*)_N(x)$ for all x; if $|f_n(P)| \leqslant |f(P)|$ and $|f_n(P)| \to |f(P)|$ as $n \to \infty$, for almost all P, then $f_n^*(x) \leqslant f^*(x)$ and $f_n^*(x) \to f^*(x)$ as $n \to \infty$ for all x; if $1 \leqslant p < \infty$, the left-continuous, non-increasing rearrangement of the function $|f(P)|^p$ is equal to $f^*(x)^p$ for all x; $(f_1 + f_2)^* \prec (f_1^* + f_2^*)$.

Theorem 3.7 (i), together with the Corollary to Theorem 4.2, now show that $[g_1 + g_2] \leqslant [g_1] + [g_2]$ and from this it follows easily that $M_{(w)}{}^q(B)$ is a linear normed space. Furthermore if $|g_n(P)| \leqslant |g(P)|$ and $|g_n(P)| \to |g(P)|$ as $n \to \infty$ for almost all P then $[g_n] \leqslant [g]$ and $[g_n] \to [g]$ as $n \to \infty$.

The hypotheses on w ensure: $|f|_p = |f_1|_p$ and $[g]_q = [g_1]_q$ if the non-negative functions $|f(P)|$, $|f_1(P)|$ and $|g(P)|$, $|g_1(P)|$ are equimeasurable, respectively.

Throughout the remainder of §§ 5 and 6, $f(P)$ and $g(P)$ will denote *numerical* valued functions not necessarily in $L_{(w)}{}^p$, $M_{(w)}{}^q$ respectively.

THEOREM 5.1.

(i) *If f varies subject to the condition $|f|_p \leqslant 1$ then*

$$\sup \int_S |f(P)|\,|g(P)|\,d\gamma(P) = [g]_q \tag{5.1}$$

and if $[g]_q < \infty$ then

$$\sup \left| \int_S f(P)_g(P)\,d\gamma(P) \right| = [g]_q.$$

(ii) *If g varies subject to the condition $[g]_q \leqslant 1$ then*

$$\sup \int_S |f(P)|\,|g(P)|\,d\gamma(P) = |f|_p \tag{5.2}$$

and if $|f|_p < \infty$ then

$$\sup \left| \int_S f(P)g(P)\,d\gamma(P) \right| = |f|_p.$$

Proof of (i). It is now clear that we need establish (5.1) only for finitely valued $g(P)$ and with f further restricted to be finitely valued. With such g and f it is easy to verify that

$$\sup \int_S |f(P)|\,|g(P)|\,d\gamma(P) = \sup \int_0^\gamma f^*(x)g^*(x)\,dx. \tag{5.3}$$

Since $w^*(x)$ is non-increasing, and g^* is a step function, the Remark 3 following Theorem 4.2 implies that the right-hand side of (5.3) is equal to $[g]_q$.

Proof of (ii). We need establish (5.2) only for finitely valued $f(P)$ and with g further restricted to be finitely valued. The argument of (i) together with (4.2) proves (ii).

Remark. The proof given for Theorem 5.1 shows that if $g(P)$ is constant on each of a countable set of disjoint e_n, then the supremum in (5.1) is not changed if f is further restricted to be constant on each e_n. Similarly, if $f(P)$ is constant on each of a countable set of disjoint e_n, then the supremum in (5.2) is not changed if g is further restricted to be constant on each e_n.

THEOREM 5.2.

(i) *If $f_1(P)$ and $f_2(P)$ are different from zero on disjoint sets,* i.e. $f_1(P)f_2(P) = 0$ *for almost all P then: if $p = \infty$, $|f_1 + f_2| = \max(|f_1|, |f_2|)$; and if $1 \leqslant p < \infty$, $|f_1 + f_2|^p \leqslant |f_1|^p + |f_2|^p$.*

(ii) *If $g_1(P)g_2(P) = 0$ for all P then:*

$$[g_1 + g_2] = [g_1] + [g_2], \qquad q = 1,$$

$$[g_1 + g_2]^q \geqslant [g_1]^q + [g_2]^q, \qquad 1 < q < \infty,$$

$$[g_1 + g_2] \geqslant \max\,([g_1], [g_2]), \qquad q = \infty.$$

Proof of (i). When $p = \infty$, $|f|$ coincides with ess. sup $|f(P)|$. When $1 \leqslant p < \infty$,

$$\begin{aligned} |f_1|^p + |f_2|^p &= \int_0^\gamma (|f_1|^{p*}(x) + |f_2|^{p*}(x))\, w^*(x)\, dx \\ &\geqslant \int_0^\gamma (|f_1|^p + |f_2|^p)^*(x)\, w^*(x)\, dx \\ &= \int_0^\gamma |f_1 + f_2|^{p*}(x) w^*(x)\, dx = |f_1 + f_2|^p. \end{aligned}$$

Proof of (ii). When $q = 1$, $[g]$ coincides with the integral of $|g(P)|$ on S. Theorem 5.1 shows that $[g_1 + g_2] \geqslant \max\,([g_1], [g_2])$ for all q, so that we need consider only the case $1 < q < \infty$ and we may clearly suppose $[g_1]$, $[g_2]$ both finite and positive. Then for any $\epsilon > 0$, Theorem 5.1 implies that there are f_1, f_2 with $|f_i| = [g_i]^{q-1}$ and

$$\int |f_i(P)|\,|g_i(P)|\, d\gamma(P) > [g_i]^q - \epsilon, \qquad i = 1, 2.$$

It may clearly be supposed further that $f_i(P) = 0$ wherever $g_i(P) = 0$ so that $f_1(P) f_2(P) = 0$ for all P. It follows that

$$\int_S |(f_1 + f_2)(P)|\,|(g_1 + g_2)(P)|\, d\gamma(P) > [g_1]^q + [g_2]^q - 2\epsilon$$

and

$$|f_1 + f_2| \leqslant (|f_1|^p + |f_2|^p)^{1/p} = ([g_1]^q + [g_2]^q)^{1/p}.$$

The validity of (ii) follows at once.

Remark. It is easy to show that the inequalities can be replaced by equalities only if $w^*(x)$ is a constant (then $L_{(w)}{}^p$, $M_{(w)}{}^q$ coincide essentially with classical L^p, L^q) or if $p = \infty$ for $L_{(w)}{}^p$ (which is actually identical with classical L^∞ for all w), or if $q = 1$ for $M_{(w)}{}^q$ (which is actually identical with classical L^1 for all w).

THEOREM 5.3.

(i) *If* $f_1(P) f_2(P) = 0$ *for all* P *and* $1 \leqslant p < \infty$ *then* $|f_1 + f_2| = |f_1| < \infty$ *implies* $f_2(P) = 0$ *for almost all* P, *in Case* (C_1) *if* $w^*(x) > 0$ *for all* $0 < x < \gamma$, *and in Case* (C_2).

(ii) *If* $g_1(P) g_2(P) = 0$ *for all* P *and* $1 \leqslant q < \infty$ *then* $[g_1 + g_2] = [g_1] < \infty$ *implies* $g_2(P) = 0$ *for almost all* P.

Proof. (ii) is an immediate corollary to Theorem 5.2 (ii).

If (i) were false there would be an $\epsilon > 0$ such that $\gamma\{|f_2(P)| > \epsilon\} > \epsilon$. Suppose $\gamma\{|f_1(P)| > \epsilon\} = A$. Then $f_1^*(x) \geqslant \epsilon$ for all $0 \leqslant x < A$ so that A must be finite in Case (C_1) or Case (C_2). Then $(f_1 + f_2)^*(x) \geqslant f_1^*(x)$ for all x and $(f_1 + f_2)^*(x) > \epsilon > f_1^*(x)$ for all $A < x < A + \epsilon$. Since $w^*(x) > 0$ for $A < x < A + \epsilon$ and $|f_1| < \infty$ we would deduce $|f_1 + f_2| > |f_1|$, contrary to the hypothesis.

THEOREM 5.4.

(i) $|f_N| \leqslant |f|$; $|f_N| \to |f|$ *as* $N \to \infty$; *if* $|f| < \infty$, *then* $|f - f_N| \to 0$ *as* $N \to \infty$.

(ii) $|f_e| \leqslant |f|$; sup $|f_e|$ *(for all* e*)* $= |f|$; $\gamma(e) \to 0$ *implies* $|f_e| \to 0$ *whenever* $|f| < \infty$ *if and only if either* $1 \leqslant p < \infty$ *or* $p = \infty$ *and for some* $A > 0$, $\gamma(e) = 0$ *whenever* $\gamma(e) < A$.

Proof. Parts of (i) and (ii) follow easily from the definition of $|f|$. To complete the proof of (i) we note: $(f - f_N)^*(x) \leqslant f^*(x)$ for all x and $= 0$ if $f^*(x) \leqslant N$; if $|f| < \infty$ and $p = \infty$ then $f = f_N$ for some N; and if $|f| < \infty$ and $1 \leqslant p < \infty$ then $A = \gamma\{|f(P)| > N\} = m\{|f^*(x)| > N\} \to 0$ as $N \to \infty$ and

$$|f - f_N|^p = \int_0^\gamma (f - f_N)^*(x)^p w^*(x) dx \leqslant \int_0^A f^*(x)^p w^*(x) dx.$$

To complete the proof of (ii) we note: $f_e^*(x) \leqslant f^*(x)$ for all x and $= 0$ for $x > \gamma(e)$; if $1 \leqslant p < \infty$ then

$$|f_e|^p \leqslant \int_0^{\gamma(e)} f^*(x)^p w^*(x)\, dx;$$

and if $p = \infty$, $|f_e| =$ ess. sup $|f(P)|$ on e.

THEOREM 5.5.

(i) $[g_N] \leqslant [g]$; $[g_N] \to [g]$ *as* $N \to \infty$; $N \to \infty$ *implies* $[g - g_N] \to 0$ *whenever* $[g] < \infty$ *if and only if either* $1 \leqslant q < \infty$ *or* $q = \infty$ *and* ess. sup $w(P)$ *on* S *is finite.*

(ii) $[g_e] \leqslant [g]$; sup $[g_e]$ *(for all* e*)* $= [g]$; *if* $[g] < \infty$ *then* $\gamma(e) \to 0$ *implies* $[g_e] \to 0$ *if and only if either* $1 \leqslant q < \infty$ *or* $q = \infty$ *but for some* $A > 0$, $\gamma(e) = 0$ *whenever* $\gamma(e) < A$.

Proof. Parts of (i) and (ii) are easy consequences of Theorem 5.1(i). To prove the rest of (i) we note: if $[g] < \infty$, and $q = \infty$ and ess. sup $w(P)$ is finite, then $g = g_N$ for some N; if $q = \infty$ and ess. sup $w(P)$ is infinite then $[w] = [w - w_N] = 1$ for all N; and if $[g] < \infty$ and $1 \leqslant q < \infty$ then

$$A(N) = \gamma\{\, |g(P)| > N\,\} \to 0$$

as $N \to \infty$ (as is easily verified) and, using Theorem 5.1, we obtain

$$[g - g_N] \leqslant \sup \int_0^\gamma u(x) g^*(x)\, dx$$

where $u(x)$ is restricted to be non-negative, non-increasing, to have $|u|_p \leqslant 1$, and to vanish for $x > A(N)$. Since $v \prec x^\circ$ for all v (Theorem 3.7 (ii)), Theorem 4.1 shows that, for all such u,

$$[g - g_N] \leqslant \sup \int_0^\gamma u(x) g^{*\circ}(x)\, dx = \sup \int_0^\gamma u(x) v(x)\, dx,$$

where $v(x) = g^{*\circ}(x)$ for $0 \leqslant x \leqslant A(N)$, $= 0$ for all other x. Theorem 4.2 now gives

$$[g - g_N]^q \leqslant [v]^q = \int_0^{A(N)} \left(\frac{g^{*\circ}(x)}{w^*(x)}\right)^q w^*(x)\, dx.$$

Thus $[g - g_N] \to 0$ since $A[N] \to 0$ when $N \to \infty$.

To prove the rest of (ii) we note: if $1 \leqslant q < \infty$,

$$[g_e] \leqslant \sup \int_0^\gamma u(x) g^*(x)\, dx = \sup \int_0^\gamma u(x) g^{*\circ}(x)\, dx$$

for all non-negative, non-increasing $u(x)$ with $|u| \leqslant 1$ and $u(x) = 0$ for $x > \gamma(e)$. Hence

$$[g_e]^q \leqslant \int_0^{\gamma(e)} \left(\frac{g^{*\circ}(x)}{w^*(x)}\right)^q w^*(x)\, dx$$

which $\to 0$ when $\gamma(e) \to 0$; if $q = \infty$,

$$[g_e] = \sup \frac{\int_{e_1} |\, g(P)\, |\, d\gamma(P)}{w^*(0, \gamma(e_1))}$$

for all $e_1 \subset e$ with $\gamma(e_1) > 0$, and hence if $\gamma(e) \to 0$ implies $[w_e] \to 0$ there must be $A > 0$ such that $\gamma(e) < A$ implies $\gamma(e) = 0$.

THEOREM 5.6. *$|f| < \infty$ implies that for any $\epsilon > 0$, there is an e for which $|f - f_e| < \epsilon$ if and only if either Case* (C_1) *holds* ($1 \leqslant p \leqslant \infty$) *or Case* ($C_2$) *holds with* $1 \leqslant p < \infty$.

Proof. In Case (C_1), S may be used for e. In Case (C_2), with $1 \leqslant p < \infty$ there is a finite A with

$$\int_A^\infty f^*(x)^p w^*(x)\, dx < \tfrac{1}{2}\epsilon.$$

Let $e = \{|f(P)|^p > \epsilon/2w^*(0, A)\}$. Then

$$|f - f_e|^p = \left(\int_0^A + \int_A^\infty\right) (f - f_e)^*(x)^p w^*(x)\, dx$$

$$\leqslant \frac{\epsilon}{2w^*(0, A)} \int_0^A w^*(x)\, dx + \int_A^\infty f^*(x)^p w^*(x)\, dx < \epsilon.$$

In Case (C_2) with $p = \infty$, and in Case (C_3) with arbitrary p, $1 \leqslant p \leqslant \infty$, the function $f(P)$ identically 1 has $(f - f_e)^*(x) = f^*(x) = 1$ for $0 < x < \infty$ and hence $|f - f_e| = |f| > 0$ for all e.

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COROLLARY. *$|f| < \infty$ implies that $f = f_E$ with E a countable union of sets of finite measure, for all p in Case* (C_1) *and for $1 \leqslant p < \infty$ in Case* (C_2).

THEOREM 5.7. *$[g] < \infty$ implies that for $\epsilon > 0$ there is an e for which $[g - g_e] < \epsilon$ except for $q = \infty$ in Case* (C_2).

Proof. In Case (C_1), $[g - g_e] = 0$ when $e = S$ for $1 \leqslant q \leqslant \infty$.

In Cases (C_2) and (C_3), if $1 \leqslant q < \infty$, Theorem 5.1 (i) implies that for some e, $[g_e]^q \geqslant [g]^q - \epsilon^q$ and Theorem 5.2(ii) implies $[g - g_e] < \epsilon$.

In Case (C_3) with $q = \infty$,

$$[g] = \sup \int_0^\infty u(x) g^*(x)\, dx \quad \text{for} \quad \int_0^\infty u(x) w^* dx \leqslant 1$$

with $u(x)$ also restricted to be non-negative and non-increasing. The particular choice $u(x) = (w^*(0, \infty))^{-1}$ for all x (we are in Case (C_3)) shows that $g^*(x)$ has a finite integral on $(0, \infty)$; and if, for $k > 0$, we set $e = \{|g(P)| > k\}$ then $A(k) = \gamma(e)$ is finite.

Clearly we need consider only the case with $A(k) \to \infty$ as $k \to 0$. Then choose B so large and then k so small that

$$\frac{1}{w^*(0, B)} \int_B^\infty g^*(x)\, dx < \tfrac{1}{2}\epsilon; \quad k < \frac{\epsilon w^*(0, B)}{2B}, \quad A(k) > B.$$

Now, when u varies so that $|u|_1 \leqslant 1$, with $u(x)$ non-negative and non-increasing,

$$\begin{aligned} [g - g_e] &\leqslant \sup \left(\int_0^B u(x) k\, dx + \int_B^\infty u(x) g^*(x)\, dx \right) \\ &\leqslant k\, [1_{(0,B)}] + (\sup u(B)) \int_B^\infty g^*(x)\, dx \\ &\leqslant \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon \end{aligned}$$

since $[1_{(0,B)}] = Bw^*(0, B)^{-1}$ and $u(B) w^*(0, B) \leqslant 1$ (use the fact that $u(x)$ is non-increasing).

In Case (C_2) with $q = \infty$, $[w - w_e] = [w] = 1$ for all e.

COROLLARY. *$[g] < \infty$ implies $g = g_E$ with E a countable union of sets of finite measure except for $q = \infty$ in Case* (C_2).

THEOREM 5.8.
(i) $|1_E|_p = w^*(0, \gamma(E))^{1/p}$.
(ii) $[1_E]_q = w^*(0, \gamma(E))^{1/q}$.

Proof. The calculations are easy.

THEOREM 5.9. *Let e_α, E_β be families of subsets of S such that for $\epsilon > 0$ and given e, E there are sets e', E', countable unions of the e_α, E_β respectively, with $\gamma(e - ee') + \gamma(e' - ee') < \epsilon$ and $\gamma(E - EE') + \gamma(E' - EE') < \epsilon$. Let T_1,*

T_2, T_3, T_4 be the sets of functions which are constant and rational on each of a finite number of the e_α, E_β, arbitrary e, arbitrary E, respectively. Then:

(i) *For $1 \leqslant p < \infty$, T_1 is dense in $L_{(w)}{}^p$ in Cases* (C_1) *and* (C_2) *and T_2 is dense in $L_{(w)}{}^p$ in all Cases* (C_1), (C_2), (C_3); *for $p = \infty$, T_4 is dense in $L_{(w)}{}^p$.*

(ii) *For $1 \leqslant q < \infty$, T_1 is dense in $M_{(w)}{}^q$ in all Cases* (C_1) (C_2), (C_3); *for $q = \infty$, if* ess. sup *$w(P)$ is finite, then T_3 is dense in $M_{(w)}{}^q$ in Cases* (C_1), (C_3) *and T_4 is dense in $M_{(w)}{}^q$ in Case* (C_2).

Proof. The proof of this theorem follows from standard arguments using the preceding theorems.

Remark. If S has a countable family of E_β as in Theorem 5.9 with $\gamma(E_\beta)$ finite for every β, for example if S is a subset of Euclidean space with positive Lebesgue measure, then $L_{(w)}{}^p$ is separable in Cases (C_1), (C_2) for $1 \leqslant p < \infty$ *and has dimensionality equal to the power of the continuum* in Case (C_3) for all $1 \leqslant p \leqslant \infty$, and in Cases ($C_1$) and ($C_2$) if $p = \infty$. To prove this, note that in Case (C_3) if $e_1, e_2, \ldots$ are disjoint, each of measure $> A > 0$, then for every infinite sequence π of increasing positive integers the function

$$
\begin{aligned}
f_\pi(P) &= 1 \qquad && \text{for } p \in e_n,\ n \in \pi,\\
&= 0 && \text{for all other } P,
\end{aligned}
$$

is in L, and for different sequences π_1, π_2,

$$|f_{\pi_1} - f_{\pi_2}| \geqslant w^*(0, A)^{1/p}.$$

On the other hand, with such S, $M_{(w)}{}^q$ is separable for $1 \leqslant q < \infty$ for all w, and has dimensionality the power of the continuum if $q = \infty$ and w is bounded on S.

6. The conjugate spaces. Let L', M' denote the closed linear subspaces spanned by the f in L and g in M respectively with $\gamma\{f(P) \neq 0\}$, $\gamma\{g(P) \neq 0\}$ finite. Then $L' = L$ in Case (C_1) with $1 \leqslant p \leqslant \infty$ and in Case (C_2) with $1 \leqslant p < \infty$; otherwise L' is not all of L. Also, $M' = M$ except for $q = \infty$ in Case (C_2); for this case, M' is not all of M.

THEOREM 6.1.

(i) *If $1 \leqslant p < \infty$, the conjugate space to L' is M, assuming, if $p = 1$, that S has property* (R).

(ii) *If $1 \leqslant q < \infty$, the conjugate space to M' is L, assuming, if $q = 1$, that S has property* (R).

Proof of (i). In view of Theorem 5.1 (i) we need only show that every bounded linear functional $\phi(f)$, $f \in L'$ has the form, for some g in M,

$$\int f(P)g(P)d\gamma(P). \tag{6.1}$$

Now $\phi(1_e)$ is defined for all e and $\phi(1_e) \to 0$ when $\gamma(e) \to 0$. The Radon-Nikodym

https://doi.org/10.4153/CJM-1953-031-3 Published online by Cambridge University Press

theorem then implies: for every E which is a countable union of sets of finite measure there is a $g(E) = g(E, P)$ vanishing outside E, with finite integral on every $e \subset E$ and such that

$$\phi(1_e) = \int_e g(E, P)\, d\gamma(P)$$

for every $e \subset E$. This implies $[g(E)]_q \leqslant |\phi|$. We can suppose E chosen to give $[g(E)]_q$ its maximum possible value.

If $p > 1$ Theorem 5.3 (ii) then shows that $g(E_1)$ is the zero function whenever E_1, E are disjoint, and we set $g(P) = g(E_1, P)$ for all P; then

$$\phi(1_e) = \int_e g(P)\, d\gamma(P)$$

for all e. Hence ϕ coincides for all f in L' with the bounded linear functional defined by this g through (6.1).

If $p = 1$ we use the decomposition of property (R) to define a single $g(P)$ to coincide on each e_α with $g(e_\alpha, P)$. For this g we have $[g]_q \leqslant |\phi|$ and the argument proceeds as before.

Proof of (ii). In view of Theorem 5.1 (ii) we need only show that every bounded linear functional $\phi(g)$, $g \in M'$ is of the form, for some f in L,

$$\int f(P) g(P)\, d\gamma(P). \tag{6.2}$$

As in the proof of (i), using Theorem 5.3(i), or property (R), we obtain an $f(P)$ with $|f|_p \leqslant |\phi|$ and such that

$$\phi(1_e) = \int f(P) g(P)\, d\gamma(P)$$

for all g in M'. This implies (ii).

COROLLARY. $(L_{(w)}{}^p)^* = M_{(w)}{}^q$ *if* $1 \leqslant p < \infty$ *in Cases* (C_1) *and* (C_2) *(assuming, if* $p = 1$, *that* S *has property* (R)*). And* $M_{(w)}{}^q$ *is part but* not *all of* $(L_{(w)}{}^p)^*$ *in Case* (C_3) *for every* p. *On the other hand,* $(M_{(w)}{}^q)^* = L_{(w)}{}^p$ *if* $1 \leqslant q < \infty$ *in* all *Cases* (C_1), (C_2), (C_2) *(assuming, if* $q = 1$, *that* S *has property* (R)*).*

It follows that $L_{(w)}{}^p$ and $M_{(w)}{}^q$ are reflexive if $1 < p < \infty$ (i.e., $1 < q < \infty$) in Cases (C_1) and (C_2), and for $p = 1$ or ∞ (i.e., $q = \infty$ or 1) in Case (C_1) if S is the union of a finite number of atoms, and that $L_{(w)}{}^p$ and $M_{(w)}{}^q$ are *not* reflexive in all other cases.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. H. W. Ellis and Israel Halperin, *Function spaces determined by a levelling length function,* to be published in a subsequent issue of this journal.
3. G. G. Lorentz, *Some new functional spaces*, Ann. Math., *51* (1950), 37–55.

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