Ergod. Th. & Dynam. Sys. (1986), **6**, 561–569
Printed in Great Britain

Multiple attractors in Newton's method

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(*Received* 29 *May* 1985)

Abstract. For each $d \geq 2$ there exists a polynomial p with real coefficients such that the associated Newton function $z - [p(z)/p'(z)]$ has $2d-2$ distinct attracting periodic orbits in the complex plane. According to a theorem of G. Julia, this is the maximal number of attracting orbits that any rational function of degree d can possess.

1. *Introduction*

Perhaps the most common example of iteration of a rational function defined on the complex plane is the use of Newton's method to find the roots of a polynomial. It is widely known that Newton's method may fail to produce a root under certain circumstances. Among the causes of non-convergence, the most robust occurs when the initial point lies in the basin of attraction of an attracting periodic orbit of the Newton function, for this situation is stable under small perturbations of both the initial point and the polynomial. A fundamental problem is to determine if the Newton function of a given polynomial has any such attractors, and if so, how many. If the degree of the polynomial is d, then a classical theorem of G. Julia gives an upper bound of $2d-2$ on the number of attracting periodic orbits (including attracting fixed points) of the Newton function. The main result of this paper is that this upper bound is precise.

THEOREM. *If $d \geq 2$, then there is a polynomial of degree d with the property that the Newton function of this polynomial has $2d-2$ distinct periodic attractors in the complex plane. The polynomial can be chosen to have only real coefficients.*

The theorem is proved by making a series of perturbations to a polynomial that has a repeated critical point of high multiplicity. Each perturbation reduces the multiplicity of the degenerate critical point by one and at the same time increases the number of attracting periodic orbits of the associated Newton function by one. The multiplicity of the critical points being dealt with is important; because of it, small changes in the coefficients of the polynomial lead to (locally) large changes in the Newton function. This, combined with the local stability of attracting periodic orbits, enables one to introduce a new attracting orbit without destroying any of the periodic attractors that had been created at previous stages in the construction.

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2. *Basic facts about Newton functions*

If $p(z)$ is a polynomial over the complex numbers, then the Newton function of p, $Np(z)$, is defined by

$$Np(z) = z - [p(z)/p'(z)] \qquad (p'(z) \neq 0).$$

The derivative of Np is $pp''/[p']^2$. Thus, if the degree of p is d, then Np is a rational function with $2d-2$ critical points. A periodic orbit

$$\gamma = \{z_0, Np(z_0), (Np)^2(z_0), \ldots, (Np)^k(z_0) = z_0\}$$

is *attracting* if

$$\left| \frac{d}{dz}(Np)^k(z_0) \right| < 1.$$

For example, every root of p is an attracting fixed point of Np. If γ is an attracting periodic orbit, its *basin of attraction* is an open set and is defined as

$$\{w \,|\, \text{dist}\,((Np)^j(w), \gamma) \to 0 \text{ as } j \to \infty\}.$$

Basic to the study of attracting orbits in Newton's method for polynomials is the following result of G. Julia [**J**], [**Bla**].

JULIA'S THEOREM. *If F is a rational mapping of the complex plane, of degree at least 2, then the basin of attraction of any periodic attractor of F contains a critical point of F. Consequently the number of attracting periodic orbits is no more than the number of critical points.*

In the case of Newton's method for a polynomial of degree d, this means that there can be at most $2d-2$ attracting periodic orbits.

In what follows, the polynomials under consideration will have only real coefficients, so the real line will be an invariant set for the corresponding Newton functions. In this situation, define a *band* for Np to be a connected component of $\mathbb{R} - \{\text{critical points of } p\}$. When p is a polynomial, then Np will have unbounded bands. Certain features of the dynamics of Np on an unbounded band will be relevant to the following discussion. We summarize these features in the following simple lemma, whose proof is an easy calculation. See [**HM**], [**H**] for more details.

LEMMA 1. *Suppose p is a real polynomial of degree d.*

(a) $(Np)'(x) \to (d-1)/d$ *as* $x \to \infty$.

(b) *Suppose that B is an unbounded band for Np whose closure contains no roots of p. Then* $Np(B) = \mathbb{R}$. *If* $B \neq \mathbb{R}$ *then under iteration by Np any point of B will eventually leave B.*

More complete descriptions of the dynamics of Newton functions on the real line can be found in a number of recent papers, including [**SaU**], [**CoM**], [**W**], [**HM**], [**H**], as well as in the original papers of Barna [**B1**]-[**B4**]. A numerical study of Newton functions on the complex plane is contained in [**CuGS**].

3. *Statement of results*

THEOREM. *Given any positive integer d,* $d \geq 2$, *there is a polynomial p with real coefficients and degree d such that Np has* $2d-2$ *distinct periodic attractors in the complex plane.*

In giving a proof of the theorem we will consider the set of polynomials g satisfying the following properties:

(3.1) All of the coefficients of g are real.

(3.2) g has no repeated roots.

(3.3) All of the roots of g' are real and lie in $[0, \infty)$; 0 is a repeated root of g', but none of the other roots of g' is repeated.

Let $\mathcal{S}$ denote the set of all polynomials satisfying (3.1)-(3.3). The theorem will follow from the following two propositions. In each of them assume that p is a polynomial of the form

(3.4) $p(z) = a_0 + \sum_{k+1}^{d} a_j z^j$, with $a_0 > 0$, $a_d \neq 0$, and $k \geq 2$.

PROPOSITION 1. *Suppose p satisfies* (3.1)-(3.4) *and that*

(1a) *k is odd*;

(1b) $a_{k+1} > 0$;

(1c) $p(x) > 0$ *for all* $x \geq 0$;

(1d) *Np is monotonic on each of its unbounded bands.*

Let $\delta > 0$ be given. Then there is a constant $a_k \in (-\delta, 0)$ such that if $f(z) = p(z) + a_k z^k$, then Nf has one more attracting periodic orbit than Np, and f has the following properties:

(1(i)) $f \in \mathcal{S}$;

(1(ii)) $f(x) > 0$ *for all* $x \geq 0$;

(1(iii)) *f has the same number of real roots as p*;

(1(iv)) *Nf has two unbounded bands, and is monotonic on each.*

PROPOSITION 2. *Suppose p satisfies* (3.1)-(3.4) *and that*

(2a) *k is even*;

(2b) $a_{k+1} < 0$;

(2c) $p(x) > 0$ *for all* $x \leq 0$;

(2d) *Np is monotonic on each of its unbounded bands.*

Let $\delta > 0$ be given. Then there is a constant $a_k \in (0, \delta)$ such that if $f(z) = p(z) + a_k z^k$, then Nf has one more attracting periodic orbit than Np, and f has the following properties:

(2(i)) $f \in \mathcal{S}$;

(2(ii)) $f(x) > 0$ *for all* $x \leq 0$;

(2(iii)) *f has the same number of real roots as p*;

(2(iv)) *Nf has two unbounded bands, and is monotonic on each.*

Proof of the theorem in the case d is even. Use the two propositions inductively; in order to satisfy both of the assumptions (1c) and (2c) it is necessary that the degree of p be even. The case $d = 2$ is trivial, so assume that d is at least 4. Define $p_0(z) = z^d + 1$ and note that Np_0 has d distinct attracting fixed points, none of which is real. Let $p_1(z) = f(z)$ where f is obtained by using proposition 1 with $p = p_0$. Np_1 has $d+1$ distinct periodic attractors and no real roots. Now let $p_2 = f$ where f is obtained by using proposition 2 with $p = p_1$, so Np_2 has $d+2$ distinct periodic attractors and no real roots. Repeating this process, we eventually obtain p_{d-2} such that Np_{d-2} has $2d-2$ distinct periodic attractors. □

Remark. The polynomial constructed in this proof satisfies the properties (3.1)-(3.3) and has no real roots.

4. *Proof of propositions* 1 *and* 2

The proof of the following lemma is straightforward.

LEMMA 2. *Suppose $p(z)$ is a polynomial and that L is a compact subset of the complex plane that is disjoint from the set of zeros of p'. Let $f_\alpha(z)=p(z)+\alpha z^k$. Then Nf_α converges C^1 uniformly on L to Np as α tends to* 0.

Proof of proposition 1. For p as in the statement of the proposition let $f_\alpha(z)=p(z)+\alpha z^k$, α in $\mathbb{R}$. For α close to 0 the roots of f_α, f'_α, and f''_α will be close to those of p, p', and p'' respectively, so (1(ii)-(iv)) will hold. By lemma 2 and the persistence of periodic attractors under local C^1 perturbations, for α sufficiently close to 0, Nf_α will have an attracting periodic orbit corresponding to each periodic attractor of Np, and of the same period. Additionally, if α is not only close to zero but is also negative, then the assumption that p is in $\mathcal{S}$ will ensure that f_α is in $\mathcal{S}$ as well: that (3.1) and (3.2) will be satisfied is obvious; to see that (3.3) is also satisfied we argue as follows. Note that

$$f'_\alpha(z)=z^{k-1}\left(k\alpha+\sum_1^{d-k} b_j z^j\right) \qquad \text{where } b_j=(j+k)(a_{j+k}),$$

so 0 is a root of f'_α of multiplicity $k-1$. Each of the remaining $d-k$ roots of f'_α has multiplicity 1; as α increases to 0 one of these roots converges to 0 and the others converge to the non-zero roots of p'. The assumptions that $\alpha<0$ and $b_1>0$ ensure that for α close enough to 0 all of the roots of f'_α are real and non-negative. Thus for such α f_α will belong to $\mathcal{S}$.

All that remains to show is that α can be chosen so that Nf_α has an attracting periodic orbit not corresponding to any of the periodic attractors of Np. Let $\delta_0>0$ be less than δ and small enough that the preceding arguments hold. For the remainder of the proof consider only values of α satisfying $-\delta_0<\alpha<0$. Let x_α denote the smallest positive root of f'_α. The argument in the preceding paragraph shows that x_α tends to 0 as α increases to 0. The restriction of Nf_α to $\mathbb{R}$ has vertical asymptotes at 0 and at x_α. Between these asymptotes f_α is positive and f'_α is negative, so that $Nf_\alpha(x)>x$. Consequently Nf_α is bounded below on $(0, x_\alpha)$ and has a turning point inside this interval. In fact, it is easy to check that

$$\inf\{Nf_\alpha(x)\,|\,0<x<x_\alpha\}\to\infty \tag{4.1}$$

as α tends to 0 from below. Let y_α denote the smallest positive critical point of Nf_α. By (1(ii)) and the fact that $(Nf_\alpha)'=f_\alpha f''_\alpha/(f'_\alpha)^2$ it follows that y_α is the least positive root of f''_α. Since f''_α is given by

$$z^{k-2}\left(\alpha k(k-1)+\sum_1^{d-k} c_j z^j\right) \qquad \text{with } c_j=(j+k)(j+k-1)a_{j+k}$$

and $c_1>0$, the map $\alpha\to y_\alpha$ is smooth for $-\delta_1\le\alpha<0$ for some δ_1 chosen to satisfy $0<\delta_1<\delta_0$, and $y_\alpha\to 0$ as α increases to 0.

Claim. There are values of α in $(-\delta_1, 0)$ for which y_α lies on an attracting periodic orbit of Nf_α.

Since $(Nf_\alpha)'(y_\alpha)=0$, a periodic orbit containing y_α is necessarily attracting.

Proof of claim. Let q_α denote the largest real root of f'_α (so $q_\alpha \geq x_\alpha > y_\alpha > 0$). The graph of Nf_α for x near 0 is depicted in figure 1. By (1(iv)) and lemma 1, Nf_α is

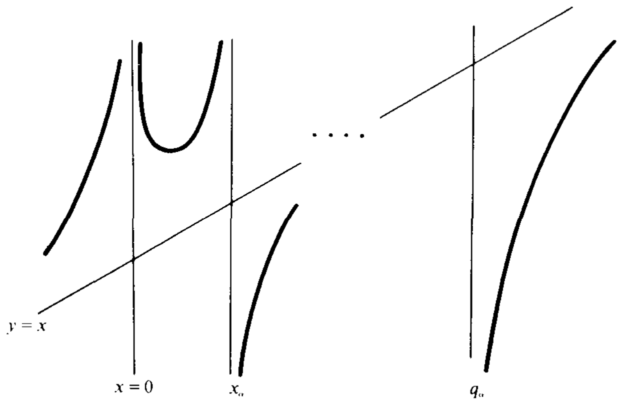

FIGURE 1

monotonic on (q_α, ∞) and maps (q_α, ∞) onto the real line. Thus there is a unique inverse image of y_α under Nf_α that lies in (q_α, ∞). We will show that for certain values of α this pre-image of y_α is also a forward iterate of y_α. Note that as α approaches 0 from below the number of consecutive iterates of $Nf_\alpha(y_\alpha)$ that are greater than q_α tends to infinity. This holds because $Nf_\alpha(y_\alpha)\to\infty$ as α increases to 0 (by (4.1)) and $|Nf_\alpha(x)-(d-1)x/d|$ has a constant limit as $x\to\infty$ (and these limits are uniform for α in a compact set). Choose α_0 in $(-\delta_1, 0)$ and $m>1$ so that for $\alpha=\alpha_0$, $Nf_\alpha(y_\alpha)>q_\alpha$ and $(Nf_\alpha)^m(y_\alpha)<q_\alpha$. Define

$$\alpha(m)=\inf\{\beta<0 \mid (Nf_\alpha)^j(y_\alpha)>q_\alpha \text{ for all } 1\leq j\leq m \text{ and } \beta<\alpha<0\}.$$

For some $j(m)\leq m$ we have $Nf^{j(m)}_{\alpha(m)}(y_{\alpha(m)})=q_{\alpha(m)}$ and $Nf^{j(m)}_{\alpha}(y_\alpha)>q_\alpha$ for $\alpha(m)<\alpha<0$. Let $T=j(m)+1$. The construction ensures that the map $G(\alpha)=(Nf_\alpha)^T(y_\alpha)$ is continuous on $(\alpha(m), 0)$, and

$$G(\alpha)\to-\infty \quad \text{as } \alpha \text{ decreases to } \alpha(m)$$

$$G(\alpha)\to\infty \quad \text{as } \alpha \text{ increases to } 0.$$

Since the curve $\alpha\to y_\alpha$ is continuous and bounded on $(-\delta_1, 0)$, this curve and the graph of $G(\alpha)$ must intersect (see figure 2). At a point of intersection we have $(Nf_\alpha)^T(y_\alpha)=y_\alpha$. □

The proof of proposition 2 is analogous to that of proposition 1. In it there is a critical point of Nf_α that is positive, near 0, and which is mapped toward $-\infty$ as $\alpha\to 0$. The graph of Nf_α in this situation is shown in figure 3.

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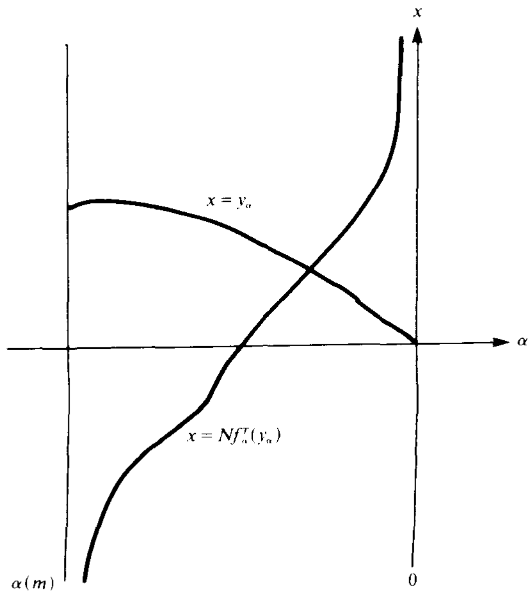

FIGURE 2

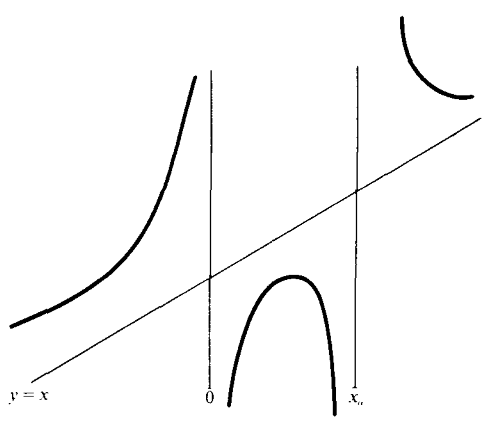

FIGURE 3

5. *Proof of the theorem when d is odd*

Suppose that d is odd and at least 3. Use the construction of § 4 to obtain a real polynomial p satisfying

(5.1) degree $(p) = d-1$.

(5.2) p satisfies (3.1)-(3.3).

(5.3) $p(x) > 0$ for all x in $\mathbb{R}$.

(5.4) Np has $2d-4$ distinct attracting periodic orbits in the complex plane.

Consider polynomials of the form

$$g_\alpha(z) = p(z) + \alpha z^d, \qquad \alpha > 0. \tag{5.5}$$

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Each g_α has a single real root, which is negative, and this root tends to $-\infty$ as α decreases to 0. By lemma 2, if $\alpha > 0$ is small enough, Ng_α will have $2d-3$ distinct attracting periodic orbits: orbits corresponding to each of the periodic attractors of Np as well as an attracting fixed point at the real root of g_α. To see that there are small positive values of α for which Ng_α has $2d-2$ distinct attractors, consider the graph of Ng_α (see figure 4).

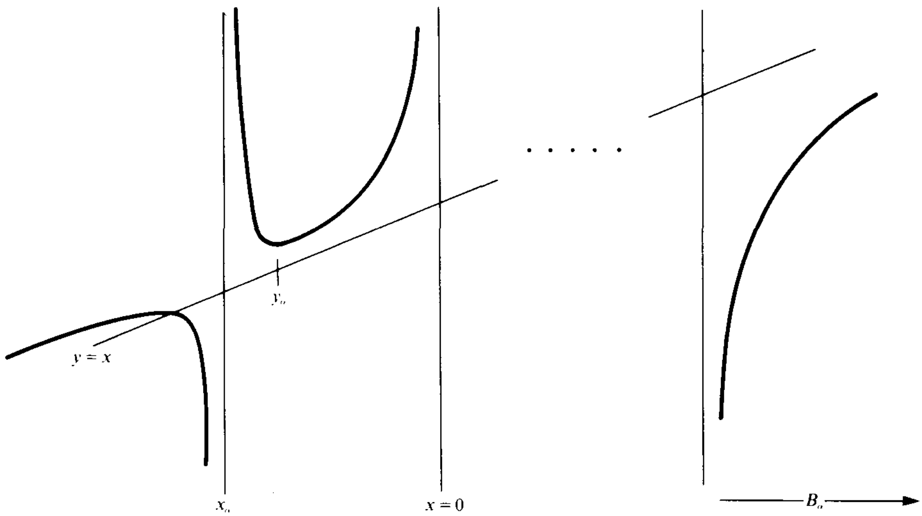

FIGURE 4

By (3.3) p' has no roots on the negative real axis, so using (5.5) it is an easy argument to show that $g'_\alpha(0) = 0$ and that g'_α has exactly one negative root at a point that we label x_α. Thus the negative real axis contains two bands for Ng_α, an unbounded one $(-\infty, x_\alpha)$ that contains the negative root of g_α, and a bounded one $(x_\alpha, 0)$ with the property that $Ng_\alpha(x) > x$ whenever $x_\alpha < x < 0$. Ng_α is bounded below on $(x_\alpha, 0)$ and has a minimum on this band, which occurs at a point y_α that is a root of g''_α. (For α small the other roots of g''_α are near those of p'', so a counting argument shows that y_α is the only turning point of Ng_α in the band $(x_\alpha, 0)$.) Note that both x_α and y_α tend to $-\infty$ as α decreases to 0.

Because $g'_\alpha(0) = 0$ there is an unbounded band for Ng_α that lies in the positive half-line; call this band B_α. Since the only real root of g_α is negative, lemma 1 shows that Ng_α maps B_α onto $\mathbb{R}$. We will show that for certain small values of α the orbit of y_α is periodic and is contained in the union of B_α and $(x_\alpha, 0)$.

Define

$$K(\alpha) = \min\{j \mid (Ng_\alpha)^{j+1}(y_\alpha) > 0\}.$$

Claim. $K(\alpha)$ goes to infinity as α decreases to 0.

This claim follows from the following facts.

(5.6) Ng_α has its minimum on $(x_\alpha, 0)$ at y_α.

(5.7) Ng_α converges uniformly to Np as α decreases to 0 on compact subsets of the negative real axis.

(5.8) $Np(x)$ tends to $-\infty$ as $x \to -\infty$.

(5.9) $|Np(x)-[(d-1)/(d-2)]x|$ has a finite limit as $x \to -\infty$.

Now choose $\delta > 0$ small enough that if $0 < \alpha < \delta$, then Ng_α has the features described above. Pick some α_1 in $(0, \delta)$ and let m_0 be a fixed integer greater than $K(\alpha_1)$. Define

$$\alpha_0 = \inf\{0 < \alpha < \delta \mid K(\alpha) \le m_0\}.$$

For any j satisfying $0 \le j \le m_0 + 1$, the map $\alpha \to (Ng^j_\alpha)(y_\alpha)$ is continuous on $0 < \alpha < \alpha_0$, and there is an integer $m \le m_0$ such that $(Ng_\alpha)^m(y_\alpha)$ tends to 0 from below as α tends to α_0 from below. Consequently, $(Ng_\alpha)^{m+1}(y_\alpha)$ goes to infinity as $\alpha \to \alpha_0$ from below. On the other hand, (5.6)-(5.9) combine to show that $(Ng_\alpha)^{m+1}(y_\alpha)$ goes to $-\infty$ as α tends to 0 from above.

Now consider inverse images of y_α. For each small positive α there is a point r_α in B_α defined by $Ng_\alpha(r_\alpha) = y_\alpha$. It is not hard to check that for $0 < \alpha < \alpha_0$ the function $\alpha \to r_\alpha$ is continuous (and nearly constant if α_0 is small). One concludes that if the two curves $\alpha \to r_\alpha$ and $\alpha \to (Ng_\alpha)^{m+1}(y_\alpha)$ are graphed in the (α, x)-plane, then the two graphs must intersect, as shown in figure 5. If $(\alpha, x) = (A, X)$ is the point of intersection, then

$$(Ng_A)^{m+2}(y_A) = Ng_A((Ng_A)^{m+1}(y_A)) = Ng_A(X) = y_A.$$

Thus Ng_A is a rational function of degree d with $2d-2$ distinct attracting periodic orbits. □

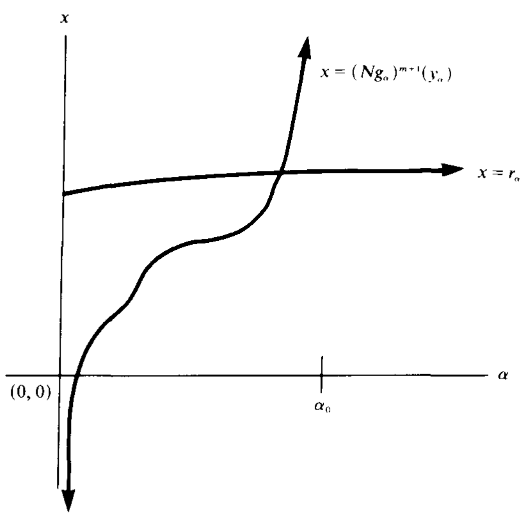

FIGURE 5

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