4

Quantum Bits

4.1 Quantum Bits

We stressed previously that although classical bits (or equivalently, bits) are not vectors, embedding them into a two-dimensional vector space $\mathcal{Q}$ gives several advantages. The first advantage is that we can represent bit operators by either dyadics or matrices, which allows an efficient encoding of bit dynamics in familiar linear algebraic terms. The second advantage is that it allows us to generalize bits to stochastic bits directly and efficiently.

A third advantage that is of fundamental importance in quantized detector networks (QDN) and which we explore in this chapter is that we can generalize bits to their quantum counterparts, known as *quantum bits* (or equivalently, *qubits*).

A qubit is a complex two-dimensional Hilbert space, denoted $\mathcal{Q}$. That is a mathematical statement, but we need more. A *signal qubit* $(\mathcal{Q}, C)$ is a qubit $\mathcal{Q}$ with an empirical context C that defines a preferred basis $B \equiv \{\mathbf{0}, \mathbf{1}\}$ for that particular Hilbert space $\mathcal{Q}$.

Normally, we shall denote a signal qubit $(\mathcal{Q}, C)$ by $\mathcal{Q}$ whenever the context C is understood and kept in mind. It is implicitly assumed that all of this discussion is relative to some observer conducting experiments in a real laboratory.

4.2 Preferred Bases

Viewed in the right empirical context, a classical bit can be identified with a preferred basis for a given signal qubit, as follows. First, we shall use the same notation $\mathbf{0}$, $\mathbf{1}$ for the two elements of a given classical bit and for their embedding in the associated qubit $\mathcal{Q}$. In that embedding, we shall take the vectors $\mathbf{0}$ and $\mathbf{1}$ in $\mathcal{Q}$ to satisfy the "inner product" rule

$$\bar{\boldsymbol{i}}\boldsymbol{j} = \delta^{ij}, \qquad i, j = 0, 1, \tag{4.1}$$

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where $\overline{\mathbf{0}}$ and $\overline{\mathbf{1}}$ are the duals of $\mathbf{0}$ and $\mathbf{1}$, respectively, and δ^{ij} is the Kronecker delta. Then the relationship between the bit $B \equiv \{\mathbf{0}, \mathbf{1}\}$ and its quantum counterpart $\mathcal{Q}$ is this: the elements of B are identified as *the* natural orthonormal basis for $\mathcal{Q}$, known as the *preferred basis*.

The existence of the preferred basis is fundamental to our approach to physics and is in no way controversial. We do not claim that there is a unique or absolute preferred frame in the Universe. Any preferred frame is preferred only by virtue of the relative context associated with a given observer. By definition of what is meant by "*observer*," each observer is assumed always to know the empirical context associated with each signal qubit that they use, being nothing other than a mathematical representation of some detector in the observer's laboratory. We can be confident that there is always going to be such a local preferred basis, because the idea that an observer could extract real information in an experiment with no knowledge about their apparatus makes no sense whatsoever.

By definition, each detector has only two possible outcome states, known as *ground state* and *signal state*. This contextually defines the preferred basis: $\mathbf{0}$ represents the ground (no signal) state of the apparatus and $\mathbf{1}$ represents the signal state of the apparatus.

We emphasize the following point. There runs throughout QM a strand of thinking, conditioned by experience with CM, that states of SUO have some sort of existence of their own. According to this logic, such states do not need any preferred bases for their mathematical representation. This line of thinking then leads to the notion that observers are not needed either.

It is along such realist lines of thinking that Hidden Variables theory (Bohm, 1952), decoherence (in its original form) (Joos, 2012), and the Multiverse (Many Worlds) (Deutsch, 1997) are based. The problem with those interpretations of QM is that they are each contextually incomplete with a generalized proposition classification[1] of zero, meaning that those theories are empirically vacuous.

4.3 Qubit Properties

The power and mystery of QM stems from the possibility of creating states in the laboratory that are represented by linear superpositions of preferred basis states. Given the preferred basis, then any pure signal qubit state $\mathbf{\Psi}$ can be written in the form

$$\mathbf{\Psi} = \alpha\mathbf{0} + \beta\mathbf{1}, \tag{4.2}$$

where α and β are complex numbers and $\mathbf{0}$, $\mathbf{1}$ are the two elements of the preferred basis.

The fact that α and β are complex and not real is of fundamental significance here. Recall that in our discussion of stochastic bits in Chapter 3, the components

[1] Generalized propositions and their classification is discussed in Section 2.12.

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of a stochastic bit state such as (3.19) are nonnegative real numbers representing conditional probabilities. Because the components of a quantum bit state are not even real, let alone nonnegative, we encounter here the first of several issues in the interpretation of QM. This particular issue is generally regarded as being resolved by Born's interpretation of the above complex components as probability amplitudes (Born, 1926), discussed in Section 4.6.

If we choose to use the matrix representation of bits (3.3), then we may write

$$\boldsymbol{\Psi} \underset{R}{=} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{4.3}$$

Given qubit state (4.2), we define its dual $\overline{\boldsymbol{\Psi}}$ by

$$\overline{\boldsymbol{\Psi}} \equiv \overline{\alpha\mathbf{0} + \beta\mathbf{1}} = \alpha^*\overline{\mathbf{0}} + \beta^*\overline{\mathbf{1}}, \tag{4.4}$$

where $\alpha^*$and β^* are the complex conjugates of α and β, respectively. Then the "inner product" $\overline{\boldsymbol{\Psi}}\boldsymbol{\Psi}$ is given, using linearity and (4.1), by

$$\begin{aligned} \overline{\boldsymbol{\Psi}}\boldsymbol{\Psi} &= (\overline{\alpha\mathbf{0} + \beta\mathbf{1}})(\alpha\mathbf{0} + \beta\mathbf{1}) = (\alpha^*\overline{\mathbf{0}} + \beta^*\overline{\mathbf{1}})(\alpha\mathbf{0} + \beta\mathbf{1}) \\ &= \alpha^*\alpha\underbrace{\overline{\mathbf{0}}\mathbf{0}}_{1} + \alpha^*\beta\underbrace{\overline{\mathbf{0}}\mathbf{1}}_{0} + \beta^*\alpha\underbrace{\overline{\mathbf{1}}\mathbf{0}}_{0} + \beta^*\beta\underbrace{\overline{\mathbf{1}}\mathbf{1}}_{1} = |\alpha|^2 + |\beta|^2. \end{aligned} \tag{4.5}$$

A *normalized* signal qubit state is one for which $\overline{\boldsymbol{\Psi}}\boldsymbol{\Psi} = 1$, that is, $|\alpha|^2 + |\beta|^2 = 1$. We shall deal extensively with normalized signal qubit states, as these are associated with probability conservation in QM.

Normalized signal qubit states have been defined in terms of their components relative to the preferred basis. However, we can discuss them more generally as qubit states, that is, drop the observational context and think of them as just elements of some qubit. A qubit is a Hilbert space, a concept that is independent of basis and therefore does not require a preferred basis. We can discuss qubits in this context in a more abstract way as follows.

Given any element Φ of a Hilbert space, then by definition it has a *norm* $\|\Phi\|$ or length given by

$$\|\Phi\| \equiv \sqrt{(\Phi, \Phi)}, \ \geq 0, \tag{4.6}$$

where (Φ, Φ) is the inner product of Φ with itself. This length is basis independent.

A normalized qubit state therefore is an element of a complex two-dimensional Hilbert space and has unit norm.

4.4 Qubit Operators

We saw in the section in Chapter 3 that there are only four bit operators, denoted $\boldsymbol{I}$, $\boldsymbol{F}$, $\boldsymbol{D}$, and $\boldsymbol{U}$, that map bit states to bit states. In contrast, there is an infinite number of qubit operators that map qubit states to qubit states. simply because qubits are vector spaces.

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We define a *qubit map* to be any map from a qubit into itself. Specifically, given such a map M, then for any element $\boldsymbol{\Psi}$ of qubit $\mathcal{Q}$, the object $M(\boldsymbol{\Psi})$ is some element of $\mathcal{Q}$. For example, the *identity map* I satisfies the rule $I(\boldsymbol{\Psi}) = \boldsymbol{\Psi}$ for any element $\boldsymbol{\Psi}$ of $\mathcal{Q}$.

This definition of qubit map makes no reference to linearity, so a qubit map need not be linear, as in the following example.

Example 4.1 Given a qubit $\mathcal{Q}$ with preferred basis $\{\mathbf{0}, \mathbf{1}\}$, define the qubit map M by $M(\boldsymbol{\Psi}) = \mathbf{0}$ for every element $\boldsymbol{\Psi}$ of $\mathcal{Q}$. Then M is a nonlinear map, as, for example, we have $M(\mathbf{0} + \mathbf{1}) = \mathbf{0}$ but $M(\mathbf{0}) + M(\mathbf{1}) = \mathbf{0} + \mathbf{0} = 2\mathbf{0}$.

The map in the above example is not unphysical. It has the interpretation of a *resetting* or preparation process that prepares a detector to be in its ground state, ready to receive a signal, regardless of the state it is currently in. We shall use such a process in later chapters.

Because we are concerned in this book with quantum processes, almost all of the qubit maps we shall deal with will be linear. Linear qubit maps will be called *qubit operators*. It is conventional in the case of linear operators to drop the round brackets of the argument; that is, we shall write $\boldsymbol{O}\boldsymbol{\Psi}$ to mean $\boldsymbol{O}(\boldsymbol{\Psi})$ whenever $\boldsymbol{O}$ is a linear operator. Then for linear operator $\boldsymbol{O}$, for any elements $\boldsymbol{\Psi}, \boldsymbol{\Phi}$ of $\mathcal{Q}$, and for any complex numbers α, β, we have the rule

$$\boldsymbol{O}\{\alpha\boldsymbol{\Psi} + \beta\boldsymbol{\Phi}\} = \alpha\boldsymbol{O}\boldsymbol{\Psi} + \beta\boldsymbol{O}\boldsymbol{\Phi}. \tag{4.7}$$

In QM, linear operators are often associated with dynamical variables and so additional mathematical structure is introduced. Given two qubit operators $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ over a qubit $\mathcal{Q}$, we define the linear combination $\alpha\boldsymbol{O}_1 + \beta\boldsymbol{O}_2$ of these two operators to satisfy the rule

$$(\alpha\boldsymbol{O}_1 + \beta\boldsymbol{O}_2)\boldsymbol{\Psi} \equiv (\alpha\boldsymbol{O}_1\boldsymbol{\Psi}) + (\beta\boldsymbol{O}_2\boldsymbol{\Psi}), \tag{4.8}$$

for arbitrary complex numbers α, β, and arbitrary elements $\boldsymbol{\Psi}$ of $\mathcal{Q}$.[2] Then we state without proof that the set $L(\mathcal{Q})$ of all qubit operators over a qubit $\mathcal{Q}$ has all the properties of a four-dimensional complex Hilbert space (Paris, 2012).

The structure of the vector space $L(\mathcal{Q})$ is relatively easily explored. We saw in the previous chapter that given the preferred basis $\{\mathbf{0}, \mathbf{1}\}$ for $\mathcal{Q}$, the four elementary transition operators (ETOs) $\boldsymbol{T}^{ij} \equiv \boldsymbol{i}\overline{\boldsymbol{j}} : i, j = 0, 1$, form a convenient basis for $L(\mathcal{Q})$, that is, any qubit operator $\boldsymbol{O}$ can be written in the form

$$\boldsymbol{O} = \sum_{i=0}^{1}\sum_{j=0}^{1} O^{ij}\boldsymbol{T}^{ij}, \tag{4.9}$$

where the coefficients O^{ij} are complex.

[2] Note that the + symbol on the left-hand side of (4.8) denotes operator addition, while the + symbol on the right-hand side denotes vector addition.

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In addition to their vectorial additive properties, it is useful to define multiplication of qubit operators. We define the multiplicative product $\boldsymbol{O}_1\boldsymbol{O}_2$ of two qubit operators $\boldsymbol{O}_1$, $\boldsymbol{O}_2$ by the rule $(\boldsymbol{O}_1\boldsymbol{O}_2)\boldsymbol{\Psi} \equiv \boldsymbol{O}_1\{\boldsymbol{O}_2\boldsymbol{\Psi}\}$ for any element $\boldsymbol{\Psi}$ of $\mathcal{Q}$. The product of any two ETOs is also an ETO, that is, ETO multiplication is closed. Specifically, the multiplication rule is

$$\boldsymbol{T}^{ij}\boldsymbol{T}^{kl} = \delta^{jk}\boldsymbol{T}^{il}. \tag{4.10}$$

This ETO multiplication rule (4.10) is associative, that is to say, that

$$(\boldsymbol{T}^{ab}\boldsymbol{T}^{cd})\boldsymbol{T}^{ef} = \boldsymbol{T}^{ab}(\boldsymbol{T}^{cd}\boldsymbol{T}^{ef}), \tag{4.11}$$

but not commutative, which means $\boldsymbol{T}^{ab}\boldsymbol{T}^{cd} \neq \boldsymbol{T}^{cd}\boldsymbol{T}^{ab}$ in general.

These properties and the existence of the identity operator $\boldsymbol{I}$ mean that qubit operators form a mathematical structure known as a *unital associative algebra*.

4.5 Signal Bit Operators

The space of operators $O(\mathcal{Q})$ over qubit $\mathcal{Q}$ contains infinitely many elements. Fortunately, the necessary existence, in our approach, of the preferred signal basis $\{\mathbf{0}, \mathbf{1}\}$ singles out a very small number of special qubit operators that we shall use extensively and refer to as *signal bit operators*. In addition to the identity operator $\boldsymbol{I}$ and the zero operator $\boldsymbol{Z}$ (it maps any vector into the zero vector) there are four important signal bit operators, defined as follows.

The Projection Operators
The qubit projection operators $\boldsymbol{P}$, $\widehat{\boldsymbol{P}}$ are defined by

$$\boldsymbol{P} \equiv \boldsymbol{T}^{00} = \mathbf{0}\overline{\mathbf{0}}, \quad \widehat{\boldsymbol{P}} \equiv \boldsymbol{T}^{11} = \mathbf{1}\overline{\mathbf{1}}. \tag{4.12}$$

The Signal (Annihilation and Creation) Operators
The signal operators $\boldsymbol{A}$, $\widehat{\boldsymbol{A}}$ are in conventional parlance adjoints of each other and are defined by

$$\boldsymbol{A} \equiv \boldsymbol{T}^{01} = \mathbf{0}\overline{\mathbf{1}}, \quad \widehat{\boldsymbol{A}} \equiv \boldsymbol{T}^{10} = \mathbf{1}\overline{\mathbf{0}}. \tag{4.13}$$

The four operators $\boldsymbol{P}, \widehat{\boldsymbol{P}}$, $\boldsymbol{A}$, and $\widehat{\boldsymbol{A}}$ are by inspection just the four ETOs $\boldsymbol{T}^{ij}$ introduced earlier. The advantage in this new designation is mainly psychological: the projection operators play one role in our formalism while the signal operators play another, and it is very helpful to distinguish between them.

In this new notation, the multiplication rule (4.10) is best expressed in the form of a table, Table 4.1, where the entries are the products $\boldsymbol{LR}$, operator $\boldsymbol{L}$ coming from the left-most column and $\boldsymbol{R}$ coming from the top-most row.

4.6 The Standard Born Interpretation

Although superficially qubits look similar to s-bits mathematically, being representable by two component column matrices, they are very different objects as

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Table 4.1 *Products of signal bit operators*

L\R	$\boldsymbol{P}$	$\widehat{\boldsymbol{P}}$	$\boldsymbol{A}$	$\widehat{\boldsymbol{A}}$
$\boldsymbol{P}$	$\boldsymbol{P}$	$\boldsymbol{Z}$	$\boldsymbol{A}$	$\boldsymbol{Z}$
$\widehat{\boldsymbol{P}}$	$\boldsymbol{Z}$	$\widehat{\boldsymbol{P}}$	$\boldsymbol{Z}$	$\widehat{\boldsymbol{A}}$
$\boldsymbol{A}$	$\boldsymbol{Z}$	$\boldsymbol{A}$	$\boldsymbol{Z}$	$\boldsymbol{P}$
$\widehat{\boldsymbol{A}}$	$\widehat{\boldsymbol{A}}$	$\boldsymbol{Z}$	$\widehat{\boldsymbol{P}}$	$\boldsymbol{Z}$

far as physics is concerned. The components of an s-bit are real and interpreted as conditional probabilities, whereas the components of a qubit relative to its preferred basis are complex amplitudes and so cannot be probabilities. It was Max Born who gave the empirically correct interpretation of such complex amplitudes (Born, 1926), as follows.

The Born Interpretation

In QM, suppose $\mathcal{H}$ is a Hilbert space with inner product of elements ψ, ϕ denoted by (ψ, ϕ). Consider two normalized elements Ψ and Φ in $\mathcal{H}$ that represent physical states of some system under observation. Then the conditional probability $Pr(\Psi|\Phi)$ of finding the system to be in state Ψ, given that the system was prepared to be in state Φ, is given by

$$Pr(\Psi|\Phi) = |(\Psi, \Phi)|^2. \tag{4.14}$$

There is a lot of implicit contextual information not given in such a definition, but physicists generally know what is meant and implied. Specifically, they would understand that the formalism refers to a statistical analysis of a sequence of runs. State Φ is prepared through one device at the start of each run, allowed to evolve undisturbed by the observer over some intermediate space-time regime, and then passed through another device that produces an outcome. Each outcome occurs randomly from a range of potential outcomes, with a countable frequency distribution over the ensemble of runs. In general, the ratios of observed outcome frequencies, when a large number of runs is performed, conform excellently to the probabilities predicted by the above Born rule.

Points to note are the following.

Symmetry

There is an inherent symmetry in the relationship between the two states in rule (4.14), referred to as the *microscopic reversibility of quantum processes* (Bohm, 1952). This is because of the mathematical equality $|(\Psi, \Phi)| = |(\Phi, \Psi)|$, which immediately leads to the physical prediction $Pr(\Psi|\Phi) = Pr(\Phi|\Psi)$. This relation has been confirmed empirically countless times.

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Complex Amplitudes

The inner product (Ψ, Φ) is complex, which means it does not have the classical interpretation of a probability. Physicists get around this by referring to such expressions as "the amplitude for Φ to go to Ψ," or similar terminology, thereby endowing it with a touch of familiarity. The fact is, however, no one understands precisely why complex amplitudes occur in QM and it remains one of the enduring mysteries of the subject. Schwinger suggested that the appearance of complex numbers in QM is associated with the existence of antiparticles (Schwinger, 1958). Other physicists have investigated the theoretical and empirical possibility of replacing complex amplitudes in QM with hypercomplex (or quaternionic) amplitudes (Adler, 1995; Procopio et al., 2016; Adler, 2016), but there is at this time no empirical evidence that such a step is necessary.

Origin of the Born Rule

There have been attempts to derive the Born interpretation of the wave function from basic principles, but so far none of these attempts has been satisfactory. An interesting variant of such attempts is the idea that the Born rule is but the first step in a possibly infinite hierarchy of terms, a so-called multiorder interference rule that is a generalization of the above Born rule. Sorkin noted that for a two-slit interference experiment, the standard Born interpretation gives for the probability Pr_{AB} of a particle landing at a point on the detecting screen the formula

$$Pr_{AB} = Pr_A + Pr_B + I_{AB}, \tag{4.15}$$

where Pr_A is the probability when slit B is blocked off, Pr_B is the probability when slit A is blocked off, and I_AB is the so-called second-order interference term. Sorkin considered a three-slit experiment with slits A, B, and C and looked at the case for a generalization of the Born rule of the form

$$Pr_{ABC} = Pr_{AB} + Pr_{BC} + Pr_{AC} - Pr_A - Pr_B - Pr_C + I_{ABC}, \tag{4.16}$$

where I_{ABC} represents some novel third-order interference term not predicted by the above Born rule (Sorkin, 1994). A recent experiment has virtually ruled out such a term (Sinha et al., 2010).

Sorkin's motivation in this discussion is directly opposite to ours in this book: he explicitly states that he does not want to base the interpretation of his generalized probabilities with "some undefined concept of 'measurement made by human observers,"' and takes "the attitude that the ontology of QM is identical to that of classical realism" (Sorkin, 1994).

Dynamics

The Born rule is usually applied to states of SUOs evolving in time. In such a case, care has to be taken with the rules for the conservation of probability. These rules are as contextual as anything else in an experiment. For instance, if a state Φ_i is prepared at initial time t_i, allowed to evolve undisturbed until

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time t_f, then the amplitude to find the system under observation in state Ψ_f at final time is given by $(\Psi_f, U_{fi}\Phi_i)$, where U_{fi} is the unitary evolution operator taking states from initial time to final time. In this scenario, total probability is conserved. On the other hand, particle decay experiments may appear to involve a loss of total probability, if the mathematical modeling is done in too basic a fashion. For example, the Schrödinger wavefunction for a decaying particle SUO is frequently asserted to be given by a function of the form

$$\Psi(t, \boldsymbol{x}) \simeq e^{-i(E-i\Gamma)t/\hbar}\Phi(\boldsymbol{x}), \tag{4.17}$$

where Γ is a real constant related to the so-called half-life of the particle. We can discuss such a scenario in QDN; the QDN approach to particle decay experiments is covered in Chapter 15.

4.7 The Born Interpretation in QDN

We now consider the Born interpretation from the QDN perspective.

Given a normalized qubit state $\boldsymbol{\Psi} = \alpha\mathbf{0} + \beta\mathbf{1}$, where $|\alpha|^2 + |\beta|^2 = 1$, then the conditional probability $Pr(\mathbf{0}|\boldsymbol{\Psi})$ of the observer finding the associated detector in its ground state $\mathbf{0}$ is given by the rule $Pr(\mathbf{0}|\boldsymbol{\Psi}) = |\overline{\mathbf{0}}\boldsymbol{\Psi}|^2 = |\alpha|^2$, while the conditional probability $Pr(\mathbf{1}|\boldsymbol{\Psi})$ of finding the detector in its signal state $\mathbf{1}$ is $Pr(\mathbf{1}|\boldsymbol{\Psi}) = |\overline{\mathbf{1}}\boldsymbol{\Psi}|^{\mathbf{2}} = |\beta|^2$. We shall discuss the generalization of this rule to collections of detectors in the chapter on quantum register dynamics, Chapter 7.

There are two notational variants that we can use to discuss these probabilities.

Standard Expectation Value Notation

We may write

$$\begin{aligned} Pr(\mathbf{0}|\boldsymbol{\Psi}) &= |\overline{\mathbf{0}}\boldsymbol{\Psi}|^2 = (\overline{\mathbf{0}}\boldsymbol{\Psi})^*(\overline{\mathbf{0}}\boldsymbol{\Psi}) = (\overline{\boldsymbol{\Psi}}\mathbf{0})(\overline{\mathbf{0}}\boldsymbol{\Psi}) \\ &= \overline{\boldsymbol{\Psi}}(\mathbf{0}\overline{\mathbf{0}})\boldsymbol{\Psi} = \overline{\boldsymbol{\Psi}}\boldsymbol{P}\boldsymbol{\Psi}, \end{aligned} \tag{4.18}$$

interpreted in words as "$Pr(\mathbf{0}|\boldsymbol{\Psi})$ *is the expectation value of the ground state projection operator* $\boldsymbol{P}$*, contextual on the prepared state* $\boldsymbol{\Psi}$." Likewise, we have the rule

$$Pr(\mathbf{1}|\boldsymbol{\Psi}) = \overline{\boldsymbol{\Psi}}\ \widehat{\boldsymbol{P}}\boldsymbol{\Psi}. \tag{4.19}$$

Density Operator Notation

Given a pure bit state $\boldsymbol{\Psi}$, first define the density operator $\boldsymbol{\varrho} \equiv \boldsymbol{\Psi}\overline{\boldsymbol{\Psi}}$. Then the probabilities $Pr(\mathbf{0}|\boldsymbol{\Psi})$, $Pr(\mathbf{1}|\boldsymbol{\Psi})$ are given by the rules

$$Pr(\mathbf{0}|\boldsymbol{\Psi}) = Tr\left\{\boldsymbol{P}\boldsymbol{\varrho}\right\}, \qquad Pr(\mathbf{1}|\boldsymbol{\Psi}) = Tr\{\widehat{\boldsymbol{P}}\boldsymbol{\varrho}\}, \tag{4.20}$$

where Tr denotes the *trace* operation, discussed in Chapter 9.

4.8 Classical and Quantum Ensembles

The crucial differences between stochastic bits and qubits are not easy to see at the rank-one level but one of them is this: in CM, a stochastic bit state represents

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an observer's epistemic uncertainty as to which bit state a detector is actually in *before* they observe it, whereas in QM an observer can be certain of which qubit state a quantized detector is in and it is only the future *outcome* of an observation that is uncertain. Moreover, the uncertainty in the quantum case is generally regarded as intrinsic, or aleatoric, uncertainty.

There are several other ways of saying much the same thing. We can discuss *state preparation*, the processes that lead up to an observer having a contextual-based belief about the signal state of their detector, before observation. A stochastic bit state represents the observer's uncertainty about the preparation processes, whereas in the quantum case, the observer need have no such uncertainty about the preparation of a qubit state. Such a qubit state is called a *pure* state. The Born rule discussed above makes no reference to state preparation and it is assumed that the qubit state is pure.

Another way of showing the difference between stochastic bits and qubits is in terms of questions and answers. Given a stochastic bit of the form $\boldsymbol{S} \equiv a\mathbf{0} + (1-a)\mathbf{0}$, then $\bar{\boldsymbol{i}}\boldsymbol{S}, i = 0, 1$, represents a stochastic answer, which is a probability. On the other hand, given a qubit $\boldsymbol{\Psi}$, then $\bar{\boldsymbol{i}}\boldsymbol{\Psi}$, $i = 0, 1$, represents a *quantum answer*, which is a complex amplitude. Quantum answers have to be processed according to the Born rule given above in order to extract outcome probabilities.

Yet another way of seeing the difference between a stochastic bit and a qubit comes from the notion of *ensemble*. These are discussed in more detail in the Appendix. An ensemble can be either a real collection of near identical systems under observation, such as atoms in a crystal, or a hypothetical collection of imagined alternative futures, only one of which is going to be realized. Stochastic bits are generally associated with the former type of ensemble, while qubits are associated with the latter. We shall refer to the former kind of ensemble as a *classical ensemble* and refer to the latter kind as a *quantum ensemble*.

The differences between stochastic bits and qubits will become more obvious when we deal with *quantum registers*, or collections of qubits, discussed in Chapter 7.

4.9 Basis Transformations

Given a particular detector, then its associated physics provides the associated empirical context: *if* they looked, the observer would recognize when that detector was in its ground state **0** and when it was in its signal state **1**. If this were not the case, we would have to ask what observation meant in this case.

This unambiguity about the context is reflected in the two questions an observer could ask of any detector: **Is this detector in its ground state?** and **Is this detector in its signal state?** A classical bit will always return an unambiguous answer of *yes* or *no*, a stochastic bit returns a probability, and a quantum bit returns a probability amplitude. This is encoded into the vector space formalism by the fact that stochastic bits and qubits are linear combinations of the classical answer states **0** and **1**.

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We return now to the fact that the definition of a Hilbert space is basis independent. Let us explore this further. Ignoring physical context and looking at a qubit strictly as a mathematical vector space, we are entitled to change our basis from the preferred basis. Consider therefore replacing each element $\boldsymbol{i}$ in our original preferred basis with some new vector $\boldsymbol{i}'$, as follows. First, we note that our preferred basis B is *orthonormal*, i.e., $\overline{\boldsymbol{i}}\boldsymbol{j} = \delta^{ij}$, $i, j = 0, 1$. Anticipating physical applications in later chapters, we shall preserve this relationship. Therefore, we shall require $\overline{\boldsymbol{i}'}\boldsymbol{j}' = \delta^{ij}$, $i, j = 0, 1$. For a complex Hilbert space, such transformations are called *unitary*.

Given our initial basis $B \equiv \{\mathbf{0}, \mathbf{1}\}$, consider a unitary transformation $B \to B' \equiv \{\mathbf{0'}, \mathbf{1'}\}$ such that $\overline{\boldsymbol{i}'}\boldsymbol{j}' = \delta_{ij}$, for $0 \leq i, j \leq 1$. For such a transformation we may write

$$\boldsymbol{i} \to \boldsymbol{i}' \equiv \boldsymbol{U}\boldsymbol{i} = \sum_{j=0}^{1} \boldsymbol{j}' \, U^{ji}, \tag{4.21}$$

where the complex coefficients U^{ij} satisfy the unitarity relations

$$\sum_{j=0}^{1} U^{ij*} U^{jk} = \delta^{ik}, \tag{4.22}$$

where U^{ij*} is the complex conjugate of U^{ij}. These unitarity relations guarantee that orthonormality is preserved.

We may always write a unitary matrix in the form

$$U = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{4.23}$$

where the coefficients α, β, γ, and δ are complex. From (4.22) we deduce the important relations

$$|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1, \quad \alpha\gamma^* + \beta\delta^* = 0. \tag{4.24}$$

4.10 The Preferred Basis Problem

At this point we come across another problem related to the fact that a unitary transformation of the elements of a Hilbert space $\mathcal{H}$ has an implied action on the elements of its dual space $\overline{\mathcal{H}}$. This is because although the original definition of the inner product (ψ, ϕ) of a Hilbert space $\mathcal{H}$ is defined as a map from the Cartesian product $\mathcal{H} \times \mathcal{H}$ into the complex field, it can also be interpreted as a mapping of the vector ϕ in $\mathcal{H}$ into the complex numbers by the action of a one-form (a dual vector) $\overline{\psi}$, which is an element of a different vector space, the dual space $\overline{\mathcal{H}}$. The implied action of the above unitary operator $\boldsymbol{U}$ on elements of the dual space is given by

$$\overline{\boldsymbol{i}} \to \overline{\boldsymbol{i}'} \equiv \overline{\boldsymbol{i}}\boldsymbol{U}^\dagger = \sum_{j=0}^{1} \overline{\boldsymbol{j}'} U^{ji*}, \tag{4.25}$$

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where $\boldsymbol{U}^\dagger$ is the *Hermitian conjugate* operator. For finite-dimensional vector spaces, the Hermitian conjugate operator is the same as the *adjoint* operator, problems arising only with nonseparable Hilbert spaces (Streater and Wightman, 1964).

The point is that according to our interpretation of the dual vectors $\overline{\mathbf{0}}$ and $\overline{\mathbf{1}}$, they are associated with questions to be asked of answer states. Given the transformation (4.25), we have to ask what "a linear combination of questions" means. While the Born interpretation gives us a meaning for an answer to a classical question asked of a linear combination of vectors, it is not immediately obvious what the interpretation of an object such as $u\overline{\mathbf{0}}+v\overline{\mathbf{1}}$ is. Our current view is that it is a mathematical artefact devoid of physical significance. What underpins this view is that observers are always sure in their minds which questions they are asking in a laboratory.

Another way of saying this is that we have not considered quantizing observers: they are always regarded as classical and this policy will be maintained throughout this book. This does not mean we shall not consider quantizing *apparatus*.

There will be situations where linear combinations of questions makes physical sense. For instance, we saw in the previous chapter that we could interpret linear combinations of questions of stochastic bits in terms of probabilities. On the quantum side, not only will such a possibility be available to us, but there will be a quantum side to this issue. Linear complex combinations of questions will have a role in the *information void*, the regime between state preparation and state outcome detection. In this regime, quantum rules apply and the concept of observer is not meaningful.

We should add at this point that the linear combination of quantum questions we have just referred to is not the same thing as a *mixed* quantum question. A mixed quantum question would be the analogue of a mixed state in QM, where an observer has an epistemic uncertainty as to which quantum state had been prepared. A mixed quantum question would likewise involve an experiment where an observer had an epistemic uncertainty as to which quantum question was being asked. This is not the same thing as a linear combination of quantum questions (which carries the implication that the observer itself is quantized).

One of the problems of interpretation that arises in the Multiverse paradigm (Deutsch, 1999) is that observers and SUOs are described by the vacuous concept of a wave function for the Universe. Since there is by assertion no primary observer, it is not clear what superposition of different observers means, if anything. In this respect, the original ideas of Everett's "Relative State" interpretation of QM seem less unattractive (Everett, 1957), because of the adjective *relative*.[3] It is not unreasonable to imagine that one observer A could describe what other observers B and C are doing by a quantum state vector. On that

[3] Everett does explicitly postulate an absolute wave function for the Universe, something QDN cannot accept, because such a postulate has a GPC of zero.

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basis, relative to A, B and C are systems under observation and not observers. QDN is fully compatible with that idea, but regards the Multiverse concept as anathema.

4.11 Rank-One Qubit Evolution

As with bits and s-bits, we can consider the dynamical evolution of a single qubit state. Given a normalized qubit state $\boldsymbol{\Psi}_n \equiv \alpha_n \mathbf{0}_n + \beta_n \mathbf{1}_n$ in $\mathcal{Q}_n$ at stage Σ_n where $|\alpha_n|^2 + |\beta_n|^2 = 1$, we consider a *linear* map $\boldsymbol{U}_{n+1,n}$ from $\mathcal{Q}_n$ to $\mathcal{Q}_{n+1}$ such that normalization is preserved; i.e., we require

$$\boldsymbol{\Psi}_n \to \boldsymbol{\Psi}_{n+1} \equiv \boldsymbol{U}_{n+1,n}\boldsymbol{\Psi}_n, \qquad \overline{\boldsymbol{\Phi}_n} \to \overline{\boldsymbol{\Phi}_{n+1}} \equiv \overline{\boldsymbol{\Phi}_n}\boldsymbol{U}^{\dagger}_{n+1,n}, \tag{4.26}$$

with $\overline{\boldsymbol{\Psi}_{n+1}}\boldsymbol{\Psi}_{n+1} = 1$.

We come now to an important point. In the conventional theory of Hilbert spaces and in its application to standard QM, unitary transformations are maps from a given Hilbert space back into itself. In our situation this is no longer the case. We are dealing with maps from a Hilbert space at time n to another Hilbert space associated with time $n+1$. For example, the transformation (4.26) gives the dyadic representation

$$\boldsymbol{U}_{n+1,n} = \sum_{i,j=0}^{1} \boldsymbol{i}_{n+1} U^{ij}_{n+1,n} \overline{\boldsymbol{j}_n}. \tag{4.27}$$

This operator is one that not only preserves the norm but also preserves inner products under the transformation from one Hilbert space to the next. Such a transformation will be called a *semi-unitary* transformation, rather than a unitary transformation, for the good reason that in a more general context, the dimensions of the Hilbert spaces need not be the same. When the dimensions of the Hilbert spaces are different, then we need to consider the *retraction* of an evolution operator, rather than its inverse. We shall discuss this important aspect of our dynamics in more detail later.

In standard QM, a dynamics that preserves magnitudes of inner products (as opposed to inner products themselves) is usually realized via either unitary transformations or *anti-unitary* transformations. Anti-unitary transformations are subtle, having a great deal to do with the concept of *time reversal* in standard QM. We shall not consider anti-unitary transformations further.

An important issue that impacts on our approach is that one way of discussing time reversal in QM is to switch bra and ket vectors. Conventionally, this does not seem to matter but it amounts to switching questions and answers in our approach, which requires great care in the interpretation.

We may readily generalize our dynamical rule (4.26) to describe evolution from initial stage S_M to final stage S_N, where $N > M$: the product of two semi-unitary transformations is also a semi-unitary transformation. We find the rule

$$\boldsymbol{\Psi}_M \to \boldsymbol{\Psi}_N = \boldsymbol{U}_{N,M}\boldsymbol{\Psi}_M, \tag{4.28}$$

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where $\boldsymbol{U}_{N,M} \equiv \boldsymbol{U}_{N,N-1}\boldsymbol{U}_{N-1,N-2}\ldots\boldsymbol{U}_{M+1,M}$. The conditional outcome probabilities $Pr(\boldsymbol{i}_N|\boldsymbol{\Psi}_M)$ as measured at stage Σ_N are then given by

$$Pr(\boldsymbol{i}_N|\boldsymbol{\Psi}_M) \equiv |\overline{\boldsymbol{i}_N}\boldsymbol{\Psi}_N|^2, \quad i = 0, 1. \tag{4.29}$$

4.12 Mixed Qubit States

The uncertainty of s-bits is different from that associated with qubits. The former is due to ignorance on the part of the observer and may be called *classical* (or epistemic) uncertainty, while the latter is considered intrinsic and so referred to as *quantum* uncertainty on that account.

It is possible to encounter situations where both types of probability occur naturally. Whenever this happens, the labstates involved are no longer referred to as *pure* but *mixed*. The following example illustrates what we mean.

Example 4.2 An observer is about to ask the question $\overline{\mathbf{1}}$ of a single detector but is not sure how the labstate was prepared. The information that they do have, however, leads them to believe that the probability of the labstate (prior to observation) being $\boldsymbol{\Psi} \equiv \alpha\mathbf{0} + \beta\mathbf{1}$ is p, where $|\alpha|^2 + |\beta|^2 = 1$, while the probability of the labstate being $\boldsymbol{\Phi} \equiv \gamma\mathbf{0} + \delta\mathbf{1}$ is $1 - p$, where $|\gamma|^2 + |\delta|^2 = 1$. What is the overall probability of finding a signal?

Solution
Consider a very large number N of runs. Of these, approximately pN will involve the labstate $\boldsymbol{\Psi}$. The signal outcome probability for each such run is $|\beta|^2$ according to the Born rule. Therefore the total number of runs that involve the labstate $\boldsymbol{\Psi}$ with a signal outcome is approximately $pN|\beta|^2$. A similar calculation for the labstate $\boldsymbol{\Phi}$ gives the number of signal outcomes as $(1-p)N|\delta|^2$. The total number of signal outcomes is therefore approximately $Np|\beta|^2 + N(1-p)|\delta|^2)$. In the limit $N \to \infty$, the signal outcome probability is therefore $p|\beta|^2 + (1-p)|\delta|^2$ (the answer).

4.13 Density Operators

An efficient and standard way of combining classical and quantum probability is through the use of dyadics called *density operators*. Suppose we have a mixed initial state consisting of k different possible states $\boldsymbol{\Psi}^a$, $a = 1, 2, \ldots, k$. The density operator $\boldsymbol{\varrho}$ is defined by the dyadic

$$\boldsymbol{\varrho} \equiv \sum_{a=1}^{k} \omega^a \boldsymbol{\Psi}^a \overline{\boldsymbol{\Psi}^a}, \tag{4.30}$$

where ω^a is the probability that the initial state is actually $\boldsymbol{\Psi}^a$. The expectation value $\langle O \rangle_{\boldsymbol{\varrho}}$ of an observable O is then given by the standard rule

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$$\langle O \rangle_{\boldsymbol{\varrho}} \equiv Tr\left\{O\boldsymbol{\varrho}\right\}, \tag{4.31}$$

where Tr denotes the trace operation, discussed in Chapter 9.

To demonstrate how this works, we apply this formalism to the example considered above.

Example 4.3 With reference to the above example, we see $k = 2$ and

$$\begin{aligned} \omega^1 &= p, & \boldsymbol{\Psi}^1 &\equiv \boldsymbol{\Psi} = \alpha\mathbf{0} + \beta\mathbf{1} \\ \omega^2 &= 1-p, & \boldsymbol{\Psi}^2 &\equiv \boldsymbol{\Phi} = \gamma\mathbf{0} + \delta\mathbf{1}. \end{aligned} \tag{4.32}$$

Hence the density matrix is

$$\begin{aligned} \boldsymbol{\varrho} &= \omega^1\boldsymbol{\Psi}^1\overline{\boldsymbol{\Psi}^1} + \omega^2\boldsymbol{\Psi}^2\overline{\boldsymbol{\Psi}^2} \\ &= p\{\alpha\mathbf{0} + \beta\mathbf{1}\}\{\alpha^*\overline{\mathbf{0}} + \beta^*\overline{\mathbf{1}}\} + (1-p)\{\gamma\mathbf{0} + \delta\mathbf{1}\}\{\gamma^*\overline{\mathbf{0}} + \delta^*\overline{\mathbf{1}}\} \\ &= \{p|\alpha|^2 + (1-p)|\gamma|^2\}\mathbf{0}\overline{\mathbf{0}} + \{p\alpha\beta^* + (1-p)\gamma\delta^*\}\mathbf{0}\overline{\mathbf{1}} \\ &\quad + \{p\beta\alpha^* + (1-p)\delta\gamma^*\}\mathbf{1}\overline{\mathbf{0}} + \{p|\beta|^2 + (1-p)|\delta|^2\}\mathbf{1}\overline{\mathbf{1}}. \end{aligned} \tag{4.33}$$

The observable to use is $\widehat{\boldsymbol{P}} \equiv \mathbf{1}\overline{\mathbf{1}}$, the projection operator associated with the signal state $\mathbf{1}$. Then we find

$$\widehat{\boldsymbol{P}}\boldsymbol{\varrho} = \{p\beta\alpha^* + (1-p)\delta\gamma^*\}\mathbf{1}\overline{\mathbf{0}} + \{p|\beta|^2 + (1-p)|\delta|^2\}\mathbf{1}\overline{\mathbf{1}}. \tag{4.34}$$

Taking the trace then gives

$$Tr\{\widehat{\boldsymbol{P}}\boldsymbol{\varrho}\} = \{p|\beta|^2 + (1-p)|\delta|^2\}, \tag{4.35}$$

which agrees with the probability found above in Example 4.2.

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