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BOUNDS FOR MODIFIED BESSEL FUNCTIONS OF THE FIRST AND SECOND KINDS

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Dedicated to my daughter, Boróka

Abstract Some new inequalities for quotients of modified Bessel functions of the first and second kinds are deduced. Moreover, some developments on bounds for modified Bessel functions of the first and second kinds, higher-order monotonicity properties of these functions and applications to a special function that arises in finite elasticity, are summarized. The key tool in our proofs is a frequently used criterion for the monotonicity of the quotient of two Maclaurin series.

Keywords: modified Bessel functions; complete monotonicity; absolute monotonicity; bounds

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1. Introduction and preliminaries

Let us consider the second-order modified Bessel differential equation

$$x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0,$$

which differs from the classical Bessel differential equation only in the coefficient of y. This differential equation frequently occurs in problems of mathematical physics and its linearly independent solutions are called modified Bessel functions of the first and second kinds. In what follows let us, as usual, denote by I_ν and K_ν the modified Bessel functions of the first and second kinds, respectively, of order ν. It is well known that the modified Bessel function of the first kind I_ν can be represented as the infinite series [**76**, p. 77]

$$I_\nu(x) = \sum_{n\geqslant 0} \frac{(\frac{1}{2}x)^{2n+\nu}}{n!\Gamma(\nu+n+1)} = \frac{(\frac{1}{2}x)^{\nu}}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1, \tfrac{1}{4}x^2), \tag{1.1}$$

where ${}_0F_1$ is the hypergeometric series (function), defined by

$${}_0F_1(a, x) = \sum_{n\geqslant 0} \frac{1}{(a)_n}\frac{x^n}{n!},$$

and $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for all $n \in \{1, 2, \dots\}$ is the well-known Pochhammer (or Appell) symbol, defined in terms of Euler's gamma function. Now, let us consider the modified Bessel function of the second kind (which is sometimes called the MacDonald or Hankel function) K_ν, defined by [**76**, p. 78]

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},$$

where the right-hand side of this equation is replaced by its limiting value if ν is an integer or zero. In the last few decades many inequalities and monotonicity properties for the functions I_ν and K_ν and their several combinations have been deduced by many authors, motivated by various problems that arise in wave mechanics, fluid mechanics, electrical engineering, quantum billiards, biophysics, mathematical physics, finite elasticity, probability and statistics, special relativity, etc. For example, an inequality for the ratio of K_ν/I_ν was used in 1962 by Rosenthal [**68**], and later in 1970 by Ross [**69**], in order to determine the stability of fluid motion. Recently, the monotonicity of the function $x \mapsto I_\nu(x)K_\nu(x)$ has been studied by Penfold *et al.* [**62**], motivated by a problem that arises in biophysics. There is also an extensive literature dealing with lower and upper bounds for the ratios $I_\nu(x)/I_\nu(y)$ and $K_\nu(x)/K_\nu(y)$, and related monotonicity problems. The interested reader is referred to [**4**, **17**, **21**, **26**, **41**, **48**, **49**, **51**, **58**, **59**, **61**, **70**, **73**, **74**] and the references therein. Although the inequalities involving the above quotients are interesting in their own right, recently the lower and upper bounds for the ratio $I_\nu(x)/I_\nu(y)$ received increasing attention, since they play an important role in the problem of bounding the generalized Marcum Q-function, which frequently arises in radar signal processing. Motivated by these applications, new lower and upper bounds have recently been deduced for the above ratio by Baricz and Sun [**17**, **21**] and the results have been applied to obtain tight bounds for the generalized Marcum Q-function. In this paper we continue the study from [**17**, **21**]. However, our aim is twofold: to present a brief summary on the known bounds of $I_\nu(x)/I_\nu(y)$ and $K_\nu(x)/K_\nu(y)$ and to compare them with the new results by applying these results to a special function that arises in finite elasticity. The detailed content is as follows: in § 2 we review the known results on the bounds for the ratio $I_\nu(x)/I_\nu(y)$ and we discuss in detail the related problems concerning higher-order monotonicity properties of several functions involving modified Bessel functions of the first kind. Furthermore, in § 2 we improve some known results from [**48**, **49**, **51**] by using a result of Biernacki and Krzyż [**23**], providing a criterion for the monotonicity of the quotient of two Maclaurin series. For the sake of completeness, in § 3 we offer a brief summary on the results concerning the bounds for the ratio $K_\nu(x)/K_\nu(y)$. In § 4 we use the results from § 2 to improve the bounds stated in [**51**] for a function that involves the modified Bessel function of the first kind. Moreover, in § 4 we discover a new Turán-type inequality involving modified Bessel functions of the first kind. It is worthwhile noting that some of the main results of this paper, namely Theorem 2.2, have important applications [**22**] to the problems of bounding the above-mentioned generalized Marcum Q-function. Finally, note that, at the end of §§ 2–4 we present certain open problems, which may be of interest for further research.

We close these preliminaries with the following definitions, which will be used in the paper. By definition, a function $f\colon (0,\infty) \to \mathbb{R}$ is said to be completely monotonic (CM) if f has derivatives of all orders and satisfies $(-1)^n f^{(n)}(x) \geqslant 0$ for all $x > 0$ and $n \in \{0, 1, 2, \dots\}$. Note that strict inequality always holds in the above inequality unless f is constant [**31**]. If for all $x > 0$ and $n \in \{0, 1, 2, \dots\}$ we have $(-1)^n f^{(n)}(x) > 0$, then f is said to be strictly completely monotonic (SCM). For more information on complete monotonicity, the interested reader is referred to [**1**, **2**, **39**, **57**, **79**] and the references therein. Similarly, a function $g\colon (0,\infty) \to \mathbb{R}$ is said to be absolutely monotonic (AM) if g has derivatives of all orders and satisfies $g^{(n)}(x) \geqslant 0$ for all $x > 0$ and $n \in \{0, 1, 2, \dots\}$. If for all $x > 0$ and $n \in \{0, 1, 2, \dots\}$ we have $g^{(n)}(x) > 0$, then g is said to be strictly absolutely monotonic (SAM). For more details on absolute monotonicity we refer the reader to [**79**].

2. Bounds for quotients of modified Bessel functions of the first kind

2.1. Overview of known results and related comments

In 1984, Paris [**61**] proved that the function $x \mapsto x^{-\nu} I_\nu(x)$ is strictly increasing on $(0,\infty)$, while the function $x \mapsto \mathrm{e}^{-x} x^{-\nu} I_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu > -\frac{1}{2}$, i.e. for all $0 < x < y$ and $\nu > -\frac{1}{2}$ we have

$$\mathrm{e}^{x-y}\left(\frac{x}{y}\right)^{\nu} < \frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^{\nu}. \tag{2.1}$$

The right-hand side of (2.1) was rediscovered in 1990 by Robert [**67**] for $\nu \geqslant 0$ and in 1991 by Joshi and Bissu [**48**] for $\nu > -\frac{1}{2}$. We note that, using the power-series expansion (1.1), we immediately have that the function $x \mapsto x^{-\nu} I_\nu(x)$ is in fact strictly increasing on $(0,\infty)$ for all $\nu > -1$, i.e. the right-hand side of (2.1) holds for all $\nu > -1$. Moreover, it is easy to see that the function $x \mapsto x^{-\nu} I_\nu(x)$ is actually SAM on $(0,\infty)$ for each $\nu > -1$.

Now let us consider the left-hand side of (2.1). Searching in the literature, we have found that this basic inequality is not as widely known as it should be, and it has been rediscovered several times. For $\nu > 0$ this result was also deduced independently by Bordelon [**26**] and Ross [**70**] in 1973. More precisely, they proved that for all $\nu > 0$ and $0 < x < y$ the following inequalities hold:

$$\mathrm{e}^{x-y}\left(\frac{x}{y}\right)^{\nu} < \frac{I_\nu(x)}{I_\nu(y)} < \mathrm{e}^{y-x}\left(\frac{x}{y}\right)^{\nu}. \tag{2.2}$$

It is worth mentioning here that the left-hand side of (2.1) (or (2.2)) was proved in 1991 by Joshi and Bissu [**48**] for $\nu > -\frac{1}{2}$ and, by using the same method independently, by Laforgia [**51**] for $\nu \geqslant -\frac{1}{2}$. We note that the method used in [**48**, **51**] is based on the inequality

$$I_\nu(x) > I_{\nu+1}(x), \tag{2.3}$$

which was proved for $\nu > -\frac{1}{2}$ and $x > 0$ by Soni [**74**] in 1965 and extended to the case when $\nu \geqslant -\frac{1}{2}$ by Nåsell [**58**] in 1974. Note that inequality (2.3) was also proved by

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Gupta [**37**] and later by de Sitter and Goovaerts [**30**]; furthermore, it was extended to Whittaker functions by Lorch [**54**]. Results that are stronger than (2.3) for $\nu \geqslant 0$ and $x > 0$ were given by Cochran [**29**] in 1967 and by Jones [**47**] and Reudink [**66**] in 1968. More precisely, Jones proved that $I_\nu(x) < I_\mu(x)$ holds for all $x > 0$ and $\nu > \mu \geqslant 0$, while Cochran and Reudink established the inequality $\partial I_\nu(x)/\partial\nu < 0$ for all $x, \nu > 0$. Moreover, tighter bounds for the ratio $I_{\nu+1}(x)/I_\nu(x)$ can be found: see, for example, [**4**, **59**, **63**].

We note here that if the function $x \mapsto \mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu \geqslant -\frac{1}{2}$, then clearly we have that

$$[\mathrm{e}^{-x}\mathcal{I}_\nu(x)]' = \mathrm{e}^{-x}[\mathcal{I}_\nu'(x) - \mathcal{I}_\nu(x)] \leqslant 0$$

for all $\nu \geqslant -\frac{1}{2}$ and $x > 0$, where for simplicity we have used the familiar notation

$$\mathcal{I}_\nu(x) = {}_0F_1(\nu+1, \tfrac{1}{4}x^2) = 2^\nu \Gamma(\nu+1)x^{-\nu}I_\nu(x) = \sum_{n\geqslant 0}\frac{x^{2n}}{4^n n!(\nu+1)_n}.$$

Now, since from the above series representation for all $\nu \neq -1, -2, \dots$ and $x \in \mathbb{R}$ we have the well-known differentiation formula

$$\mathcal{I}_\nu'(x) = \frac{x}{2(\nu+1)}\mathcal{I}_{\nu+1}(x),$$

it follows that the inequality $I_\nu(x) \geqslant I_{\nu+1}(x)$ holds for all $\nu \geqslant -\frac{1}{2}$ and $x > 0$. However, it is easy to see from (1.1) that in the above inequality we cannot have equality when $x > 0$ and $\nu \geqslant -\frac{1}{2}$. More precisely, if we were have equality, then this would imply that $\mathcal{I}_\nu(x) = \mathrm{e}^x$ for positive x, which would contradict the inequality $\mathcal{I}_\nu(x) < \mathrm{e}^x$, where $x > 0$ and $\nu \geqslant -\frac{1}{2}$. Here we have used once again that $x \mapsto \mathrm{e}^{-x}\mathcal{I}_\nu(x)$ is strictly decreasing on $(0,\infty)$ and $\mathcal{I}_\nu(0) = 1$. The above argument actually yields that the left-hand side of (2.1) is equivalent to the inequality (2.3) of Soni [**74**].

Taking into account the above discussion, a natural question that arises is the following: what happens to the left-hand side of (2.1) when $\nu \in (-1, -\frac{1}{2})$? Using some known upper bounds for the ratio $I_{\nu+1}(x)/I_\nu(x)$, it is easy to see that for small x the function $x \mapsto \mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ is still strictly decreasing when $\nu \in (-1, -\frac{1}{2})$. However, for large values of x, the derivative of $x \mapsto \mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ changes sign.

It is also important to note here that the function $x \mapsto \mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ is in fact SCM on $(0,\infty)$ for all $\nu \geqslant -\frac{1}{2}$. For $\nu > -\frac{1}{2}$ this statement was proved by Nåsell [**59**] in 1978 by using the Schläfli integral representation [**76**, p. 79]

$$I_\nu(x) = \frac{(\frac{1}{2}x)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\mathrm{e}^{-xt}\,\mathrm{d}t,$$

which implies that the function $x \mapsto \mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ is a Laplace transform, and hence by the classical Bernstein–Widder Theorem it is SCM. For $\nu > -\frac{1}{2}$, another approach to prove the above complete monotonicity is to use the integral representation [**28**]

$$\mathcal{I}_\nu(x) = \frac{\Gamma(2\nu+1)}{\Gamma^2(\nu+\frac{1}{2})}\int_0^1 (t-t^2)^{\nu-1/2}\mathrm{e}^{x(2t-1)}\,\mathrm{d}t.$$

This was done recently in [**21**] by Baricz and Sun, and it was pointed out that for $\nu = -\frac{1}{2}$ the function $x \mapsto \mathrm{e}^{-x}\mathcal{I}_\nu(x)$ is also SCM on $(0,\infty)$. From the above discussion it follows that when $\nu \in (-1,-\frac{1}{2})$ the function $x \mapsto \mathrm{e}^{-x}\mathcal{I}_\nu(x)$ is not SCM on $(0,\infty)$. A strongly related result was proved in 1990 by Ismail [**43**], who proved, by using the Weierstrassian infinite product representation of the modified Bessel function of the first kind, that if $\nu > \frac{1}{2}$, then the function $x \mapsto 2^\nu\Gamma(\nu+1)x^{-\nu/2}I_\nu(\sqrt{x})\mathrm{e}^{-\sqrt{x}}$ is the Laplace transform of an infinitely divisible distribution. Clearly, in particular we have that $x \mapsto 2^\nu\Gamma(\nu+1)x^{-\nu/2}I_\nu(\sqrt{x})\mathrm{e}^{-\sqrt{x}}$ is SCM on $(0,\infty)$ for all $\nu > \frac{1}{2}$. We point out here that, in fact, the above statement holds for all $\nu \geqslant -\frac{1}{2}$. More precisely, it is known [**25**] that if f is SCM and g is non-negative with an SCM derivative, then the composite function $f \circ g$ is also SCM. Now, since the function $x \mapsto \mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ is SCM on $(0,\infty)$ for all $\nu \geqslant -\frac{1}{2}$ and $x \mapsto [\sqrt{x}]'$ is SCM on $(0,\infty)$, the required result immediately follows. This completes the result of Ismail in the case when $|\nu| \leqslant \frac{1}{2}$.

Furthermore, it is also worth noting a significant consequence of Nåsell's result. Since an SCM function is strictly log-convex, from the above discussion we immediately have that the function $x \mapsto \mathrm{e}^{-x}\mathcal{I}_\nu(x)$ and hence the function $x \mapsto \mathcal{I}_\nu(x)$ is strictly log-convex on $(0,\infty)$ for all $\nu \geqslant -\frac{1}{2}$. Now, since the function $x \mapsto \mathcal{I}_\nu(x)$ is even, we obtain that $x \mapsto \mathcal{I}_\nu(x)$ is strictly log-convex on $\mathbb{R}$ for all $\nu \geqslant -\frac{1}{2}$. For $\nu = 0$, this result was proved implicitly in 1967 by Boyd [**27**] and for $\nu > -\frac{1}{2}$ it was proved explicitly in 1992 by Neuman [**60**] by using the above integral representation of the function $\mathcal{I}_\nu$. Not knowing about the above results, Kayumov [**50**] reproduced the log-convexity of $\mathcal{I}_0 = I_0$ in 2005 by using the classical Hölder–Rogers inequality. We note that Neuman's result was slightly improved recently in [**19**] and it was conjectured that the function $x \mapsto \mathcal{I}_\nu(x)$ is strictly log-convex on $\mathbb{R}$ for all $\nu > -1$. In [**15**] the case when $x \in [-j_{\nu,1}, j_{\nu,1}]$ was settled by Baricz. However, the strict log-convexity of $x \mapsto \mathcal{I}_\nu(x)$ on $\mathbb{R} \setminus [-j_{\nu,1}, j_{\nu,1}]$ for all $\nu \in (-1,-\frac{1}{2})$ is still unproved. Here $j_{\nu,1}$ is the first positive zero of the νth-order Bessel function of the first kind. Moreover, in [**13**] it was proved that, for all $\nu > -1$ and $0 < x \leqslant \sqrt{2(\nu+1)(\nu+2)}$, the function $x \mapsto \mathcal{I}_\nu(x)$ is strictly log-convex. Finally, we note that, in 1996, Giordano *et al*. [**35**] presented an alternative proof of the log-convexity of $\mathcal{I}_\nu$ for $\nu > -\frac{1}{2}$ by using the classical Hölder–Rogers inequality and claimed that all derivatives of $x \mapsto \mathcal{I}_\nu(x)$ with respect to x are strictly log-convex on $\mathbb{R}$ for all $\nu > -\frac{1}{2}$. However, in 1997 Ismail [**44**] pointed out that in fact just the derivatives of even order of the function $\mathcal{I}_\nu$ would be strictly log-convex. Recall that $\mathcal{I}_{-1/2}(x) = \cosh x$; thus, in this case it is clear that for all natural numbers k the function

$$x \mapsto \mathcal{I}_{-1/2}^{(2k)}(x) = \cosh x$$

is strictly log-convex on $\mathbb{R}$ and

$$x \mapsto \mathcal{I}_{-1/2}^{(2k+1)}(x) = \sinh x$$

is strictly log-concave on $(0,\infty)$. Motivated by the above particular cases, Baricz [**13**] recently conjectured the following: for each natural number k the function $x \mapsto \mathcal{I}_\nu^{(2k)}(x)$ is strictly log-convex on $\mathbb{R}$ for all $\nu > -1$, while the function $x \mapsto \mathcal{I}_\nu^{(2k+1)}(x)$ is strictly log-concave on $(0,\infty)$ for all $\nu > -1$.

Now let us focus on the right-hand side of the inequality (2.2), which states that the function $x \mapsto e^x x^{-\nu} I_\nu(x)$ is strictly increasing on $(0, \infty)$ for all $\nu > 0$. We observe that using (1.1) in fact the function $x \mapsto e^x x^{-\nu} I_\nu(x)$ is not only strictly increasing on $(0, \infty)$ for all $\nu > -1$, but even SAM.

Another upper bound for the ratio $I_\nu(x)/I_\nu(y)$ which may be of interest is due to Laforgia [**51**], who showed, by using Soni's inequality (2.3), that for all $\nu \geqslant \frac{1}{2}$ and $0 < x < y$ we have

$$\frac{I_\nu(x)}{I_\nu(y)} < e^{x-y}\left(\frac{y}{x}\right)^\nu. \tag{2.4}$$

An alternative way to deduce (2.4) is the following: owing to Gronwall [**36**], it is known that the inequality

$$\frac{xI'_\nu(x)}{I_\nu(x)} > x - \tfrac{1}{2} \tag{2.5}$$

holds for all $x > 0$ and $\nu \geqslant \frac{1}{2}$. This in turn implies that for all $x > 0$ and $\nu \geqslant \frac{1}{2}$ we have

$$\frac{[e^{-x} x^\nu I_\nu(x)]'}{e^{-x} x^{\nu-1} I_\nu(x)} = \frac{xI'_\nu(x)}{I_\nu(x)} + \nu - x > x - \tfrac{1}{2} + \nu - x \geqslant 0,$$

i.e. the function $x \mapsto e^{-x} x^\nu I_\nu(x)$ is strictly increasing on $(0, \infty)$ for all $\nu \geqslant \frac{1}{2}$. Furthermore, it is important to note here that by using the idea from [**48, 51**], i.e. dividing by $x > 0$, changing x to s, and after integrating on $[x, y]$ both sides of Gronwall's inequality (2.5), we obtain that for all $\nu \geqslant \frac{1}{2}$ and $0 < x < y$

$$\frac{I_\nu(x)}{I_\nu(y)} < e^{x-y}\left(\frac{y}{x}\right)^{1/2}, \tag{2.6}$$

i.e. the function $x \mapsto e^{-x} x^{1/2} I_\nu(x)$ is strictly increasing on $(0, \infty)$ for all $\nu \geqslant \frac{1}{2}$. However, this follows also directly by observing that, by using Gronwall's inequality (2.5) for all $\nu \geqslant \frac{1}{2}$ and $x > 0$, we obtain

$$\frac{[e^{-x} x^{1/2} I_\nu(x)]'}{e^{-x} x^{-1/2} I_\nu(x)} = \frac{xI'_\nu(x)}{I_\nu(x)} + \frac{1}{2} - x > 0.$$

Observe that the new upper bound in (2.6) is tighter than the upper bound of Laforgia in (2.4).

When $\nu \in (0, \frac{1}{2})$, graphs and numerical computations suggest that the function $x \mapsto e^{-x} x^\nu I_\nu(x)$ is not strictly increasing on $(0, \infty)$, but is strictly increasing only on $(0, \epsilon)$ for sufficiently small ϵ. However, since the product of two SCM functions is also SCM, it can easily be seen that the function $x \mapsto e^{-x} x^\nu I_\nu(x)$ is SCM, and hence strictly decreasing on $(0, \infty)$ for all $\nu \in [-\frac{1}{2}, 0]$. Here we have used that $x \mapsto e^{-x} x^{-\nu} I_\nu(x)$ is SCM on $(0, \infty)$ for all $\nu \geqslant -\frac{1}{2}$ and $x \mapsto x^{2\nu}$ is also SCM on $(0, \infty)$ for all $\nu \in [-\frac{1}{2}, 0)$. Consequently, when $\nu \in [-\frac{1}{2}, 0]$, we have that the inequality (2.4) is reversed. However, this reversed inequality is weaker than the left-hand side of (2.1). The remaining part, when $\nu \in (-1, -\frac{1}{2})$, is stated at the end of this section as an open problem. Similarly,

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computer-generated pictures suggest that for $\nu \in (-1, \frac{1}{2})$ the derivative of the function $x \mapsto e^{-x}x^{1/2}I_\nu(x)$ changes sign.

A strongly related result to inequality (2.4), deduced recently by Baricz and Sun [**21**], is as follows: the function $x \mapsto x^{\nu+1}e^{-x}I_\nu(x)$ is strictly increasing on $(0, \infty)$ for all $\nu \geqslant 0$, i.e. for each $0 < x < y$ and $\nu \geqslant 0$ one has

$$\frac{I_\nu(x)}{I_\nu(y)} < e^{x-y}\left(\frac{y}{x}\right)^{\nu+1}. \tag{2.7}$$

Clearly, Laforgia's upper bound in (2.4) is tighter than the upper bound in (2.7). However, the inequality (2.7) is valid for all $\nu \geqslant 0$, not just for $\nu \geqslant \frac{1}{2}$. Numerical experiments suggest that the derivative of the function $x \mapsto x^{\nu+1}e^{-x}I_\nu(x)$ changes sign on $(0, \infty)$ when $\nu \in (-\frac{1}{2}, 0)$; however, the case when $\nu \in (-1, -\frac{1}{2}]$ is stated below as another open problem. It is also worth mentioning here that (2.7) can be improved as follows [**17**]: for all $\nu \geqslant 0$ and $0 < x < y$ we have

$$\frac{I_\nu(x)}{I_\nu(y)} < \frac{\sinh x}{\sinh y}\left(\frac{y}{x}\right)^{\nu+1}, \tag{2.8}$$

i.e. the function $x \mapsto (\sinh x)^{-1}x^{\nu+1}I_\nu(x)$ is strictly increasing on $(0, \infty)$ for all $\nu \geqslant 0$. However, since the function $x \mapsto e^{-x}\mathcal{I}_{1/2}(x) = e^{-x}x^{-1}\sinh x$ is strictly decreasing on $(0, \infty)$ we observe that for $\nu \geqslant \frac{1}{2}$ the upper bound in (2.8) is still weaker than the upper bound in (2.4). Analogously, numerical experiments suggest that the derivative of the function $x \mapsto (\sinh x)^{-1}x^{\nu+1}I_\nu(x)$ changes sign on $(0, \infty)$ for $\nu \in (-\frac{1}{2}, 0)$, while the case when $\nu \in (-1, -\frac{1}{2}]$ is stated below as an open problem.

Now, let us reconsider the left-hand side of (2.1). Tighter lower bounds of this kind were established in 1991 by Joshi and Bissu [**48**], by using Nåsell's inequality [**58**], which states that

$$\left(1 + \frac{\nu}{x}\right)I_{\nu+1}(x) < I_\nu(x) \tag{2.9}$$

for all $\nu > -1$ and $x > 0$. Observe that for $\nu > 0$ Nåsell's inequality (2.9) improves Soni's inequality (2.3). The result stated in [**48**] is as follows: the function $x \mapsto x^{-\nu}(x + \nu)^\nu e^{-x}I_\nu(x)$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$, i.e. for all $0 < x < y$ and $\nu > -1$ we have

$$e^{x-y}\left(\frac{y+\nu}{x+\nu}\right)^\nu\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)}. \tag{2.10}$$

This lower bound is clearly tighter than the lower bound in (2.1) if $\nu > 0$. Another lower bound which is tighter than the lower bound in (2.1) was deduced recently by Baricz [**17**], who showed that the function $x \mapsto (e^x + \lambda_\nu)^{-1}x^{-\nu}I_\nu(x)$ is decreasing on $(0, \infty)$ for all $\nu \geqslant 0$, where $\lambda_\nu = f(\rho_\nu)$ is the largest positive constant and $f_\nu(x) = e^x[I_\nu(x)/I_{\nu+1}(x) - 1]$, while ρ_ν is the unique positive root of the equation $(x + 2\nu + 1)I_{\nu+1}(x) = xI_\nu(x)$. Equivalently, we have that for all $\nu \geqslant 0$ and $0 < x < y$ the following inequality holds:

$$\frac{e^x + \lambda_\nu}{e^y + \lambda_\nu}\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)}. \tag{2.11}$$

We note that in (2.11) the constant λ_ν is the best possible, i.e. cannot be changed by any larger constant.

Finally, let us discuss some other types of bounds that we have found in the literature. An interesting approach was given in 1990 by Ifantis and Siafarikas [**41**], who proved by using operator techniques that the right-hand side of (2.1) can be improved as follows:

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^\nu \left(\frac{x^2 + j_{\nu,1}^2}{y^2 + j_{\nu,1}^2}\right)^{j_{\nu,1}^2/4(\nu+1)}, \tag{2.12}$$

where $j_{\nu,1}$ is the first positive zero of the νth-order Bessel function of the first kind, $\nu > -1$ and $0 < x < y$. Moreover, in 1996 Joshi and Bissu [**49**] proved that, for all $\nu > -1$ and $0 < x < y$,

$$\exp\left\{\frac{x^2 - y^2}{4(\nu+1)}\right\}\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)} < \exp\left\{\frac{x^2 - y^2}{4(\nu+1)} - \frac{x^4 - y^4}{32(\nu+1)^2(\nu+2)}\right\}\left(\frac{x}{y}\right)^\nu. \tag{2.13}$$

However, these bounds are tighter than those presented in (2.1) only for a strong assumption on the order ν, and hence they are not very useful. All the same, an alternative idea for deducing the left-hand side of (2.13) will be useful in order to deduce new lower and upper bounds, as we can see in the next subsection. More precisely, the left-hand side of (2.13) can be reformulated as follows: the function $x \mapsto \exp\{-x^2/[4(\nu+1)]\}x^{-\nu}I_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu > -1$, and this can be proved directly by using the following old result of Biernacki and Krzyż [**23**].

Lemma 2.1. *Consider the power series* $f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ *and* $g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots$, *where, for all* $n \geqslant 0$, *integer* $a_n \in \mathbb{R}$ *and* $b_n > 0$, *and suppose that both converge on* $\mathbb{R}$. *If the sequence* $\{a_n/b_n\}_{n\geqslant 0}$ *is (strictly) increasing (decreasing), then the function* $x \mapsto f(x)/g(x)$ *is also (strictly) increasing (decreasing) on* $(0,\infty)$.

For different proofs and various applications of this result the interested reader, is referred to [**3**, **5**, **7**–**10**, **12**, **40**, **60**, **65**] and the references therein. We note that we can see easily that the above result remains true if we get even and odd functions, i.e.

(i) $f(x) = a_0 + a_1 x^2 + \cdots + a_n x^{2n} + \cdots$ and $g(x) = b_0 + b_1 x^2 + \cdots + b_n x^{2n} + \cdots$,

(ii) $f(x) = a_0 x + a_1 x^3 + \cdots + a_n x^{2n+1} + \cdots$ and $g(x) = b_0 x + b_1 x^3 + \cdots + b_n x^{2n+1} + \cdots$.

Now, using the power-series representation of the functions $x \mapsto \exp\{x^2/[4(\nu+1)]\}$ and $x \mapsto \mathcal{I}_\nu(x)$, i.e.

$$\frac{\mathcal{I}_\nu(x)}{\exp\{x^2/[4(\nu+1)]\}} = \sum_{n\geqslant 0} \frac{x^{2n}}{4^n n!(\nu+1)_n} \bigg/ \sum_{n\geqslant 0} \frac{x^{2n}}{4^n n!(\nu+1)^n},$$

in view of Lemma 2.1, to prove the left-hand side of (2.13) it is sufficient to prove that the sequence $\{\alpha_n\}_{n\geqslant 0} = \{(\nu+1)^n/(\nu+1)_n\}_{n\geqslant 0}$ is strictly decreasing. Since $\alpha_0 = \alpha_1 = 1$ and, for all integers $n \geqslant 2$, the condition $\alpha_n > \alpha_{n+1}$ is equivalent to $\nu + n + 1 > \nu + 1$,

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the required result follows easily. Observe that by using the monotonicity of the function $x \mapsto \exp\{-x^2/[4(\nu+1)]\}\mathcal{I}_\nu(x)$ we immediately have an upper bound for the modified Bessel function of the first kind, i.e. for all $\nu > -1$ and $x > 0$ one has

$$I_\nu(x) < \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \mathrm{e}^{x^2/[4(\nu+1)]}.$$

However, this result can be rewritten as follows:

$$\sum_{n \geqslant 1} \log\left(1 + \frac{x^2}{j_{\nu,n}^2}\right) < \sum_{n \geqslant 1} \frac{x^2}{j_{\nu,n}^2},$$

which follows easily by using the well-known inequality $\mathrm{e}^x > x + 1$. Here $j_{\nu,n}$ is the nth positive zero of the νth-order Bessel function of the first kind and we have used the known formulae [**76**]

$$\mathcal{I}_\nu(x) = \prod_{n \geqslant 1} \left(1 + \frac{x^2}{j_{\nu,n}^2}\right) \quad \text{and} \quad \sum_{n \geqslant 1} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}.$$

A similar result for Bessel functions of the first kind was proved for a real argument by Ismail and Muldoon [**46**] in 1986, for a complex argument in the open disc $\{z \in \mathbb{C}\colon |z| < j_{\nu,1}\}$ by Sitnik [**73**] in 1995 and, recently, for a complex argument in the closed Cassinian oval by Pogány [**64**]. It is also worthwhile noting that András and Baricz [**6**] recently deduced a similar result for the complex argument in the open unit disc.

We close this subsection with the following results of Sitnik [**73**]:

$$\left(\frac{x}{y}\right)^\nu \left[\frac{x^2 + 2(\nu+1)(\nu+3)}{y^2 + 2(\nu+1)(\nu+3)}\right]^{(\nu+3)^2/4(\nu+2)} \exp\left\{\frac{x^2 - y^2}{8(\nu+2)}\right\} < \frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^\nu \left[\frac{x^2 + 4(\nu+1)(\nu+2)}{y^2 + 4(\nu+1)(\nu+2)}\right]^{\nu+2},$$

where $0 < x < y$ and $\nu > -1$. For more general inequalities of this kind we refer also to [**73**]. We note that for $\nu > -1$ the above upper bound improves the Paris upper bound in (2.1).

2.2. Some new results and related comments

By using the idea of the proof of the left-hand side of (2.13), we obtain the following result, which is mainly based on Lemma 2.1.

Theorem 2.2. *Let $\nu, \mu > -1$ and let k be a natural number. The following assertions are then true.*

(i) *If $\nu < \mu$ ($\nu > \mu$), then the function $x \mapsto \mathcal{I}_\nu(x)/\mathcal{I}_\mu(x)$ is strictly increasing (decreasing) on $(0, \infty)$.*

(ii) *The function $x \mapsto \mathcal{I}_\nu^{(2k)}(x)/\cosh x$ is strictly increasing on $(0, \infty)$ for all $\nu < -\frac{1}{2}$ and strictly decreasing on $(0, \infty)$ for all $\nu > -\frac{1}{2}$.*

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(iii) *The function* $x \mapsto \mathcal{I}_\nu^{(2k+1)}(x)/\sinh x$ *is strictly increasing on* $(0,\infty)$ *for all* $\nu < -\frac{1}{2}$ *and strictly decreasing on* $(0,\infty)$ *for all* $\nu > -\frac{1}{2}$.

(iv) *The function* $x \mapsto x^{-\nu} I_\nu(x)/\cosh x$ *is strictly increasing on* $(0,\infty)$ *for all* $\nu < -\frac{1}{2}$ *and strictly decreasing on* $(0,\infty)$ *for all* $\nu > -\frac{1}{2}$.

(v) *The function* $x \mapsto x^{1-\nu} I_\nu(x)/\sinh x$ *is strictly increasing on* $(0,\infty)$ *for all* $\nu < \frac{1}{2}$ *and strictly decreasing on* $(0,\infty)$ *for all* $\nu > \frac{1}{2}$.

(vi) *The following inequalities hold for all* $x > 0$*:*

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} < \tanh x < 1 \quad \text{for all } \nu > -\tfrac{1}{2}, \tag{2.14}$$

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} > \tanh x > 0 \quad \text{for all } \nu < -\tfrac{1}{2}, \tag{2.15}$$

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} < \coth x - \frac{1}{x} < \tanh x < 1 \quad \text{for all } \nu > \tfrac{1}{2}, \tag{2.16}$$

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} > \coth x - \frac{1}{x} > 0 \quad \text{for all } \nu < \tfrac{1}{2}. \tag{2.17}$$

Proof. (i) By using the power-series representation of the functions $\mathcal{I}_\nu$ and $\mathcal{I}_\mu$, in view of Lemma 2.1 we need only study the monotonicity of the sequence $\{\alpha_n\}_{n\geqslant 0} = \{(\mu+1)_n/(\nu+1)_n\}_{n\geqslant 0}$. However, by the ascending factorial notation we have that $(\nu+n)\alpha_n = (\mu+n)\alpha_{n-1}$ for all $n \geqslant 1$. Consequently, we obtain that $\alpha_n < \alpha_{n-1}$ for all $n \geqslant 1$ if and only if $\mu < \nu$, and $\alpha_n > \alpha_{n-1}$ for all $n \geqslant 1$ if and only if $\mu > \nu$.

(ii), (iii) Using the power-series representation of the function $\mathcal{I}_\nu$ it is easy to show that

$$\mathcal{I}_\nu^{(2k)}(x) = \sum_{n\geqslant 0} (2n+1)_{2k} \frac{(\frac{1}{4})^{n+k}}{(\nu+1)_{n+k}(n+k)!} x^{2n},$$
$$\mathcal{I}_\nu^{(2k+1)}(x) = \sum_{n\geqslant 0} (2n+2)_{2k+1} \frac{(\frac{1}{4})^{n+k+1}}{(\nu+1)_{n+k+1}(n+k+1)!} x^{2n+1}.$$

On the other hand, we know that

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n\geqslant 0} \frac{x^{2n}}{(2n)!} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n\geqslant 0} \frac{x^{2n+1}}{(2n+1)!}.$$

In view of Lemma 2.1 we need to study the monotonicity of sequences $\{\alpha_n\}_{n\geqslant 0}$ and $\{\beta_n\}_{n\geqslant 0}$, defined by

$$\alpha_n = (2n+1)_{2k} \frac{(\frac{1}{4})^{n+k}}{(\nu+1)_{n+k}(n+k)!}(2n)! = \frac{(\frac{1}{4})^{n+k}(2n+2k)!}{(\nu+1)_{n+k}(n+k)!},$$
$$\beta_n = (2n+2)_{2k+1} \frac{(\frac{1}{4})^{n+k+1}}{(\nu+1)_{n+k+1}(n+k+1)!}(2n+1)! = \frac{(\frac{1}{4})^{n+k+1}(2n+2k+2)!}{(\nu+1)_{n+k+1}(n+k+1)!}$$

for all natural numbers n and k. Observe first that for all $n \in \mathbb{N}$ we have $\beta_n = \alpha_{n+1}$, which implies that it is sufficient to study the monotonicity of the sequence $\{\alpha_n\}_{n\geqslant 0}$. By the ascending factorial notation for all $n, k \in \mathbb{N}$ and $\nu > -1$ we have

$$\alpha_{n+1} = \frac{2n+2k+1}{2n+2k+2\nu+2}\alpha_n.$$

Consequently, $\alpha_{n+1} < \alpha_n$ holds for all $n \in \mathbb{N}$ if and only if $\nu > -\frac{1}{2}$, and $\alpha_{n+1} > \alpha_n$ holds for all $n \in \mathbb{N}$ if and only if $\nu < -\frac{1}{2}$. In view of Lemma 2.1 these complete the proof of parts (ii) and (iii).

(iv) This follows from part (i) by taking $\mu = -\frac{1}{2}$ and noticing that $\mathcal{I}_{-1/2}(x) = \cosh x$. Furthermore, this statement can also be deduced from part (ii) by choosing $k = 0$.

(v) Similarly, this follows from part (i) by taking $\mu = \frac{1}{2}$ and noticing that $\mathcal{I}_{1/2}(x) = x^{-1}\sinh x$. However, by using the well-known recurrence relation [**76**, p. 79] $xI_{\nu+1}(x) = xI'_\nu(x) - \nu I_\nu(x)$, we obtain the formula

$$\mathcal{I}'_\nu(x) = 2^\nu \Gamma(\nu+1)x^{-\nu}I_{\nu+1}(x), \tag{2.18}$$

which in turn yields from part (iii) that the function $x \mapsto x^{-\nu}I_{\nu+1}(x)/\sinh x$ is strictly increasing on $(0,\infty)$ for all $\nu < -\frac{1}{2}$ and is strictly decreasing on $(0,\infty)$ for all $\nu > -\frac{1}{2}$. Now, replacing ν by $\nu - 1$ we obtain the second proof of part (v).

(vi) Using (2.18), we obtain the relation

$$[\cosh^2 x][\mathcal{I}_\nu(x)/\cosh x]' = 2^\nu\Gamma(\nu+1)x^{-\nu}[I_{\nu+1}(x)\cosh x - I_\nu(x)\sinh x],$$

and from part (iv) we immediately obtain the inequalities (2.14) and (2.15). On the other hand, we know that

$$\begin{aligned}\mathcal{I}''_\nu(x) &= 2^\nu\Gamma(\nu+1)[-\nu x^{-\nu-1}I_{\nu+1}(x) + x^{-\nu}I'_{\nu+1}(x)]\\ &= 2^\nu\Gamma(\nu+1)x^{-\nu}\left[I'_{\nu+1}(x) - \frac{\nu}{x}I_{\nu+1}(x)\right]\\ &= 2^\nu\Gamma(\nu+1)x^{-\nu}\left[I_{\nu+2}(x) + \frac{1}{x}I_{\nu+1}(x)\right],\end{aligned}$$

which in turn implies that

$$\left[\frac{\mathcal{I}'_\nu(x)}{\sinh x}\right]' = \frac{2^\nu\Gamma(\nu+1)}{x^\nu\sinh^2 x}\left\{\left[I_{\nu+2}(x) + \frac{1}{x}I_{\nu+1}(x)\right]\sinh x - I_{\nu+1}(x)\cosh x\right\}.$$

Using part (iii) for $k = 0$, these relations imply the inequalities (2.16) and (2.17). All that it remains to prove is the inequality $\coth x - 1/x < \tanh x$ for $x > 0$. But this inequality can be rewritten as $x < (\sinh x)(\cosh x)$, i.e. $\mathcal{I}_{-1/2}(x)\mathcal{I}_{1/2}(x) > 1$, which clearly holds, since for all $x > 0$ and $\nu > -1$ we have $\mathcal{I}_\nu(x) > 1$. With this the proof is complete. □

First let us focus on the first part of Theorem 2.2. By (2.18) we get that

$$\left[\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_\mu(x)}\right]' = \left[\frac{2^\nu\Gamma(\nu+1)x^{-\nu}}{2^\mu\Gamma(\mu+1)x^{-\mu}}\right]\left[\frac{I_{\nu+1}(x)I_\mu(x)-I_{\mu+1}(x)I_\nu(x)}{I_\mu^2(x)}\right],$$

and by using Theorem 2.2 (i) we obtain that, for all $\mu > \nu > -1$ and $x > 0$, the Turán-type inequality $I_{\nu+1}(x)I_\mu(x) - I_{\mu+1}(x)I_\nu(x) > 0$ holds. Now, choosing $\mu = \nu + 1$ in the above inequality, we obtain the well-known Turán-type inequality $I_{\nu+1}^2(x) - I_{\nu+2}(x)I_\nu(x) > 0$, where $\nu > -1$ and $x > 0$. This inequality is known in literature as Amos's inequality [**4**] and it was deduced in 1974. However, its equivalent form had already been deduced in 1932 by Gronwall [**36**], as was pointed out recently by Baricz [**16**]. For more details on Turán-type inequalities on Bessel, modified Bessel and hypergeometric functions and related results, the interested reader is referred to [**11**, **14**, **15**, **18**, **54**].

Recall that from the second (or fourth) part of Theorem 2.2 we obtain the result that the function $x \mapsto \mathcal{I}_\nu(x)/\cosh x$ is strictly decreasing on $(0,\infty)$ for all $\nu > -\frac{1}{2}$. Consequently, we immediately have that for all $x > 0$ and $\nu > -\frac{1}{2}$ the inequality $\mathcal{I}_\nu(x) < \cosh x$ holds. This inequality is actually also an immediate consequence of the result [**15**, Theorem 1] that $\nu \mapsto \mathcal{I}_\nu(x)$ is decreasing on $(-1,\infty)$ for each fixed $x > 0$. However, an alternative derivation of the above inequality was given in 1972 by Luke [**56**, p. 63], by using a completely different approach.

Clearly, Theorem 2.2 (iv) can be rewritten as follows:

$$\frac{\cosh x}{\cosh y}\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)}, \tag{2.19}$$

where $0 < x < y$ and $\nu > -\frac{1}{2}$. Moreover, the above inequality is reversed when $\nu \in (-1, -\frac{1}{2})$. It is worth mentioning that since the function $x \mapsto \mathrm{e}^{-x}\mathcal{I}_{-1/2}(x) = \mathrm{e}^{-x}\cosh x$ is decreasing on $(0,\infty)$, clearly the bound in (2.19) is tighter than the bound in the left-hand side of (2.1) or (2.2). Similarly, since the function $x \mapsto \mathcal{I}_{-1/2}(x) = \cosh x$ is strictly increasing on $(0,\infty)$, we obtain that, when $\nu \in (-1, -\frac{1}{2})$, the reversed form of (2.19) improves the right-hand side of (2.1). All the same, the inequalities (2.11) and (2.19) cannot be compared on the whole interval $(0,\infty)$, due to their different natures.

Analogously, the fifth part of Theorem 2.2 can be rewritten as follows:

$$\frac{\sinh x}{\sinh y}\left(\frac{x}{y}\right)^{\nu-1} < \frac{I_\nu(x)}{I_\nu(y)}, \tag{2.20}$$

where $0 < x < y$ and $\nu > \frac{1}{2}$. Furthermore, the above inequality is reversed, when $\nu \in (-1, \frac{1}{2})$. We note that, since the function $x \mapsto \mathrm{e}^{-x}\mathcal{I}_{1/2}(x) = \mathrm{e}^{-x}x^{-1}\sinh x$ is decreasing on $(0,\infty)$, clearly the bound in (2.20) is tighter than the bound in the left-hand side of (2.1) or (2.2). Similarly, since the function $x \mapsto \mathcal{I}_{1/2}(x) = x^{-1}\sinh x$ is strictly increasing on $(0,\infty)$, we obtain that when $\nu \in (-1, \frac{1}{2})$ the reversed form of (2.20) improves the right-hand side of (2.1) and completes (2.8).

Now, let us compare the inequalities (2.19) and (2.20). From the first part of Theorem 2.2 we know that the function $x \mapsto \mathcal{I}_{-1/2}(x)/\mathcal{I}_{1/2}(x)$ is strictly increasing on $(0,\infty)$.

This in turn implies that for $\nu > \frac{1}{2}$ the lower bound in (2.20) is tighter than the lower bound in (2.19). However, by using the above argument for $\nu < -\frac{1}{2}$ the reversed form of (2.19) is better than the reversed form of (2.20).

Observe that for $\nu > -\frac{1}{2}$ the inequality (2.14) improves Soni's inequality (2.3). Moreover, for $\nu > \frac{1}{2}$ the inequality (2.16) improves the inequality (2.14). Although the bounds given in (2.14) and (2.16) are weaker than the more accurate bounds given in [**4**, **59**, **63**], they have the advantage of being easily evaluated numerically, since they do not depend on ν. Furthermore, numerical experiments suggest that for $\nu \in (-\frac{1}{2}, 0)$ the upper bound in (2.14) is very tight and, similarly, for $\nu \in (\frac{1}{2}, 1)$ the upper bound in (2.16) is also tight. This is in agreement with the fact that, as the order ν increases, sharper upper bounds than in (2.14) and (2.16) for the ratio $I_{\nu+1}(x)/I_\nu(x)$ can be obtained using higher even- and odd-order derivatives of the function $x \mapsto \mathcal{I}_\nu(x)$ together with parts (ii) and (iii) of Theorem 2.2. We note that it is easy to see that, from the inequality (2.14), by using the recurrence formula [**76**, p. 79] $I'_\nu(x)/I_\nu(x) = \nu/x + I_{\nu+1}(x)/I_\nu(x)$, after integration we obtain the inequality (2.19). Analogously, from the inequality (2.16), after integration we get the inequality (2.20). Taking into account the proof of Theorem 2.2 (vi), these in turn imply that in fact the inequality (2.14) is equivalent to (2.19), while the inequality (2.16) is equivalent to (2.20).

Finally, we propose some open problems which are related to the problems discussed in this section.

(a) Is the function $x \mapsto x^\nu e^{-x} I_\nu(x)$ SCM on $(0, \infty)$ for all $\nu \in (-1, -\frac{1}{2})$?

(b) Is the function $x \mapsto x^{\nu+1} e^{-x} I_\nu(x)$ SCM on $(0, \infty)$ for all $\nu \in (-1, -\frac{1}{2}]$?

(c) Is the function $x \mapsto (\sinh x)^{-1} x^{\nu+1} I_\nu(x)$ SCM on $(0, \infty)$ for all $\nu \in (-1, -\frac{1}{2}]$?

3. Bounds for quotients of modified Bessel functions of the second kind

In 1984 Paris [**61**] proved the inequality

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x} \left(\frac{x}{y}\right)^\nu, \tag{3.1}$$

which holds for all $\nu > -\frac{1}{2}$ and $0 < x < y$. However, this result also appeared in [**26**, **70**] in 1973. Inequality (3.1) was also derived by Laforgia [**51**] in 1991. The above result actually states that the function $x \mapsto x^{-\nu} e^x K_\nu(x)$ is strictly decreasing on $(0, \infty)$ for all $\nu > -\frac{1}{2}$. We note that it is well known [**57**] that the function $x \mapsto e^x K_\nu(x)$ is SCM on $(0, \infty)$ for all $\nu \in \mathbb{R}$. Now, since the function $x \mapsto x^{-\nu}$ is SCM on $(0, \infty)$ for all $\nu > 0$, it follows that the function $x \mapsto x^{-\nu} e^x K_\nu(x)$ for all $\nu \geqslant 0$ is not only strictly decreasing but even SCM on $(0, \infty)$. Moreover, since, due to Miller and Samko [**57**], the function $x \mapsto x^\nu e^x K_\nu(x)$ is SCM on $(0, \infty)$ for all $\nu \in (0, \frac{1}{2})$, we easily obtain that the function $x \mapsto x^{-\nu} e^x K_\nu(x)$ is SCM on $(0, \infty)$ for all $\nu \in (-\frac{1}{2}, 0)$ and hence for all $\nu > -\frac{1}{2}$. Here we have used the well-known fact that the function $\nu \mapsto K_\nu(x)$ is even. Observe that, since the function $x \mapsto e^x K_\nu(x)$ is strictly decreasing, as we have mentioned above, we get the following inequality [**51**]:

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x}, \tag{3.2}$$

where $0 < x < y$ and ν is an arbitrary real number. If $\nu > 0$, then (3.2) is more stringent than (3.1). Now, using the fact that the function $x \mapsto K_\nu(x)/I_\nu(x)$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$, as a product of two strictly decreasing functions, in view of (2.1) Joshi and Bissu [**48**] in 1991 deduced the inequality

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{x-y}\left(\frac{x}{y}\right)^\nu,$$

where $\nu > -\frac{1}{2}$ and $0 < x < y$. Clearly, this lower bound is weaker than the lower bound in (3.1). We note that the function $x \mapsto e^{-x}x^{-\nu}K_\nu(x)$ is not only strictly decreasing on $(0, \infty)$ for all $\nu > -\frac{1}{2}$ but is actually SCM, as a product of the SCM functions $x \mapsto e^{-2x}$ and $x \mapsto x^{-\nu}e^x K_\nu(x)$.

Another result related to the inequalities (3.1) and (3.2) was proved in 1991 by Laforgia [**51**], who showed that

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x}\left(\frac{y}{x}\right)^\nu, \tag{3.3}$$

where $\nu \in (0, \frac{1}{2})$ and $0 < x < y$. Observe that for $\nu \in (0, \frac{1}{2})$ the inequality (3.3) improves (3.1). As we have pointed out above, the function $x \mapsto x^{-\nu}e^x K_\nu(x)$ is SCM on $(0, \infty)$ for all $\nu > -\frac{1}{2}$. From this we easily obtain that, in fact, the function $x \mapsto x^\nu e^x K_\nu(x)$ is also SCM on $(0, \infty)$ for all $\nu < \frac{1}{2}$. This completes the above-mentioned result from [**57**] for the case when $\nu \leqslant 0$, and in particular implies that the inequality (3.3) holds for all $\nu < \frac{1}{2}$.

It is important to note here that Ismail [**43**] proved in 1990 that $x \mapsto e^{\sqrt{x}}x^{\nu/2}K_\nu(\sqrt{x})$ is the Laplace transform of a generalized gamma convolution for each $\nu \in (0, \frac{1}{2})$. Clearly, in particular, this yields that $x \mapsto e^{\sqrt{x}}x^{\nu/2}K_\nu(\sqrt{x})$ is SCM on $(0, \infty)$ for all $\nu \in (0, \frac{1}{2})$. However, since $x \mapsto x^\nu e^x K_\nu(x)$ is SCM on $(0, \infty)$ for all $\nu < \frac{1}{2}$, we immediately obtain that $x \mapsto e^{\sqrt{x}}x^{\nu/2}K_\nu(\sqrt{x})$ is SCM on $(0, \infty)$ for all $\nu < \frac{1}{2}$. This completes the result of [**43**]. Surprisingly, when $\nu > \frac{1}{2}$, the function $x \mapsto x^\nu e^x K_\nu(x)$ becomes strictly increasing on $(0, \infty)$, and hence for $\nu > \frac{1}{2}$ the inequality (3.3) is reversed, as was stated in [**51**]. Furthermore, using again the recursive relation $K_\nu(x) = K_{-\nu}(x)$, from this we deduce that the function $x \mapsto x^{-\nu}e^x K_\nu(x)$ is strictly increasing on $(0, \infty)$ for all $\nu < -\frac{1}{2}$. In other words, for $\nu < -\frac{1}{2}$ we have that the inequality (3.1) is reversed. We note that the reversed form of (3.3) was deduced by Laforgia from Soni's inequality $K_{\nu+1}(x) > K_\nu(x)$, which holds for all $\nu > -\frac{1}{2}$ and $x > 0$. However, as was observed in [**51**], the function $\nu \mapsto K_\nu(x)$ is strictly increasing on $(0, \infty)$ for all fixed $x > 0$. In addition, since it is even, the function $\nu \mapsto K_\nu(x)$ is strictly decreasing on $(-\infty, 0)$ for all fixed $x > 0$.

In 1996 Joshi and Bissu [**49**] deduced the inequality

$$\frac{K_\nu(x)}{K_\nu(y)} > \left(\frac{y}{x}\right)^\nu, \tag{3.4}$$

where $\nu > -\frac{1}{2}$ and $0 < x < y$. We note that, clearly, for $|\nu| < \frac{1}{2}$, the lower bound in (3.3) is more stringent than the lower bound in (3.4). However, the inequality (3.4) holds for all $\nu \in \mathbb{R}$. To see this, just recall that the function $x \mapsto x^{\nu}e^{x}K_{\nu}(x)$ is also SCM on $(0, \infty)$ for all $\nu < \frac{1}{2}$ and the function $x \mapsto e^{-x}$ is SCM on $(0, \infty)$. From these we obtain that $x \mapsto x^{\nu}K_{\nu}(x)$ is SCM on $(0, \infty)$ for all $\nu < \frac{1}{2}$, and hence it is strictly decreasing. We note that the monotonicity of $x \mapsto x^{\nu}K_{\nu}(x)$ for all $\nu \in \mathbb{R}$ is actually an immediate consequence of the well-known derivative formula [**76**, p. 79] $[x^{\nu}K_{\nu}(x)]' = -x^{\nu}K_{\nu-1}(x)$. Notice also that, in order to prove the infinite divisibility of Student's t-distribution, in 1978 Hartman [**38**] proved that the function $x \mapsto x^{\nu/2}K_{\nu}(\sqrt{x})$ is SCM on $(0, \infty)$ for all $\nu > -\frac{1}{2}$. Appealing again to the fact that if a function f is SCM and g is non-negative with an SCM derivative, then the composite function $f \circ g$ is also SCM, and by using the result that $x \mapsto x^{\nu}K_{\nu}(x)$ is SCM on $(0, \infty)$ for all $\nu < \frac{1}{2}$, we obtain that $x \mapsto x^{\nu/2}K_{\nu}(\sqrt{x})$ is actually SCM on $(0, \infty)$ for all $\nu \in \mathbb{R}$. This completes the result of Hartman [**38**].

Another result which may be of interest appears in [**78**]:

$$K_{\nu}(x) = \frac{x^{1/2}e^{-x}}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2} - \nu)} \int_0^{\infty} \frac{t^{-1/2}}{x + t} e^{-t} K_{\nu}(t)\, dt,$$

where $x > 0$ and $\nu \in (0, \frac{1}{2})$. This formula is actually a special case of a more general result for Whittaker functions obtained by Gargantini and Henrici [**34**] and states that the function $x \mapsto x^{-1/2}e^{x}K_{\nu}(x)$ is a Stieltjes transform and, in particular, it is CM on $(0, \infty)$ for all $\nu \in (0, \frac{1}{2})$. We note that since the function $x \mapsto x^{-1/2}$ is SCM on $(0, \infty)$ we obtain that $x \mapsto x^{-1/2}e^{x}K_{\nu}(x)$ is SCM on $(0, \infty)$ for all $\nu \in \mathbb{R}$, as a product of two SCM functions. This completes the result from [**78**]. In particular, for all $\nu \in \mathbb{R}$ and $0 < x < y$ one has

$$\frac{K_{\nu}(x)}{K_{\nu}(y)} > e^{y-x}\left(\frac{x}{y}\right)^{1/2}. \tag{3.5}$$

Observe that for $\nu > \frac{1}{2}$ the lower bound in (3.5) is tighter than the lower bound in (3.1), but is weaker for $|\nu| < \frac{1}{2}$. However, for $\nu < -\frac{1}{2}$, inequality (3.5) completes the reversed form of (3.1).

Let us mention that [**57**] the function $x \mapsto x^{1/2}e^{x}K_{\nu}(x)$ is also SCM on $(0, \infty)$ for all $\nu > \frac{1}{2}$. By using the recursive relation $K_{\nu}(x) = K_{-\nu}(x)$, we obtain that $x \mapsto x^{1/2}e^{x}K_{\nu}(x)$ is also SCM on $(0, \infty)$ for all $\nu < -\frac{1}{2}$. In particular, for all $|\nu| > \frac{1}{2}$ and $0 < x < y$ we have

$$\frac{K_{\nu}(x)}{K_{\nu}(y)} > e^{y-x}\left(\frac{y}{x}\right)^{1/2}. \tag{3.6}$$

The right-hand side of (3.6) is more stringent than the lower bound in (3.3) for $\nu < -\frac{1}{2}$. However, in the case when $\nu > \frac{1}{2}$ the inequality (3.6) completes the inequality (3.3) and resembles the inequality (2.6). Furthermore, for all $|\nu| > \frac{1}{2}$ the lower bound in (3.6) is better than the lower bound in (3.5). In the case when $|\nu| < \frac{1}{2}$, based on some numerical experiments, we conjecture the following: the function $x \mapsto x^{1/2}e^{x}K_{\nu}(x)$ for all $|\nu| < \frac{1}{2}$ is a Bernstein function, i.e. its derivative with respect to x is SCM on $(0, \infty)$ for all $|\nu| < \frac{1}{2}$. If this result were true, then this would imply that for all $|\nu| < \frac{1}{2}$ the inequality (3.6) is reversed, and this reversed inequality would complete the reversed version of (3.1) and (3.3).

4. Bounds for quotients of some special functions that arise in finite elasticity

4.1. The monotonicity and convexity of w_ν and related comments

Let us consider the function $w_\nu\colon (0,\infty)\to\mathbb{R}$, defined by

$$w_\nu(x) = xI_\nu(x)/I_{\nu+1}(x),$$

which is of special interest in finite elasticity [**71**, **72**]. In 1984 Simpson and Spector [**71**] proved that w_ν is strictly increasing and strictly convex for all $\nu \geqslant 0$. The function w_0 and a special inequality were used to prove that a nonlinearly elastic cylinder eventually becomes unstable in uniaxial compression. Moreover, they proved that for any $\nu > 0$ the function w_ν has application in the buckling and necking of such cylinders. For more details the interested reader is referred to [**72**] and the references therein. Other inequalities for the function w_ν were deduced in [**19**, **60**]. We note that the monotonicity property of w_ν was actually also deduced in 1983 by Watson [**77**], by showing that $x \mapsto 1/w_\nu(x)$ is decreasing on $(0,\infty)$ for all $\nu \geqslant 0$, and this result was used by Robert [**67**] in 1990 in the study of a Bayesian estimation problem. It is worth mentioning here that, in fact, w_ν is strictly increasing for all $\nu > -1$ and consequently the function $x \mapsto 1/w_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu > -1$. To prove this, observe that by using (1.1) we have

$$w_\nu(x) = \sum_{n\geqslant 0}\frac{x^{2n}}{n!\Gamma(n+\nu+1)2^{2n+\nu}}\bigg/\sum_{n\geqslant 0}\frac{x^{2n}}{n!\Gamma(n+\nu+2)2^{2n+\nu+1}},$$

and in view of Lemma 2.1 it is enough to show that the sequence $\{\alpha_n\}_{n\geqslant 0}$, defined by

$$\alpha_n = \frac{n!\Gamma(n+\nu+2)2^{2n+\nu+1}}{n!\Gamma(n+\nu+1)2^{2n+\nu}} = 2(n+\nu+1)$$

for all $n \in \{0,1,2,\dots\}$, is strictly increasing for all $\nu > -1$, which is clearly true. This was stated in 2007 by Baricz and Neuman [**19**]. However, another way to deduce the above monotonicity property involves the series expansion of $x \mapsto 1/w_\nu(x)$. More precisely, by using the Mittag–Leffler expansion [**33**, Equation (7.9.3)]

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} = \sum_{n\geqslant 1}\frac{2x}{x^2+j_{\nu,n}^2},$$

one has

$$\left[\frac{1}{w_\nu(x)}\right]' = \left[\sum_{n\geqslant 1}\frac{2}{x^2+j_{\nu,n}^2}\right]' = \sum_{n\geqslant 1}\left[\frac{2}{x^2+j_{\nu,n}^2}\right]' = -\sum_{n\geqslant 1}\frac{4x}{(x^2+j_{\nu,n}^2)^2} < 0$$

for all $\nu > -1$ and $x > 0$, as we required. These remarks complete the results from [**71**,**77**].

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Moreover, we note that the monotonicity of $x \mapsto w_\nu(x)$ can also be deduced directly from the first part of Theorem 2.2. To be more precise, recall that the function $x \mapsto x^{\mu-\nu}I_\nu(x)/I_\mu(x)$ is strictly increasing on $(0,\infty)$ for all $\mu > \nu > -1$. Choosing $\mu = \nu+1$, the asserted result immediately follows. A related, much more deep result was deduced in 1979 by Ismail and Kelker [**45**], who, in order to introduce a new continuous infinitely divisible probability distribution, showed that the function $x \mapsto x^{(\nu-\mu)/2}I_\mu(\sqrt{x})/I_\nu(\sqrt{x})$ is SCM on $(0,\infty)$ for all $\mu > \nu > -1$. We note that the first-order monotonicity of $x \mapsto x^{(\nu-\mu)/2}I_\mu(\sqrt{x})/I_\nu(\sqrt{x})$ also follows from the first part of Theorem 2.2 and, as in [**71**], it was pointed out that this in particular yields that $x \mapsto 1/w_\nu(\sqrt{x})$, and hence $x \mapsto 1/w_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu > -1$. Motivated by the above result of Ismail and Kelker, we conjecture that *the function* $x \mapsto x^{\nu-\mu}I_\mu(x)/I_\nu(x)$ *is SCM on* $(0,\infty)$ *for all* $\mu > \nu > -1$. From this we would again obtain Theorem 2.2 (i) and the above result in [**45**].

Another interesting fact here is that for $x > 0$ the inequality $w'_\nu(x) > 0$ is equivalent to $xI'_\nu(x)/I_\nu(x) > x-1$. The latter inequality was proved recently by Baricz and Sun [**21**] for $\nu \geqslant 0$, by using some results of Gronwall [**36**] and Nåsell [**58**]. From the above discussion we deduce that the inequality $xI'_\nu(x)/I_\nu(x) > x-1$ actually holds for all $\nu > -1$ and $x > 0$. This inequality was the crucial fact in the proof of (2.7) and implies that the function $x \mapsto \mathrm{e}^{-x}xI_\nu(x)$ is strictly increasing on $(0,\infty)$ for all $\nu > -1$. Indeed, since

$$\frac{[\mathrm{e}^{-x}xI_\nu(x)]'}{\mathrm{e}^{-x}I_\nu(x)} = \frac{xI'_\nu(x)}{I_\nu(x)} + 1 - x > 0,$$

the asserted result follows. Consequently, we have that, for all $\nu > -1$ and $0 < x < y$,

$$\frac{I_\nu(x)}{I_\nu(y)} < \mathrm{e}^{x-y}\left(\frac{y}{x}\right). \tag{4.1}$$

Observe that this upper bound for $\nu \geqslant 1$ improves Laforgia's upper bound (2.4); for $\nu \geqslant \frac{1}{2}$ (4.1) is weaker than (2.6) and for $\nu \geqslant 0$ (4.1) is more stringent than the upper bound in (2.7). Moreover, since the function $x \mapsto x^\nu/\sinh x$ is strictly decreasing on $(0,\infty)$ for all $\nu \in [0,1]$, we conclude that for $\nu \in [0,1]$ the upper bound in (2.8) is more stringent than the upper bound in (4.1). However, for $\nu > 1$ the bounds in (2.8) and (4.1) are not directly comparable, since for $\nu > 1$ the derivative of the function $x \mapsto x^\nu/\sinh x$ changes sign on $(0,\infty)$. All the same, the inequality (4.1) has the virtue that it holds for all $\nu > -1$, while the others do not.

We note that, in the proof of (2.11), the key tool [**17**] was the fact that the function w_ν is strictly increasing and strictly convex for all $\nu \geqslant 0$. Now, since the function w_ν is actually strictly increasing for all $\nu > -1$, it is natural to ask whether it is strictly convex for all $\nu > -1$. Based on numerical experiments, we conjecture the following: *the function* w_ν *is strictly convex for all* $\nu > -1$. If this conjecture were true, then following the proof from [**17**] would imply that (2.11) holds for all $\nu > -1$.

Now, let us study the monotonicity of w_ν with respect to ν. Recall that by using Theorem 2.2 (i) we obtain that, for all $\mu > \nu > -1$ and $x > 0$, the Turán-type inequality $I_{\nu+1}(x)I_\mu(x) - I_{\mu+1}(x)I_\nu(x) > 0$ holds. Clearly, this is equivalent to $w_\mu(x) > w_\nu(x)$,

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where $\mu > \nu > -1$ and $x > 0$, i.e. the function $\nu \mapsto w_\nu(x)$ is strictly increasing on $(-1, \infty)$ for each fixed $x > 0$. See also the proof of (2.8) in [**17**] for further details. Actually, this can also be deduced from the above Mittag–Leffler expansion in view of

$$\frac{\partial}{\partial\nu}\left[\frac{1}{w_\nu(x)}\right] = \frac{\partial}{\partial\nu}\left[\sum_{n\geqslant 1}\frac{2}{x^2+j_{\nu,n}^2}\right] = \sum_{n\geqslant 1}\frac{\partial}{\partial\nu}\left[\frac{2}{x^2+j_{\nu,n}^2}\right] = -\sum_{n\geqslant 1}\frac{4j_{\nu,n}\partial j_{\nu,n}/\partial\nu}{(x^2+j_{\nu,n}^2)^2} < 0,$$

where $\nu > -1$ and $x > 0$. Here we have used the well-known fact that the function $\nu \mapsto j_{\nu,n}$ is strictly increasing [**24**] on $(-1, \infty)$ for each $n \in \{1, 2, 3, \dots\}$ fixed. Moreover, by using the above idea in what follows, we show that the function $\nu \mapsto w_\nu(x)$ is log-concave on $(-1, \infty)$ for all $x > 0$ fixed. This can be verified by proving that $\nu \mapsto 1/w_\nu(x)$ is log-convex on $(-1, \infty)$ for all $x > 0$ fixed. More precisely, due to [**32**] it is known that $\nu \mapsto j_{\nu,n}$ is concave on $(-n, \infty)$ for all $n \in \{1, 2, 3 \dots\}$ and, consequently, we have that $\nu \mapsto j_{\nu,n}$ and $\nu \mapsto \log j_{\nu,n}$ are concave on $(-1, \infty)$ for all $n \in \{1, 2, 3, \dots\}$. This in turn implies that for each $n \in \{1, 2, 3, \dots\}$ and $x > 0$ the function $\nu \mapsto x^2 + j_{\nu,n}^2$ is log-concave, and hence each term in the above expansion of $1/w_\nu(x)$ is log-convex with respect to ν. Because the sum of log-convex functions is log-convex too, it follows that indeed $\nu \mapsto 1/w_\nu(x)$ is log-convex on $(-1, \infty)$ for all $x > 0$ fixed. Since, $\nu \mapsto w_\nu(x)$ is log-concave, it follows that the following new interesting Turán-type inequality:

$$w_{\nu+1}^2(x) > w_\nu(x)w_{\nu+2}(x) \quad\Longleftrightarrow\quad I_{\nu+1}^3(x)I_{\nu+3}(x) > I_\nu(x)I_{\nu+2}^3(x)$$

holds for all $\nu > -1$ and $x > 0$. This completes the results from [**12, 15, 16, 48, 55**].

4.2. Bounds for quotients of the function w_ν

Now, let us focus on some inequalities involving the ratio $w_\nu(x)/w_\nu(y)$, where $0 < x < y$. By using the left-hand side of (2.1) or (2.2), the inequality (2.4) and the relation

$$\frac{w_\nu(x)}{w_\nu(y)} = \frac{x}{y}\frac{I_\nu(x)}{I_\nu(y)}\frac{I_{\nu+1}(y)}{I_{\nu+1}(x)},$$

Laforgia [**51**] deduced the following result:

$$\frac{w_\nu(x)}{w_\nu(y)} > \left(\frac{x}{y}\right)^{2(\nu+1)}, \tag{4.2}$$

which holds for all $\nu \geqslant -\frac{1}{2}$ and $0 < x < y$. By using a similar approach, Joshi and Bissu [**49**] proved another result, i.e.

$$\frac{w_\nu(x)}{w_\nu(y)} > \left(\frac{y}{x}\right)^{\nu}\frac{I_\nu(x)}{I_\nu(y)}, \tag{4.3}$$

which holds for all $\nu > -\frac{1}{2}$ and $0 < x < y$.

Furthermore, Joshi and Bissu [**49**] claimed that for all $\nu > -\frac{1}{2}$ and $0 < x < y$ their lower bound in (4.3) is more stringent than the lower bound in (4.2). We point out here that the above statement is not true. More precisely, observe that it is enough to show

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that the function $x \mapsto x^{-3\nu-2}I_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu > -\frac{1}{2}$. Recall that for all $\nu > -\frac{1}{2}$ the function $\mathrm{e}^{-x}x^{-\nu}I_\nu(x)$ is strictly decreasing on $(0,\infty)$. Now, since the function $x \mapsto \mathrm{e}^{x}x^{-2(\nu+1)}$ is also strictly decreasing for all $\nu > -1$ and $x < 2(\nu+1)$, we obtain that indeed the function $x \mapsto x^{-3\nu-2}I_\nu(x)$ is strictly decreasing for all $\nu > -\frac{1}{2}$ and $x < 2(\nu+1)$. Consequently, if $0 < x < y < 2(\nu+1)$, then (4.3) does indeed improve (4.2). However, for larger values of x and y the situation is reversed. This is justified by the following fact: the function $x \mapsto x^{-2(\nu+1)}\mathcal{I}_\nu(x)$ maps $(0,\infty)$ into $(0,\infty)$, is strictly log-convex and hence is strictly convex on $(0,\infty)$ for all $\nu > -\frac{1}{2}$. Moreover, it is easy to see that

$$\lim_{x\to 0} x^{-2(\nu+1)}\mathcal{I}_\nu(x) = \lim_{x\to\infty} x^{-2(\nu+1)}\mathcal{I}_\nu(x) = \infty.$$

Hence, there exists an $x_0 \in (0,\infty)$ such that the function $x \mapsto x^{-3\nu-2}I_\nu(x)$ becomes strictly increasing on (x_0,∞), and this means that in this domain actually (4.2) improves (4.3). Here we have used the well-known fact that $x \mapsto x^{-2(\nu+1)}$ is strictly log-convex on $(0,\infty)$ for all $\nu > -1$ and $x \mapsto \mathcal{I}_\nu(x)$ is strictly log-convex on $(0,\infty)$ for all $\nu > -\frac{1}{2}$ [**60**]. All the same, by combining the inequalities (2.6) and (2.19) we obtain that

$$\frac{w_\nu(x)}{w_\nu(y)} > \frac{\mathrm{e}^{y}\cosh x}{\mathrm{e}^{x}\cosh y}\left(\frac{x}{y}\right)^{\nu+3/2} \tag{4.4}$$

holds for all $\nu > -\frac{1}{2}$ and $0 < x < y$. Since the function $x \mapsto x^{-\nu-1/2}\mathrm{e}^{-x}\cosh x$ is strictly decreasing on $(0,\infty)$ for all $\nu > -\frac{1}{2}$, as a product of the strictly decreasing functions $x \mapsto x^{-\nu-1/2}$ and $x \mapsto \mathrm{e}^{-x}\cosh x$, we obtain that for $\nu > -\frac{1}{2}$ the lower bound in (4.4) is tighter than the lower bound in (4.2). Furthermore, by using (2.6) and (2.20) for $\nu > \frac{1}{2}$ the inequality (4.4) can be improved in the following way:

$$\frac{w_\nu(x)}{w_\nu(y)} > \frac{\mathrm{e}^{y}\sinh x}{\mathrm{e}^{x}\sinh y}\left(\frac{x}{y}\right)^{\nu+1/2},$$

where $\nu > \frac{1}{2}$ and $0 < x < y$. Here we have used that from the first part of Theorem 2.2 we know that the function $x \mapsto \mathcal{I}_{1/2}(x)/\mathcal{I}_{-1/2}(x)$ is strictly decreasing on $(0,\infty)$.

Another improvement of (4.2) reads as follows:

$$\frac{w_\nu(x)}{w_\nu(y)} > \frac{\mathrm{e}^{y}(\mathrm{e}^{x}+\lambda_\nu)}{\mathrm{e}^{x}(\mathrm{e}^{y}+\lambda_\nu)}\left(\frac{x}{y}\right)^{\nu+3/2}, \tag{4.5}$$

where $\nu \geqslant 0$ and $0 < x < y$. This can be deduced by combining the inequalities (2.11) and (2.19). Now, since the function $x \mapsto (1+\lambda_\nu\mathrm{e}^{-x})x^{-\nu-1/2}$ is strictly decreasing on $(0,\infty)$ for all $\nu \geqslant 0$, we deduce that for $\nu \geqslant 0$ the inequality (4.5) improves (4.2) significantly. Here we have used that for all $\nu \geqslant 0$ the constant λ_ν is strictly positive [**17**]. Moreover, if the above conjecture on the convexity of w_ν is true, then (4.5) holds for all $\nu > -\frac{1}{2}$ and still improves (4.2).

Now, combining once again the left-hand side of (2.1) or (2.2) with the inequality (2.4) we obtain [**51**]

$$\frac{w_\nu(x)}{w_\nu(y)} < \left(\frac{y}{x}\right)^{2\nu} \tag{4.6}$$

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for all $\nu \geqslant \frac{1}{2}$ and $0 < x < y$. In [**49**] Joshi and Bissu showed that (4.6) can be improved as follows:

$$\frac{w_\nu(x)}{w_\nu(y)} < \mathrm{e}^{y-x}\left(\frac{y}{x}\right)^{\nu}\frac{I_\nu(x)}{I_\nu(y)}, \tag{4.7}$$

where $\nu > -\frac{1}{2}$ and $0 < x < y$. By using the inequality (2.4) we obtain immediately that for $\nu \geqslant \frac{1}{2}$ the upper bound in (4.7) is more stringent than the upper bound in (4.6). It is worth mentioning that by combining the inequalities (2.6) and (2.19) we get the inequality

$$\frac{w_\nu(x)}{w_\nu(y)} < \frac{\mathrm{e}^x \cosh y}{\mathrm{e}^y \cosh x}\left(\frac{y}{x}\right)^{\nu-1/2}, \tag{4.8}$$

which holds for all $\nu \geqslant \frac{1}{2}$ and $0 < x < y$. Since the function $x \mapsto x^{\nu+1/2}\mathrm{e}^x/\cosh x$ is strictly increasing on $(0,\infty)$ we obtain that the upper bound in (4.8) is tighter than the upper bound in (4.6). However, since the derivative of the function $x \mapsto \mathrm{e}^{-2x}x^{-1/2}I_\nu(x)\cosh x$ changes sign for $\nu \geqslant \frac{1}{2}$, the inequalities (4.7) and (4.8) are not directly comparable on $(0,\infty)$. Observe that by using the inequalities (2.6) and (2.20) the inequality (4.8) can be improved as follows:

$$\frac{w_\nu(x)}{w_\nu(y)} < \frac{\mathrm{e}^x \sinh y}{\mathrm{e}^y \sinh x}\left(\frac{y}{x}\right)^{\nu-3/2}, \tag{4.9}$$

where $\nu \geqslant \frac{1}{2}$ and $0 < x < y$. Similarly, since the derivative of the function $x \mapsto \mathrm{e}^{-2x}x^{-3/2}I_\nu(x)\sinh x$ changes sign for $\nu \geqslant \frac{1}{2}$, the inequalities (4.7) and (4.9) are not directly comparable on $(0,\infty)$.

Combining (2.6) with (2.11) we obtain the inequality

$$\frac{w_\nu(x)}{w_\nu(y)} < \frac{\mathrm{e}^x(\mathrm{e}^y + \lambda_{\nu+1})}{\mathrm{e}^y(\mathrm{e}^x + \lambda_{\nu+1})}\left(\frac{y}{x}\right)^{\nu+1/2}, \tag{4.10}$$

where $\nu \geqslant \frac{1}{2}$ and $0 < x < y$. Since the function $x \mapsto (1 + \lambda_{\nu+1}\mathrm{e}^{-x})^{-1}x^{\nu-1/2}$ is strictly increasing on $(0,\infty)$ for all $\nu \geqslant \frac{1}{2}$, as a product of the increasing function $x \mapsto x^{\nu-1/2}$ and the strictly increasing function $x \mapsto (1 + \lambda_{\nu+1}\mathrm{e}^{-x})^{-1}$, we deduce that the upper bound in (4.10) is tighter than the Laforgia upper bound in (4.6). However, since the derivative of the function $x \mapsto (\mathrm{e}^x + \lambda_{\nu+1}\mathrm{e}^{-2x})x^{1/2}I_\nu(x)$ changes sign for $\nu \geqslant \frac{1}{2}$, the inequalities (4.7) and (4.10) are not directly comparable on $(0,\infty)$.

Finally, we note that some other bounds were deduced in [**49**] by using the inequality (2.13). Moreover, more lower and upper bounds that are tighter than in [**49**, **51**] can be deduced for the ratio $w_\nu(x)/w_\nu(y)$ in the same manner by using the inequality (4.1) instead of (2.6).

4.3. Further open problems and related comments

As in the section above we close this discussion with an interesting open problem. For this recall that, by definition [**75**], a sequence $\{\alpha_n\}_{n\geqslant 0}$ of real numbers is called completely monotone (CM) if $\Delta^k\alpha_n \geqslant 0$ for all $n, k \in \{0, 1, 2, \dots\}$, where $\Delta^k\alpha_n$ is defined

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inductively by $\Delta^0\alpha_n = \alpha_n$ and $\Delta^{k+1}\alpha_n = \Delta^k\alpha_n - \Delta^k\alpha_{n+1}$. Note that for a CM sequence $\{\alpha_n\}_{n\geqslant 0}$ we always have $\Delta^k\alpha_n > 0$ unless $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. A sequence $\{\alpha_n\}_{n\geqslant 0}$ that satisfies $\Delta^k\alpha_n > 0$ for all $n, k \in \{0, 1, 2, \dots\}$ is said to be strictly completely monotone (SCM). Note that a sequence $\{\alpha_n\}_{n\geqslant 0}$ is said to be convex if, for all $n \in \{0, 1, 2, \dots\}$, we have $\Delta^2\alpha_n \geqslant 0$ and is said to be concave if $\Delta^2\alpha_n \leqslant 0$. Similarly, a sequence $\{\beta_n\}_{n\geqslant 0}$ is said to be log-convex if $\beta_{n+1}^2 \leqslant \beta_n\beta_{n+2}$ for all $n \in \{0, 1, 2, \dots\}$, and is log-concave if the above inequality is reversed. If the above inequalities are strict, then we say that the corresponding sequence is strictly convex (concave) and log-convex (log-concave), respectively. Now, let us consider the *Maclaurin series* $f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n + \cdots$, where, for all integers $n \geqslant 0$, $a_n \in \mathbb{R}$ and $b_n > 0$, and suppose that both converge on $(-r, r)$, where $0 < r \leqslant \infty$. Motivated by the previous open problems of this paper we ask the following questions.

(a) If the sequence $\{a_n/b_n\}_{n\geqslant 0}$ is (strictly) convex (concave), is the function $x \mapsto f(x)/g(x)$ also (strictly) convex (concave) on $(0, r)$?

(b) If the sequence $\{a_n/b_n\}_{n\geqslant 0}$ is (strictly) log-convex (log-concave), is the function $x \mapsto f(x)/g(x)$ also (strictly) log-convex (log-concave) on $(0, r)$?

(c) If the sequence $\{a_n/b_n\}_{n\geqslant 0}$ is SCM (CM), is the function $x \mapsto f(x)/g(x)$ also SCM (CM) on $(0, r)$?

These problems seem to be quite difficult to solve and we note that by solving any one of them many other interesting open and known problems in the theory of special functions could be proved simply and uniquely. Clearly, part (a) and the first-order monotonicity, stated in Lemma 2.1, are included in part (c). Thus, part (c) actually asks whether or not the lemma of Biernacki and Krzyż can be generalized in a natural way. If this part were to be true, then we would easily have that the function $x \mapsto x^{\nu-\mu}I_\mu(x)/I_\nu(x)$ is indeed SCM on $(0, \infty)$ for all $\mu > \nu > -1$. It would be enough to show that the sequence $\{\beta_n\}_{n\geqslant 0}$, defined by $\beta_n = (\nu + 1)_n/(\mu + 1)_n$, is SCM. However, for all $n \in \{0, 1, 2, \dots\}$ we have $(\mu + n + 1)\beta_{n+1} = (\nu + n + 1)\beta_n$ and, by using mathematical induction, this in turn implies that

$$\Delta^k\beta_n = \frac{(\mu - \nu)_k}{(\mu + n + 1)_k}\beta_n > 0$$

for all $n, k \in \{0, 1, 2, \dots\}$ and $\mu > \nu > -1$. Similarly, question (b) would imply that the function w_ν is strictly log-concave and strictly convex for all $\nu > -1$.

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Note added in proof.

After this paper was already in the press we found the papers [**42**, **52**, **53**], which are closely related to this paper. In these papers some other interesting bounds were obtained for the quotients $I_\nu(x)/I_\nu(y)$ and $K_\nu(x)/K_\nu(y)$. The inequalities (2.14) and (2.19) also appear in [**42**]; however, the proofs given there are completely different from those in this paper. Finally, we note that the monotonicity property of the function $x \mapsto \exp\{-x^2/[4(\nu+1)]\}\mathcal{I}_\nu(x)$, proved here, has recently been used by Baricz and Ponnusamy [**20**] in the study of the starlikeness and convexity of generalized Bessel functions.

References

1. H. Alzer and C. Berg, Some classes of completely monotonic functions, *Annales Acad. Sci. Fenn. Math.* **27**(2) (2002), 445–460.
2. H. Alzer and C. Berg, Some classes of completely monotonic functions, II, *Ramanujan J.* **11**(2) (2006), 225–248.
3. H. Alzer and S.-L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, *J. Computat. Appl. Math.* **172**(2) (2004), 289–312.
4. D. E. Amos, Computation of modified Bessel functions and their ratios, *Math. Comput.* **28** (1974), 239–251.
5. G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Analysis Applic.* **335**(2) (2007), 1294–1308.
6. S. András and Á. Baricz, Monotonicity property of generalized and normalized Bessel functions of complex order, *Complex Variables Ellipt. Eqns* **54**(7) (2009), 689–696.
7. R. Balasubramanian, S. Ponnusamy and M. Vuorinen, Functional inequalities for the quotients of hypergeometric functions, *J. Math. Analysis Applic.* **218**(1) (1998), 256–268.
8. Á. Baricz, Landen-type inequality for Bessel functions, *Computat. Meth. Funct. Theory* **5**(2) (2005), 373–379.
9. Á. Baricz, Functional inequalities involving special functions, *J. Math. Analysis Applic.* **319**(2) (2006), 450–459.
10. Á. Baricz, Functional inequalities involving special functions, II, *J. Math. Analysis Applic.* **327**(2) (2007), 1202–1213.
11. Á. Baricz, Turán-type inequalities for generalized complete elliptic integrals, *Math. Z.* **256**(4) (2007), 895–911.
12. Á. Baricz, Functional inequalities for Galué's generalized modified Bessel functions, *J. Math. Inequal.* **1**(2) (2007), 183–193.
13. Á. Baricz, Generalized Bessel functions of the first kind, PhD Thesis, Babeş-Bolyai University, Cluj-Napoca (2008).
14. Á. Baricz, Turán-type inequalities for hypergeometric functions, *Proc. Am. Math. Soc.* **136**(9) (2008), 3223–3229.
15. Á. Baricz, Functional inequalities involving Bessel and modified Bessel functions of the first kind, *Expo. Math.* **26**(3) (2008), 279–293.
16. Á. Baricz, On a product of modified Bessel functions, *Proc. Am. Math. Soc.* **137**(1) (2009), 189–193.
17. Á. Baricz, Tight bounds for the generalized Marcum Q-function, *J. Math. Analysis Applic.* **360**(1) (2009), 265–277.
18. Á. Baricz, Turán-type inequalities for some probability density functions, *Studia Sci. Math. Hungar.* **47**(2) (2010), 175–189.
19. Á. Baricz and E. Neuman, Inequalities involving modified Bessel functions of the first kind, II, *J. Math. Analysis Applic.* **332**(1) (2007), 265–271.
20. Á. Baricz and S. Ponnusamy, Starlikeness and convexity of generalized Bessel functions, *Integ. Trans. Special Funct.*, in press (doi: 10.1080/10652460903516736).

21. Á. Baricz and Y. Sun, New bounds for the generalized Marcum Q-function, *IEEE Trans. Inform. Theory* **55**(7) (2009), 3091–3100.
22. Á. Baricz and Y. Sun, Bounds for the generalized Marcum Q-function, *Appl. Math. Computat.*, in press.
23. M. Biernacki and J. Krzyż, On the monotonity of certain functionals in the theory of analytic functions, *Annales Univ. Mariae Curie-Skłodowska* A **9** (1955), 135–147.
24. M. Bôcher, On certain methods of Sturm and their application to the roots of Bessel's functions, *Bull. Am. Math. Soc.* **3** (1987), 205–213.
25. S. Bochner, *Harmonic analysis and the theory of probability* (University of California Press, Berkeley, CA, 1960).
26. D. J. Bordelon, Problem 72-15, inequalities for special functions, *SIAM Rev.* **15** (1973), 665–668.
27. A. V. Boyd, The monotonicity of $I_1(x)/I_0(x)$, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **209** (1967), 35–36.
28. B. C. Carlson, *Special functions of applied mathematics* (Academic Press, New York, 1977).
29. J. A. Cochran, The monotonicity of modified Bessel functions with respect to their order, *J. Math. Phys.* **46** (1967), 220–222.
30. J. M. De Sitter and M. J. Goovaerts, On two inequalities satisfied by Bessel functions, *Bull. Belg. Math. Soc. Simon Stevin* **46** (1972), 159–164.
31. J. Dubourdieu, Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace–Stieltjes, *Compositio Math.* **7** (1939), 96–111.
32. Á. Elbert, Concavity of the zeros of Bessel functions, *Studia Sci. Math. Hungar.* **12** (1977), 81–88.
33. A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher transcendental functions*, Volume 2 (McGraw-Hill, New York, 1954).
34. I. Gargantinia and P. Henrici, A continued fraction algorithm for the computation of higher transcendental functions in the complex plane, *Math. Comp.* **21** (1967), 18–29.
35. C. Giordano, A. Laforgia and J. Pečarić, Supplements to known inequalities for some special functions, *J. Math. Analysis Applic.* **200**(1) (1996), 34–41.
36. T. H. Gronwall, An inequality for the Bessel functions of the first kind with imaginary argument, *Annals Math.* **33** (1932), 275–278.
37. S. L. Gupta, On an inequality for modified Bessel functions, *Univ. Roorkee Res. J.* **9** (1966), 17–19.
38. P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, *Trans. Am. Math. Soc.* **246** (1978), 1–30.
39. F. Hausdorff, Summationsmethoden und Momentfolgen, I, *Math. Z.* **9** (1921), 74–109.
40. V. Heikkala, M. K. Vamanamurthy and M. Vuorinen, Generalized elliptic integrals, *Computat. Meth. Funct. Theory* **9**(1) (2009), 75–109.
41. E. K. Ifantis and P. D. Siafarikas, Inequalities involving Bessel and modified Bessel functions, *J. Math. Analysis Applic.* **147**(1) (1990), 214–227.
42. E. K. Ifantis and P. D. Siafarikas, Bounds for modified Bessel functions, *Rend. Circ. Mat. Palermo* **40**(2) (1991), 347–356.
43. M. E. H. Ismail, Complete monotonicity of modified Bessel functions, *Proc. Am. Math. Soc.* **108**(2) (1990), 353–361.
44. M. E. H. Ismail, Remarks on a paper by Giordano, Laforgia and Pečarić, *J. Math. Analysis Applic.* **211**(2) (1997), 621–625.
45. M. E. H. Ismail and D. H. Kelker, Special functions, Stieltjes transforms and infinite divisibility, *SIAM J. Math. Analysis* **10**(2) (1979), 884–901.
46. M. E. H. Ismail and M. E. Muldoon, Certain monotonicity properties of Bessel functions, *J. Math. Analysis Applic.* **118**(1) (1986), 145–150.

https://doi.org/10.1017/S0013091508001016 Published online by Cambridge University Press

47. A. L. Jones, An extension of an inequality involving modified Bessel functions, *J. Math. Phys.* **47** (1968), 220–221.
48. C. M. Joshi and S. K. Bissu, Some inequalities of Bessel and modified Bessel functions, *J. Austral. Math. Soc.* A **50**(2) (1991), 333–342.
49. C. M. Joshi and S. K. Bissu, Inequalities for some special functions, *J. Computat. Appl. Math.* **69**(2) (1996), 251–259.
50. I. R. Kayumov, The spectrum of integral means and a modified Bessel function of zero order, *Algebra I Analiz* **17**(3) (2005), 107–123 (in Russian; English translation: *St Petersburg Math. J.* **17**(3) (2006), 453–463).
51. A. Laforgia, Bounds for modified Bessel functions, *J. Computat. Appl. Math.* **34**(3) (1991), 263–267.
52. A. Laforgia and M. L. Mathis, Bounds for Bessel functions, *Rend. Circ. Mat. Palermo* **38**(2) (1989), 319–328.
53. A. Laforgia and P. Natalini, Some inequalities for modified Bessel functions, *J. Inequal. Appl.* **2010** (2010), 253035.
54. L. Lorch, Inequalities for some Whittaker functions, *Arch. Math. (Brno)* **3** (1967), 1–9.
55. L. Lorch, Monotonicity of the zeros of a cross product of Bessel functions, *Meth. Applic. Analysis* **1**(1) (1994), 75–80.
56. Y. L. Luke, Inequalities for generalized hypergeometric functions, *J. Approx. Theory* **5** (1972), 41–65.
57. K. S. Miller and S. G. Samko, Completely monotonic functions, *Integ. Transf. Special Funct.* **12**(4) (2001), 389–402.
58. I. Nåsell, Inequalities for modified Bessel functions, *Math. Comp.* **28** (1974), 253–256.
59. I. Nåsell, Rational bounds for ratios of modified Bessel functions, *SIAM J. Math. Analysis* **9**(1) (1978), 1–11.
60. E. Neuman, Inequalities involving modified Bessel functions of the first kind, *J. Math. Analysis Applic.* **171**(2) (1992), 532–536.
61. R. B. Paris, An inequality for the Bessel function $J_\nu(\nu x)$, *SIAM J. Math. Analysis* **15**(1) (1984), 203–205.
62. R. Penfold, J.-M. Vanden-Broeck and S. Grandison, Monotonicity of some modified Bessel function products, *Integ. Transf. Special Funct.* **18** (2007), 139–144.
63. E. Petropoulou, Bounds for ratios of modified Bessel functions, *Integ. Transf. Special Funct.* **9**(4) (2000), 293–298.
64. T. Pogány, Further results on generalized Kapteyn-type expansions, *Appl. Math. Lett.* **22**(2) (2009), 192–196.
65. S. Ponnusamy and M. Vuorinen, Aymptotic expansions and inequalities for hypergeometric functions, *Mathematika* **44**(2) (1997), 43–64.
66. D. O. Reudink, On the signs of the ν-derivatives of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$, *J. Res. Nat. Bur. Standards* B **72** (1968), 279–280.
67. C. Robert, Modified Bessel functions and their applications in probability and statistics, *Statist. Prob. Lett.* **9**(2) (1990), 155–161.
68. D. K. Rosenthal, The shape and stability of a bubble at the axis of a rotating liquid, *J. Fluid Mech.* **12** (1962), 358–366.
69. D. K. Ross, The stability of a column of liquid to torsial oscillations, *Z. Angew. Math. Phys.* **21** (1970), 137–140.
70. D. K. Ross, Problem 72-15, inequalities for special functions, *SIAM Rev.* **15** (1973), 668–670.
71. H. C. Simpson and S. J. Spector, Some monotonicity results for ratios of modified Bessel functions, *Q. Appl. Math.* **42**(1) (1984), 95–98.
72. H. C. Simpson, S. J. Spector, On barelling for a special material in finite elasticity, *Q. Appl. Math.* **42**(1) (1984), 99–105.

73. S. M. Sitnik, Inequalities for Bessel functions, *Dokl. Akad. Nauk SSSR* **340**(1) (1995), 29–32.
74. R. P. Soni, On an inequality for modified Bessel functions, *J. Math. Phys.* **44** (1965), 406–407.
75. H. S. Wall, *Analytic theory of continued fractions* (Van Nostrand, New York, 1948).
76. G. N. Watson, *A treatise on the theory of Bessel functions* (Cambridge University Press, 1944).
77. G. S. Watson, *Statistics on spheres* (John Wiley & Sons, New York, 1983).
78. E. J. Weniger and J. Čížek, Rational approximations for the modified Bessel function of the second kind, *Comput. Phys. Commun.* **59**(3) (1990), 471–493.
79. D. V. Widder, *The Laplace transform* (Princeton University Press, 1941).

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