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UNSTABLE NEUTRAL FUCTIONAL DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. Let y be the solution of the equation

$$y'(z) = Ay(z) + By(\lambda z) + Cy'(\eta z),\ z \in C,$$

where A, B, C, λ and η are complex numbers and $0 \leq |\lambda| < 1, 0 \leq |\eta| < 1$. It is shown that y has exponential order equal to one if $A \neq 0$ and if y is not a polynomial; otherwise, y has exponential order equal to zero. In the latter case, y and all of its derivatives are unbounded on any ray.

1. **Introduction.** This note investigates the exponential order of the solutions to the neutral functional differential equation

$$y'(z) = Ay(z) + By(\lambda z) + Cy'(\eta z),\ z \in C, \tag{1}$$

where A, B, C, λ and η are complex parameters and $0 \leq |\lambda| < 1, 0 \leq |\eta| < 1$. Special cases of this equation (that is, $C = 0$) have been considered by Feldstein and Grafton [4], by Kato and McLeod [7], by Fox, Mayers, Ockendon and Tayler [5], and by Morris, Feldstein and Bowen [8], as well as second order variations by Waltman [10] and by Bélair [1].

The existence of stable or unstable solutions to Eqn. (1), while of considerable interest in its own right, is of particular importance in numerical analysis because of its applicability to the development of stiffly stable numerical methods for neutral equations. See, for example, Dahlquist [3], Gear [6], and Bellen, Jackiewicz and Zennaro [2].

2. **Representation of solutions.** We seek solutions of Eqn. (1) in the form of the power series

$$y(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{2}$$

There are several cases to consider, depending upon whether or not $1 - C\eta^k = 0$ for some non-negative integer k. Theorem 1 gives $y(z)$ for the case where there is no such k. Theorem 2 covers the alternative and is divided into several special sub-cases. The results are obtained by substituting Eqn. (2) into Eqn. (1) and then equating coefficients of successive powers of z. The ratio test readily shows that the resulting powers series are absolutely convergent on the whole complex plane. The details of all those proofs are omitted. Throughout this paper, empty products are equal to one.

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THEOREM 1. *Assume that* $1 - C\eta^k \neq 0$ *for any* $k = 0, 1, \ldots$. *Then Eqn. (1) has an entire solution given by*

$$y(z) = a_0 \sum_{n=0}^{\infty} \left(\prod_{j=0}^{n-1} \frac{A + B\lambda^j}{1 - C\eta^j} \right) \frac{z^n}{n!}, \ z \in C,$$

where a_0 *is an arbitrary constant. Every non-trivial solution reduces to a polynomial if and only if there exists an integer* $l \geq 0$ *such that* $A + B\lambda^l = 0$.

THEOREM 2. *Let* $\eta \neq 0$. *Assume there exists some fixed integer* $k \geq 0$ *such that* $1 - C\eta^k = 0$.

(I) Suppose A *and* $B\lambda$ *do not both vanish.*

a) If $A + B\lambda^m \neq 0$ *for all* $m = 0, 1, \ldots$, *then Eqn. (1) has an entire solution given by*

$$y(z) = (k+1)!\ a_{k+1} \sum_{n=k+1}^{\infty} \left(\prod_{j=k+1}^{n-1} \frac{A + B\lambda^j}{1 - C\eta^j} \right) \frac{z^n}{n!}, \ z \in C,$$

where a_{k+1} *is an arbitrary constant.*

b) Assume that there exists some fixed integer l, *where* $0 \leq l \leq k$, *such that* $A + B\lambda^l = 0$. *Then the solution of Eqn. (1) is given by*

$$y(z) = l!\ a_l \sum_{n=0}^{l} \left(\prod_{j=n}^{l-1} \frac{1 - C\eta^j}{A + B\lambda^j} \right) \frac{z^n}{n!} + (k+1)!\ a_{k+1} \sum_{n=k+1}^{\infty} \left(\prod_{j=k+1}^{n-1} \frac{A + B\lambda^j}{1 - C\eta^j} \right) \frac{z^n}{n!}, \ z \in C,$$

where a_l *and* a_{k+1} *are arbitrary constants.*

c) Assume that there exists some fixed integer l, *where* $l \geq k+1$, *such that* $A + B\lambda^l = 0$. *Then Eqn. (1) has a polynomial solution given by*

$$y(z) = (k+1)!\ a_{k+1} \sum_{n=k+1}^{l} \left(\prod_{j=k+1}^{n-1} \frac{A + B\lambda^j}{1 - C\eta^j} \right) \frac{z^n}{n!}, \ z \in C,$$

where a_{k+1} *is an arbitrary constant.*

(II) Suppose that $A = B\lambda = 0$.

a) If $B \neq 0$ *and* $k \neq 0$ *(that is,* $C \neq 1$*), then* $\lambda = 0$ *and*

$$y(z) = a_0 \left(1 + \frac{B}{1 - C} z \right) + a_{k+1} z^{k+1}.$$

b) If $B \neq 0$ *and* $k = 0$ *(that is,* $C = 1$*), then* $\lambda = 0$ *and*

$$y(z) = a_1 z.$$

c) If $B = 0$, *then*

$$y(z) = a_0 + a_{k+1} z^{k+1}.$$

Here, a_0, a_1, *and* a_{k+1} *are arbitrary constants, and* $z \in C$.

Theorem 2 gives conditions which ensure that Eqn. (1) has polynomial solutions. For the equal delay case, the following corollary codifies and summarizes such results in terms of a necesssary and suffficient condition on the parameters A, B, C, and λ.

COROLLARY 1. *Let* $\lambda = \eta$. *Suppose that there are non-negative integers k and l such that* $0 = 1 - C\lambda^k = A + B\lambda^l$ *(which implies that* $|C| \geq 1$*). Then Eqn. (1) has a polynomial solution that is not identically zero if and only if*

$$|AC| \leq |B\lambda|.$$

PROOF. Since $0 = 1 - C\lambda^k = A + B\lambda^l$ for some non-negative integers k and l, it follows from Theorem 2 that Eqn. (1) has nontrivial polynomial solutions if and only if $l \geq k + 1$ or $A = B\lambda = 0$. If $A = B\lambda = 0$, then clearly $|AC| = |B\lambda| = 0$, and the conclusion follows. If $B\lambda \neq 0$, then $A + B\lambda^l = 0$ implies that

$$l - 1 = \frac{\ln|A/B\lambda|}{\ln|\lambda|}.$$

Similarly, $1 - C\lambda^k = 0$ implies that

$$k = \frac{\ln|1/C|}{\ln|\lambda|}.$$

It follows from these two expressions for k and $l - 1$ that the condition $l \geq k + 1$ is equivalent to

$$\ln|A/B\lambda| \leq \ln|1/C|.$$

The desired conclusion follows from this inequality.

It is interesting to contrast Corollary 1 with some results from Kato and McLeod [7], where they proved (in the notation of this paper) that if $C = 0$, then for real t,

$$|A| < |B| \text{ implies } |y(t)| \longrightarrow \infty \text{ as } t \longrightarrow \infty,$$
$$|A| > |B| \text{ implies } y(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

3. **Exponential order of solutions.** To investigate the exponential order of solutions to Eqn. (1), recall that the order ρ of an entire function $f(z)$ is defined for $z \in C$ by

$$\rho = \inf\{w : f(z) = 0(\exp(|z|^w)), |z| \longrightarrow \infty\};$$

see Titchmarsh [9], p. 248. Then

THEOREM 3. *A solution* $y(z)$ *of Eqn. (1), and all of its derivatives, have finite order* ρ *equal to*

a) zero, if $A = 0$ *or if* $y(z)$ *is polynomial.*

b) one, if $A \neq 0$ *and if* $y(z)$ *is not polynomial.*

PROOF. Assume first that $1 - C\eta^m \neq 0$ for any $m = 0, 1, \ldots$ and that $A + B\lambda^l \neq 0$ for any $l = 0, 1, \ldots$. Apply Theorem 1. Then the *kth* derivative of a solution $y(z)$ of Eqn. (1) is given by

$$y^{(k)}(z) = a_0 \sum_{n=0}^{\infty} \left(\prod_{j=0}^{n+k-1} \frac{A + B\lambda^j}{1 - C\eta^j} \right) \frac{z^n}{n!}, \; z \in C, \; k = 0, 1, 2, \ldots.$$

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It is known (see Titchmarsh [9], p. 253) that an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a finite order ρ if and only if

$$\liminf_{m\to\infty} \frac{\ln(1/|a_m|)}{m \ln m} = \frac{1}{\rho},$$

where ln denotes the natural logarithm. Applying this criterion to the function $y^{(k)}(z)$ yields $\frac{1}{\rho} = L_1 + L_2 - L_3$, where

$$\begin{aligned} L_1 &= \liminf_{m\to\infty} \frac{\ln(m!)}{m \ln m} = 1, \\ L_2 &= \liminf_{m\to\infty} \frac{\sum_{j=0}^{m+k-1} \ln|1 - C\eta^j|}{m \ln m}, \\ L_3 &= \liminf_{m\to\infty} \frac{\sum_{j=0}^{m+k-1} \ln|A + B\lambda^j|}{m \ln m}. \end{aligned}$$

Consider L_2. Given $0 < \varepsilon < 1$. Since $|\eta| < 1$, there exists an integer μ such that

$$|C\eta^j| < \varepsilon, \quad \text{for } j \geq \mu,$$

and

$$1 - \varepsilon \leq 1 - |C\eta^j| \leq |1 - C\eta^j| \leq 1 + |C|, \quad \text{for } j \geq \mu.$$

Hence,

$$\frac{(m + k - \mu)\ln(1 - \varepsilon)}{m \ln m} \leq \frac{\sum_{j=\mu}^{m+k-1} \ln|1 - C\eta^j|}{m \ln m} \leq \frac{(m + k - \mu)\ln(1 + |C|)}{m \ln m},$$

and if follows that

$$L_2 = \liminf_{m\to\infty} \frac{\sum_{j=0}^{\mu-1} \ln|1 - C\eta^j| + \sum_{j=\mu}^{m+k-1} \ln|1 - C\eta^j|}{m \ln m} = 0.$$

To compute L_3 one must distinguish two cases: $A = 0$ and $A \neq 0$. If $A = 0$, then $B\lambda \neq 0$ and (recall that $|\lambda| < 1$)

$$\begin{aligned} L_3 &= \liminf_{m\to\infty} \frac{\sum\limits_{j=0}^{m+k-1} (\ln|B| + j \ln|\lambda|)}{m \ln m} \\ &= \liminf_{m\to\infty} \frac{(m + k)\ln|B| + (m + k)(m + k - 1)(\ln|\lambda|)/2}{m \ln m} \\ &= -\infty. \end{aligned}$$

Consider L_3 when $A \neq 0$. Given $0 < \varepsilon < |A|$. Since $|\lambda| < 1$, there exists an integer μ such that

$$|B\lambda^j| \leq \varepsilon, \quad \text{for } j \geq \mu,$$

and

$$|A| - \varepsilon \le |A| - |B\lambda^j| \le |A + B\lambda^j| \le |A| + |B|, \quad \text{for } j \ge \mu.$$

Hence,

$$\frac{(m+k-\mu)\ln(|A| - \varepsilon)}{m \ln m} \le \frac{\sum_{j=\mu}^{m+k-1} \ln|A + B\lambda^j|}{m \ln m} \le \frac{(m+k-\mu)\ln(|A| + |B|)}{m \ln m},$$

and it follows for $A \neq 0$ that

$$L_3 = \liminf_{m \to \infty} \frac{\sum_{j=0}^{\mu-1} \ln|A + B\lambda^j| + \sum_{j=\mu}^{m+k-1} \ln|A + B\lambda^j|}{m \ln m} = 0.$$

A summary of the above discussion is the following: If $1 - C\eta^m \neq 0$ for any $m = 0, 1, \ldots$, and if $A + B\lambda^l \neq 0$ for any $l = 0, 1, \ldots$, then the order ρ of the kth derivative $y^{(k)}(z)$, for $k = 0, 1, \ldots$, is

$$\rho = \frac{1}{L_1 + L_2 - L_3} = \begin{cases} 0, & \text{if } A = 0, \\ 1, & \text{if } A \neq 0. \end{cases}$$

On the other hand, if $1 - C\eta^m \neq 0$ for any $m = 0, 1, \ldots$, yet $A + B\lambda^l = 0$ for some integer $l \ge 0$, then by Theorem 1 $y(z)$ is a polynomial and clearly has order $\rho = 0$.

Finally, if $1 - C\eta^m = 0$ for any $m \ge 0$, then Theorem 2 may be applied, and, with slight modifications in the proof, it yields the same conclusions about the order ρ.

Applying the Phragmén-Lindelöf principle to a sector of opening 2π (see Titchmarsh [9] p. 273), yields the corollary below, which is a generalization of Theorem 5 in Morris, Feldstein and Bowen [8]. However, that paper also contains results about the oscillation of unbounded solutions when $A = C = 0$ (pure delay equations). The situation concerning the possible oscillation of unbounded solutions when $C \neq 0$ (neutral equations) is much more complicated and cannot be addressed here.

COROLLARY 2. *If $B \neq 0$, then every solution of the equation*

$$y'(z) = By(\lambda z) + Cy'(\eta z), \quad z \in C,$$

$0 < |\lambda| < 1, 0 < |\eta| < 1$, *and all of its derivatives, are unbounded on any ray.*

Various numerical experiments suggest the following conjecture.

CONJECTURE. Suppose that the parameters A, B, C, λ, and η are all real. Suppose further that $A = 0$ and $-1 < C < 1$.

a) If $B < 0$, then every nontrivial solution to Eqn. (1) oscillates (unboundedly).

b) If $B > 0$, then every nontrivaial solution ot Eqn. (1) is monotonic.

REFERENCES

1. J. Bélair, *Sur une équation différentielle fonctionnelle analytique*, Canad. Math. Bull. **24**(1981), 43–46.

2. A. Bellen, Z. Jackiewicz, and M. Zennaro, *Stability analysis of one-step methods for neutral delay-differential equations*, Numer. Math.**52** (1988), 605–619.

3. G. Dahlquist, *A special stability problem for linear multistep methods*, BIT **3** (1963), 27–43.

4. A. Feldstein, and C. K. Grafton, *Experimental mathematics: an application to retarded ordinary differential equations with infinite lag*, Proc. ACM National Conf., Brandon Systems Press (1968) 67–71.

5. L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler, *On a functional differential equation*, J. Inst. Maths. Applics. **8** (1971), 271–307.

6. C. W. Gear, *The automatic integration of stiff ordinary differential equations*, In: Information Processing (1969) **68** (A.J.H. Morrel, Ed.), North Holland, Amsterdam, 187–193.

7. T. Kato, and J. B. McLeod, *The functional-differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. **77** (1971), 891–937.

8. G. R. Morris, A. Feldstein, and E. W. Bowen, *The Phragmén-Lindelöf Principle and a class of functional-differential equations*, In: Ordinary differential equations (L. Weiss, Ed.), Academic Press, New York, (1972) 513–540.

9. E. C. Titchmarsh, *The theory of functions*, Clarendon Press, Oxford (1932).

10. P. Waltman, *A note on an oscillation criterion for an equation with a retarded argument*, Canad. Math. Bull. **24** (1968), 593–595.

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