Can. J. Math., Vol. XXIII, No. 5, 1971, pp. 896–906

A CHARACTERIZATION OF SPIN REPRESENTATIONS

ROBERT B. BROWN

1. Introduction. Associated with a non-degenerate symmetric bilinear form on a vector space is a Clifford algebra and various Clifford groups, which have spin representations on minimal right ideals of the Clifford algebra. Several invariants for these representations have been known for some time. In this paper the forms are assumed to be "split", and several relations between the invariants are derived and promoted to the status of axioms. Then it is shown that any system satisfying the axioms comes from a minimal right ideal in a Clifford algebra and that the automorphism groups associated with the system are the Clifford groups. Hence, the axioms characterize spin representations.

A description of split forms and spin representations is in section two. In section three the invariants and their properties are described. The axiomatic characterization is developed in section four, and in the last section the automorphism groups are identified as the Clifford groups.

Acknowledgement. I am grateful to D. Z. Djoković for simplifications to the proof of Theorem 3.

2. Description of spin representations. Let (v, w) be a non-degenerate symmetric bilinear form on a vector space $\mathfrak{V}$ of dimension $k \geqq 2$ over a field of characteristic not two. If $\mathfrak{T}$ is the tensor algebra of $\mathfrak{V}$ and $\mathfrak{J}$ is the ideal generated by all of the elements $v \otimes v - (v, v)1$, then $\mathfrak{C} = \mathfrak{T}/\mathfrak{J}$ is the Clifford algebra of the bilinear form [**1**, pp. 186–193; **2**, p. 37; **3**, pp. 33–34].

With $\mathfrak{V}$ identified with the homogeneous component of degree one in $\mathfrak{T}$, the natural map of $\mathfrak{T}$ onto $\mathfrak{C}$ carries $\mathfrak{V}$ injectively onto a linear subspace of $\mathfrak{C}$ which shall also be denoted by $\mathfrak{V}$ [**2**, p. 39; **3**, p. 40, Theorem 3.4]. Denote by $\mathfrak{C}^+$ the subalgebra of $\mathfrak{C}$ spanned by products of an even number of elements in $\mathfrak{V}$.

An important feature of $\mathfrak{C}$ is its main anti-automorphism α. This is the map induced by the anti-automorphism of $\mathfrak{T}$ which carries $v_1 \ldots v_j$ to $v_j \ldots v_1$ for all j and all v_i in $\mathfrak{V}$ [**1**, p. 192; **2**, pp. 37–38; **3**, p. 42, Theorem 3.5].

The Clifford group Γ associated with the bilinear form is the group of elements c in $\mathfrak{C}$ which have inverses and for which $c\mathfrak{V}c^{-1} \subseteq \mathfrak{V}$ [**2**, p. 49; **3**, p. 46]. If c is in Γ, $\alpha(c)$ is in Γ also. Let Γ_0 be the subgroup of those c in Γ for which $c\alpha(c) = 1$ [**2**, p. 52]. We define the special Clifford group to be $\Gamma^+ = \Gamma \cap \mathfrak{C}^+$ and the reduced Clifford group to be $\Gamma_0{}^+ = \Gamma_0 \cap \mathfrak{C}^+$ [**2**, pp. 51–52; **3**, p. 47].

Received June 4, 1971. This research was partially supported by National Science Foundation Grant GP-14066 and National Research Council Grant A7536.

Definition. A *split form* is a non-degenerate symmetric bilinear form (v, w) on a vector space $\mathfrak{V}$ such that (i) $\dim \mathfrak{V} = 2l$ and there is a basis

$$u_1, \ldots, u_l, w_1, \ldots, w_l$$

for $\mathfrak{V}$ such that for $i, j = 1, \ldots, l$,

$$(u_i, u_j) = (w_i, w_j) = 0 \quad \text{and} \quad (u_i, w_j) = \delta_{ij};$$

or (ii) $\dim \mathfrak{V} = 2l + 1$ and there is a basis $t, u_1, \ldots, u_l, w_1, \ldots, w_l$ for $\mathfrak{V}$ such that for $i, j = 1, \ldots, l$,

$$(t, t) = 1,\ (t, u_i) = (t, w_i) = 0,$$
$$(u_i, u_j) = (w_i, w_j) = 0, \quad \text{and} \quad (u_i, w_j) = \delta_{ij}.$$

Bases satisfying these conditions will be called splitting bases. In $\mathfrak{C}$ the elements of a splitting basis satisfy

$$u_i^2 = w_i^2 = 0,\ u_i w_i + w_i u_i = 2,$$
$$u_i u_j = -u_j u_i,\ w_i w_j = -w_j w_i,\ u_i w_j = -w_j u_i, \text{ if } i \neq j,$$
$$t^2 = 1,\ u_i t = -t u_i,\ w_i t = -t w_i.$$

If $P = \{\alpha, \beta, \ldots, \gamma\}$ with $1 \leqq \alpha < \beta < \ldots < \gamma \leqq l$, define $u(P) = u_\alpha u_\beta \ldots u_\gamma$ and $w(P) = w_\alpha w_\beta \ldots w_\gamma$. Further, define $u = u_1 u_2 \ldots u_l$ if $\dim \mathfrak{V} = 2l$ and $u = u_1 u_2 \ldots u_l(1 + t)$ if $\dim \mathfrak{V} = 2l + 1$. The span of the 2^l elements $uw(P)$ is a minimal right ideal in $\mathfrak{C}$. Any element c of a Clifford group acts on $\mathfrak{N}$ via the right multiplication $R(c) : n \to nc$. This action defines the spin representations of the Clifford groups [**2**, pp. 55–57; **3**, p. 48; **4**, pp. 228–234].

3. Invariants on spin representations. The invariants we define and study for the actions of various Clifford groups on the minimal right ideal $\mathfrak{N}$ of the Clifford algebra are

(i) a linear transformation $S : \mathfrak{N} \otimes \mathfrak{N} \to \mathfrak{N} \otimes \mathfrak{N}$,
(ii) a non-degenerate bilinear form B on $\mathfrak{N}$.

A third invariant J will be defined in section five, where it is first used.

The transformation S is discussed for Euclidean spaces by Schouten [**5**, pp. 343–344]. He defines it as acting on $\mathfrak{C} \otimes \mathfrak{C}$ by choosing an orthonormal basis $v_1, \ldots, v_k$ of $\mathfrak{V}$ and setting $S(c \otimes d) = \sum_{i=1}^{k} cv_i \otimes dv_i$ for every c and d in $\mathfrak{C}$. He then shows that the Clifford group Γ is the group of non-singular linear transformations T on $\mathfrak{N}$ such that $S(T \otimes T) = (T \otimes T)S$. We generalize the definition slightly.

Definition. Let $v_1, \ldots, v_k$ be an arbitrary basis for a vector space $\mathfrak{V}$ with a non-degenerate symmetric bilinear form (v, w). Let $B = ||(v_i, v_j)||$ and let $||\alpha_{ij}|| = B^{-1}$. Define the linear transformation $S : \mathfrak{C} \otimes \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}$ by setting

$$S(c \otimes d) = \sum_{i,j=1}^{k} \alpha_{ij} cv_i \otimes dv_j$$

for all c and d in $\mathfrak{C}$. The invariant on $\mathfrak{N} \otimes \mathfrak{N}$ is the restriction of this S to $\mathfrak{N} \otimes \mathfrak{N}$. Notice that S is present on $\mathfrak{C} \otimes \mathfrak{C}$ whether or not the bilinear form on $\mathfrak{V}$ is split.

LEMMA 1. *S is independent of the choice of basis for $\mathfrak{V}$.*

Proof. The proof is a straightforward calculation and is omitted here.

In terms of a splitting basis, for all n and m in $\mathfrak{N}$,

$$S(n \otimes m) = \sum_{i=1}^{l} (nu_i \otimes mw_i + nw_i \otimes mu_i)$$

if $\dim \mathfrak{V} = 2l$; whereas if $\dim \mathfrak{V} = 2l + 1$, then

$$S(n \otimes m) = nt \otimes mt + \sum_{i=1}^{l} (nu_i \otimes mw_i + nw_i \otimes mu_i).$$

A scalar multiple of the bilinear form B is studied by Chevalley [**2**, pp. 77–80, pp. 108–111], who denotes it by β. The bilinear form B is the one guaranteed by the following lemma.

LEMMA 2. *On $\mathfrak{N}$ there is a unique non-degenerate bilinear form B such that*
(i) $B(u, uw_1 \ldots w_l) = 1$,
(ii) $B(u, uw(P)) = 0$ *if* $P \neq \{1, \ldots, l\}$,
(iii) $B(nv, m) = (-1)^l B(n, mv)$,
for all n and m in $\mathfrak{N}$, all v in $\mathfrak{V}$, and where $\dim \mathfrak{V}$ *is $2l$ or $2l + 1$.*

Proof. We define $B(uw(Q), uw(P))$ by induction on the number of indices in Q as follows. If Q is empty, use as definition (i) or (ii). Next let $uw(Q)$ be $uw_\alpha \ldots w_\beta w_\gamma$ where $\alpha < \beta < \ldots < \gamma$. Then let $B(uw(Q), uw(P)) = (-1)^l B(uw_\alpha \ldots w_\beta, uw(P)w_\gamma)$. It is easy to check that $B(uw(Q), uw(P)) = 0$ unless $\{1, \ldots, l\}$ is a disjoint union of Q and P. The verification of (iii) is omitted because it is a straightforward calculation, conveniently executed by examining three cases: $v = w_i$, $v = u_i$, and $v = t$.

The quadruple $\langle \mathfrak{N}, S, B, (-1)^l \rangle$ will be called a *spin system for* $\mathfrak{V}$ and the bilinear form on $\mathfrak{V}$. Some of the relations between the invariants are conveniently written using the linear transformation $\omega : \mathfrak{N} \otimes \mathfrak{N} \rightarrow \operatorname{Hom}(\mathfrak{N}, \mathfrak{N})$ defined by

$$\omega(\mathbf{z})n = (B \otimes 1)(1 \otimes S)(\mathbf{z} \otimes n)$$

for all $\mathbf{z}$ in $\mathfrak{N} \otimes \mathfrak{N}$ and all n in $\mathfrak{N}$ and the bilinear form $q(\mathbf{z}, \mathbf{w})$ on $\mathfrak{N} \otimes \mathfrak{N}$ defined by

$$q(\mathbf{z}, \mathbf{w}) = (B \otimes B)(1 \otimes S \otimes 1)(\mathbf{z} \otimes \mathbf{w}).$$

Here the identity map is denoted by 1. Recall that $\operatorname{rad} q$ denotes the set of all $\mathbf{z}$ in $\mathfrak{N} \otimes \mathfrak{N}$ for which $q(\mathbf{z}, \mathfrak{N} \otimes \mathfrak{N}) = q(\mathfrak{N} \otimes \mathfrak{N}, \mathbf{z}) = 0$ [**1**, p. 115]. Also let τ denote the linear transformation of $\mathfrak{N} \otimes \mathfrak{N} \rightarrow \mathfrak{N} \otimes \mathfrak{N}$ which sends every $n \otimes m$ to $m \otimes n$. The following theorem describes those properties of a spin system which will later be shown to characterize spin representations.

THEOREM 1. *Let $\mathfrak{V}$ be a vector space of dimension $k \geqq 2$ over a field of characteristic not two and possessing a split form. A spin system $\langle \mathfrak{N}, S, B, (-1)^l \rangle$ for $\mathfrak{V}$ has the following properties. For all* $\mathbf{z}$ *in* $\mathfrak{N} \otimes \mathfrak{N}$,

(1) $B(\omega(\mathbf{z}) \otimes 1) = (-1)^l B(1 \otimes \omega(\mathbf{z}))$,

(2) $S(1 \otimes \omega(\mathbf{z})) + (1 \otimes \omega(\mathbf{z}))S = 2\omega(\mathbf{z}) \otimes 1$,

(3) $S\tau = \tau S$,

(4) *q is symmetric,*

(5) *the factor space $\mathfrak{N} \otimes \mathfrak{N}/\mathrm{rad}\, q$ has dimension k and the bilinear form induced on it by q is a non-zero scalar multiple of a split form.*

Proof. Direct calculation will verify (1), (2), and (4). (3) follows obviously from the definition of S.

The verification of (5) is by induction on l. For $l = 1$, (5) follows directly from the following facts. First,

$$q(u \otimes u, u \otimes u) = q(uw_1 \otimes uw_1, uw_1 \otimes uw_1) = 0.$$

If $\dim \mathfrak{V} = 2$, then $q(u \otimes u, uw_1 \otimes uw_1) = 2$ and $\mathrm{rad}\, q$ is spanned by $u \otimes uw_1$ and $uw_1 \otimes u$. If $\dim \mathfrak{V} = 3$, then $q(u \otimes u, uw_1 \otimes uw_1) = 2$,

$$q(u \otimes u, u \otimes uw_1) = q(uw_1 \otimes uw_1, u \otimes uw_1) = 0,$$

$q(u \otimes uw_1, u \otimes uw_1) = -1$ and $\mathrm{rad}\, q$ is spanned by $u \otimes uw_1 - uw_1 \otimes u$.

Next suppose that $l > 1$. Then $\mathfrak{V}$ is the orthogonal direct sum $\mathfrak{V} = \mathfrak{V}' \perp (u_l, w_l)$. Here let $\mathfrak{V}'$ be the span of $u_1, w_1, \ldots, u_{l-1}, w_{l-1}$ (and t if $\dim \mathfrak{V}$ is odd), which comprise a splitting basis for $\mathfrak{V}'$ and yields a spin system $\langle \mathfrak{N}', S', B', (-1)^{l-1} \rangle$ for $\mathfrak{V}'$. Define two linear transformations ϕ_1 and ϕ_2 of $\mathfrak{N}'$ into $\mathfrak{N}$ as follows. In $\mathfrak{N}'$ let $u' = u_1 \ldots u_{l-1}$ if k is even and $u' = u_1 \ldots u_{l-1}(1 + t)$ if k is odd. For all $P \subseteq \{1, \ldots, l-1\}$ let

$$\begin{aligned} \phi_1(u'w(P)) &= (-1)^{l-1} u_l u' w(P) = uw(P), \\ \phi_2(u'w(P)) &= (-1)^{l-1} u_l u' w(P) w_l = uw(P) w_l, \end{aligned} \tag{6}$$

and extend by linearity. Then $\mathfrak{N} = \phi_1(\mathfrak{N}') \oplus \phi_2(\mathfrak{N}')$. It is a straightforward calculation to show that for all n, m, p, r in $\mathfrak{N}'$, for $i = 1, 2$, and for $j \neq i$

$$\begin{aligned} B(\phi_i n, \phi_i m) &= 0, \\ B(\phi_1 n, \phi_2 m) &= B'(n, m), \\ S(\phi_i n \otimes \phi_i m) &= (\phi_i \otimes \phi_i) S'(n \otimes m), \\ S(\phi_i n \otimes \phi_j m) &= -(\phi_i \otimes \phi_j) S'(n \otimes m) + 2\phi_j n \otimes \phi_i m, \\ q(\phi_i n \otimes \phi_i m, \phi_j p \otimes \phi_j r) &= 2B'(n, m) B'(p, r), \\ q(\phi_i n \otimes \phi_j m, \phi_i p \otimes \phi_j r) &= -q'(n \otimes m, p \otimes r), \\ q(\phi_i n \otimes \phi_j m, \phi_j p \otimes \phi_i r) &= (-1)^l q'(n \otimes m, p \otimes r). \end{aligned} \tag{7}$$

Also, q is zero whenever more than two of its arguments are images of the same ϕ_i. These equations show that for every $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathfrak{N}' \otimes \mathfrak{N}'$, $(\phi_i \otimes \phi_i)\mathbf{x}$ is in $\mathrm{rad}\, q$ except when $B'(\mathbf{x}) \neq 0$, and $(\phi_1 \otimes \phi_2)\mathbf{y} + (\phi_2 \otimes \phi_1)\mathbf{z}$

https://doi.org/10.4153/CJM-1971-099-8 Published online by Cambridge University Press

is in rad q if and only if $\mathbf{y} - (-1)^l\mathbf{z}$ is in rad q'. Then a complementary subspace to rad q will be spanned by $(\phi_1 \otimes \phi_2)\mathbf{y}$ where $\mathbf{y}$ runs through a complementary subspace to rad q' in $\mathfrak{N}' \otimes \mathfrak{N}'$ and the two elements $\mathbf{b}_i = (\phi_i \otimes \phi_i)\mathbf{a}$ where $\mathbf{a}$ is a fixed element in $\mathfrak{N}' \otimes \mathfrak{N}'$ for which $B'(\mathbf{a}, \mathbf{a}) \neq 0$. Then by induction $\mathfrak{N} \otimes \mathfrak{N}/\mathrm{rad}\, q$ has dimension k and the bilinear form induced on it by q is a scalar multiple of a split form because the elements $\mathbf{b}_i$ are orthogonal to the elements $(\phi_1 \otimes \phi_2)\mathbf{y}$ and $q(\mathbf{b}_i, \mathbf{b}_i) = 0$ and $q(\mathbf{b}_1, \mathbf{b}_2) \neq 0$.

4. Characterization of spin representations. In this section we define generalized spin systems by taking some of the properties of Theorem 1 as axioms. Some properties of generalized spin systems are derived, and then an abstract characterization of spin representations is given by showing that a generalized spin system which is "split" is isomorphic to a spin system for a vector space $\mathfrak{V}$ with a split form.

Definition. A *generalized spin system* is a four-tuple $\langle \mathfrak{N}, S, B, \epsilon \rangle$ consisting of a vector space $\mathfrak{N}$ of dimension at least two, a linear transformation $S : \mathfrak{N} \otimes \mathfrak{N} \to \mathfrak{N} \otimes \mathfrak{N}$, a bilinear form B on $\mathfrak{N}$, and a fixed scalar $\epsilon = +1$ or -1 such that for every $\mathbf{z}$ in $\mathfrak{N} \otimes \mathfrak{N}$

(i) $B(\omega(\mathbf{z}) \otimes 1) = \epsilon B(1 \otimes \omega(\mathbf{z}))$,

(ii) $S(1 \otimes \omega(\mathbf{z})) + (1 \otimes \omega(\mathbf{z}))S = 2\omega(\mathbf{z}) \otimes 1$,

(iii) $S\tau = \tau S$.

Here $\omega(\mathbf{z})n = (B \otimes 1)(1 \otimes S)(\mathbf{z} \otimes n)$ for every n in $\mathfrak{N}$ and τ is the linear transformation on $\mathfrak{N} \otimes \mathfrak{N}$ for which $\tau(n \otimes m) = m \otimes n$.

In any generalized spin system q will denote the bilinear form on $\mathfrak{N} \otimes \mathfrak{N}$ for which $q(\mathbf{z}, \mathbf{w}) = (B \otimes B)(1 \otimes S \otimes 1)(\mathbf{z} \otimes \mathbf{w})$ for every $\mathbf{z}$ and $\mathbf{w}$ in $\mathfrak{N} \otimes \mathfrak{N}$. Notice that

$$q(\mathbf{z}, \mathbf{w}) = B(\omega(\mathbf{z}) \otimes 1)\mathbf{w}. \tag{8}$$

LEMMA 3. *Let $\langle \mathfrak{N}, S, B, \epsilon \rangle$ be a generalized spin system. Then for every $\mathbf{z}$ and $\mathbf{w}$ in $\mathfrak{N} \otimes \mathfrak{N}$.*

(i) $S(\omega(\mathbf{z}) \otimes 1) + (\omega(\mathbf{z}) \otimes 1)S = 2(1 \otimes \omega(\mathbf{z}))$,

(ii) $\omega(\mathbf{z})\omega(\mathbf{w}) + \omega(\mathbf{w})\omega(\mathbf{z}) = 2\epsilon q(\mathbf{z}, \mathbf{w})1$, *and q is a symmetric bilinear form,*

(iii) $S(\omega(\mathbf{z}) \otimes \omega(\mathbf{z})) = (\omega(\mathbf{z}) \otimes \omega(\mathbf{z}))S$.

Furthermore, if the characteristic of the field is not two and if $\dim \mathfrak{N} \otimes \mathfrak{N}/\mathrm{rad}\, q$ *is $2l$ or $2l + 1$ for $l \geqq 1$, then 2^l divides the dimension of $\mathfrak{N}$.*

Proof. For (i), calculate

$$\begin{aligned} S(\omega(\mathbf{z}) \otimes 1) + (\omega(\mathbf{z}) \otimes 1)S &= S\tau(1 \otimes \omega(\mathbf{z}))\tau + \tau(1 \otimes \omega(\mathbf{z}))\tau S \\ &= \tau S(1 \otimes \omega(\mathbf{z}))\tau + \tau(1 \otimes \omega(\mathbf{z}))S\tau \\ &= \tau 2(\omega(\mathbf{z}) \otimes 1)\tau = 2(1 \otimes \omega(\mathbf{z})). \end{aligned}$$

https://doi.org/10.4153/CJM-1971-099-8 Published online by Cambridge University Press

To verify (ii), calculate $\omega(\mathbf{w})\omega(\mathbf{z})$. For all n in $\mathfrak{N}$,

$$\begin{aligned}\omega(\mathbf{w})\omega(\mathbf{z})n &= (B \otimes 1)(1 \otimes S)[\mathbf{w} \otimes \omega(\mathbf{z})n] \\ &= (B \otimes 1)(1 \otimes S(1 \otimes \omega(\mathbf{z})))\mathbf{w} \otimes n \\ &= (B \otimes 1)[1 \otimes (2\omega(\mathbf{z}) \otimes 1 - (1 \otimes \omega(\mathbf{z}))S)]\mathbf{w} \otimes n \\ &= 2(B \otimes 1)(1 \otimes \omega(\mathbf{z}) \otimes 1)\mathbf{w} \otimes n - (B \otimes \omega(\mathbf{z}))(1 \otimes S)\mathbf{w} \otimes n \\ &= 2\epsilon[B(\omega(\mathbf{z}) \otimes 1)\mathbf{w}]n - \omega(\mathbf{z})(B \otimes 1)(1 \otimes S)\mathbf{w} \otimes n \\ &= 2\epsilon q(\mathbf{z}, \mathbf{w})n - \omega(\mathbf{z})\omega(\mathbf{w})n.\end{aligned}$$

Further, (iii) is established as follows.

$$\begin{aligned}S(\omega(\mathbf{z}) \otimes \omega(\mathbf{z})) &= S(1 \otimes \omega(\mathbf{z}))(\omega(\mathbf{z}) \otimes 1) \\ &= [2(\omega(\mathbf{z}) \otimes 1) - (1 \otimes \omega(\mathbf{z}))S](\omega(\mathbf{z}) \otimes 1) \\ &= 2\omega(\mathbf{z})^2 \otimes 1 - (1 \otimes \omega(\mathbf{z}))[2(1 \otimes \omega(\mathbf{z})) - (\omega(\mathbf{z}) \otimes 1)S] \\ &= 2\epsilon q(\mathbf{z}, \mathbf{z})1 \otimes 1 - 1 \otimes 2\epsilon q(\mathbf{z}, \mathbf{z})1 + (\omega(\mathbf{z}) \otimes \omega(\mathbf{z}))S \\ &= (\omega(\mathbf{z}) \otimes \omega(\mathbf{z}))S.\end{aligned}$$

The final assertion is established by induction on l. There is no harm in assuming that the field is algebraically closed. Choose $\mathbf{x}$ and $\mathbf{y}$ in $\mathfrak{N} \otimes \mathfrak{N}$ so that $q(\mathbf{x}, \mathbf{x}) = q(\mathbf{y}, \mathbf{y}) = 0$ and $q(\mathbf{x}, \mathbf{y}) = \epsilon/2$. Then $\omega(\mathbf{x})^2 = \omega(\mathbf{y})^2 = 0$ and $\omega(\mathbf{x})\omega(\mathbf{y}) + \omega(\mathbf{y})\omega(\mathbf{x}) = 1$, the identity transformation. Let $\mathfrak{N}[\mathbf{x}] =$ kernel $\omega(\mathbf{x})$ and $\mathfrak{N}[\mathbf{y}] =$ kernel $\omega(\mathbf{y})$. Then $\mathfrak{N} = \mathfrak{N}[\mathbf{x}] \oplus \mathfrak{N}[\mathbf{y}]$ and the restrictions $\omega(\mathbf{x}) : \mathfrak{N}[\mathbf{y}] \to \mathfrak{N}[\mathbf{x}]$ and $\omega(\mathbf{y}) : \mathfrak{N}[\mathbf{x}] \to \mathfrak{N}[\mathbf{y}]$ are nonsingular. Hence $\dim \mathfrak{N} = 2 \dim \mathfrak{N}[\mathbf{x}]$.

We now make $\mathfrak{N}[\mathbf{x}]$ into a generalized spin system. Let $S[\mathbf{x}]$ be the restriction of S to $\mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]$. Since $\mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]$ is the image of $\omega(\mathbf{x}) \otimes \omega(\mathbf{x})$, $S[\mathbf{x}](\mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]) = S(\omega(\mathbf{x}) \otimes \omega(\mathbf{x}))\mathfrak{N} \otimes \mathfrak{N} = (\omega(\mathbf{x}) \otimes \omega(\mathbf{x}))S(\mathfrak{N} \otimes \mathfrak{N}) \subseteq \mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]$. Also $B(\mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]) = B(\omega(\mathbf{x})\mathfrak{N}, \omega(\mathbf{x})\mathfrak{N}) = \epsilon B(\omega(\mathbf{x})^2\mathfrak{N}, \mathfrak{N}) = 0$. Define $B[\mathbf{x}] = B(1 \otimes \omega(\mathbf{y}))$ and $\epsilon[\mathbf{x}] = -\epsilon$.

Let $\mathbf{z}$ be an element of $\mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]$. On $\mathfrak{N}[\mathbf{x}]$ the transformation $\omega(\mathbf{z})$ is replaced by $\omega(\mathbf{x}; \mathbf{z})$, where for all n in $\mathfrak{N}[\mathbf{x}]$,

$$\begin{aligned}\omega(\mathbf{x}; \mathbf{z})n &= (B[\mathbf{x}] \otimes 1)(1 \otimes S[\mathbf{x}])\mathbf{z} \otimes n \\ &= (B \otimes 1)(1 \otimes \omega(\mathbf{y}) \otimes 1)(1 \otimes S)\mathbf{z} \otimes n \\ &= 2(B \otimes 1)(1 \otimes 1 \otimes \omega(\mathbf{y}))\mathbf{z} \otimes n \\ &\qquad - (B \otimes 1)(1 \otimes S)(1 \otimes \omega(\mathbf{y}) \otimes 1)\mathbf{z} \otimes n \\ &= \omega(\mathbf{z}')n,\end{aligned}$$

where $\mathbf{z}' = -(1 \otimes \omega(\mathbf{y}))\mathbf{z}$. Then

$$\begin{aligned}B[\mathbf{x}](\omega(\mathbf{x}; \mathbf{z}) \otimes 1) - \epsilon[\mathbf{x}]B[\mathbf{x}](1 \otimes \omega(\mathbf{x}; \mathbf{z}))& \\ &= B(\omega(\mathbf{z}') \otimes \omega(\mathbf{y})) + \epsilon B(1 \otimes \omega(\mathbf{y})\omega(\mathbf{z}')) \\ &= \epsilon B(1 \otimes (\omega(\mathbf{z}')\omega(\mathbf{y}) + \omega(\mathbf{y})\omega(\mathbf{z}')) \\ &= 2q(\mathbf{y}, \mathbf{z}')B = 0,\end{aligned}$$

because by (8), $q(\mathbf{y}, \mathbf{z}') = -B(\omega(\mathbf{y}) \otimes 1)(1 \otimes \omega(\mathbf{y})\mathbf{z}) = -\epsilon B(1 \otimes \omega(\mathbf{y})^2)\mathbf{z} = 0$. Finally,

$$\begin{aligned} S[\mathbf{x}](1 \otimes \omega(\mathbf{x};\mathbf{z})) + (1 \otimes \omega(\mathbf{x};\mathbf{z}))S[\mathbf{x}] &= S(1 \otimes \omega(\mathbf{z}')) + (1 \otimes \omega(\mathbf{z}'))S \\ &= 2\omega(\mathbf{z}') \otimes 1 = 2\omega(\mathbf{x};\mathbf{z}) \otimes 1. \end{aligned}$$

Hence $\langle \mathfrak{N}[\mathbf{x}], S[\mathbf{x}], B[\mathbf{x}], \epsilon[\mathbf{x}]\rangle$ is a generalized spin system.

Next we express q in terms of $q[\mathbf{x}]$. Let n, m, p, r be elements of $\mathfrak{N}[\mathbf{x}]$. Then

$$\begin{aligned} q(n \otimes m, \omega(\mathbf{y})p \otimes \omega(\mathbf{y})r) &= (B \otimes B)(1 \otimes S \otimes 1)(n \otimes m \otimes \omega(\mathbf{y})p \otimes \omega(\mathbf{y})r) \\ &= 2(B \otimes B)(n \otimes \omega(\mathbf{y})m \otimes p \otimes \omega(\mathbf{y})r) \\ &= 2B[\mathbf{x}](n, m)B[\mathbf{x}](p, r). \end{aligned}$$

Similarly,

$$\begin{aligned} q(n \otimes \omega(\mathbf{y})m, p \otimes \omega(\mathbf{y})r) &= -q[\mathbf{x}](n \otimes m, p \otimes r), \\ q(n \otimes \omega(\mathbf{y})m, \omega(\mathbf{y})p \otimes r) &= \epsilon q[\mathbf{x}](n \otimes m, p \otimes r), \\ q(\omega(\mathbf{y})n \otimes m, \omega(\mathbf{y})p \otimes r) &= -q[\mathbf{x}](n \otimes m, p \otimes r). \end{aligned}$$

These last four formulae are analogous to those in (7) which express q in terms of q' and B'. As in the proof of Theorem 1, they show that rank $q = 2 +$ rank $q[\mathbf{x}]$. Therefore, by induction 2^{l-1} divides dim $\mathfrak{N}[\mathbf{x}]$ and 2^l divides dim $\mathfrak{N}$. This finishes the proof of Lemma 3.

Definition. A *split spin system* is a generalized spin system $\langle \mathfrak{N}, S, B, \epsilon\rangle$ such that

(i) B is non-degenerate,
(ii) if dim $\mathfrak{N} \otimes \mathfrak{N}/\text{rad } q$ is $2l$ or $2l + 1$, then dim $\mathfrak{N} = 2^l$,
(iii) the bilinear form induced by q on $\mathfrak{N} \otimes \mathfrak{N}/\text{rad } q$ has maximal index,
(iv) $\epsilon = (-1)^l$.

If $\langle \mathfrak{N}, S, B, \epsilon\rangle$ is a generalized spin system and κ is a non-zero scalar, then $\langle \mathfrak{N}, S, \kappa B, \epsilon\rangle$ is a generalized spin system and is split if and only if the original system is split.

If $\langle \mathfrak{N}, S, B, \epsilon\rangle$ and $\langle \mathfrak{N}_1, S_1, B_1, \epsilon_1\rangle$ are generalized spin systems, an isomorphism from $\mathfrak{N}$ to $\mathfrak{N}_1$ is defined to be a linear bijection φ from $\mathfrak{N}$ to $\mathfrak{N}_1$ such that for every n and m in $\mathfrak{N}$, $B_1(\varphi n, \varphi m) = B(n, m)$ and $S_1(\varphi n \otimes \varphi m) = (\varphi \otimes \varphi) S(n \otimes m)$. Clearly, if an isomorphism exists then $\epsilon = \epsilon_1$, unless $B = 0$.

A spin system for a vector space $\mathfrak{V}$ with split form is a split spin system and clearly any two spin systems for $\mathfrak{V}$ are isomorphic.

Here is the main result of this section.

THEOREM 2. *Let $\langle \mathfrak{N}, S, B, (-1)^l\rangle$ be a split spin system over a field of characteristic not two. Then there is a non-zero field element κ such that $\langle \mathfrak{N}, S, \kappa B, (-1)^l\rangle$ is isomorphic to a spin system for a vector space $\mathfrak{V}$ with a split form. Furthermore,* dim $\mathfrak{V}$ = dim $\mathfrak{N} \otimes \mathfrak{N}/\text{rad } q$.

Proof. Use induction on l. Since q induces on $\mathfrak{N} \otimes \mathfrak{N}/\text{rad } q$ a form of maximal index, we can choose $\mathbf{x}$ and $\mathbf{y}$ in $\mathfrak{N} \otimes \mathfrak{N}$ as in the proof of Lemma 3. We use the notation from the proof of Lemma 3.

If $l = 1$ and $\dim \mathfrak{N} \otimes \mathfrak{N}/\mathrm{rad}\, q = 2$, then $\dim \mathfrak{N}[\mathbf{x}] = \dim \mathfrak{N}[\mathbf{y}] = 1$. Fix $a \neq 0$ in $\mathfrak{N}[\mathbf{x}]$ and choose b in $\mathfrak{N}[\mathbf{y}]$ so that $B(a, b) = 1$. Then $b = \kappa\omega(y)a$ with $\kappa \neq 0$. Replace B by κB so as to assume that $B[\mathbf{x}](a, a) = B(a, \omega(\mathbf{y})a) = 1$. Then $B(\omega(\mathbf{y})a, a) = -1$. As in the proof of Lemma 3, $B(a, a) = B(\omega(\mathbf{y})a, \omega(\mathbf{y})a) = 0$. To calculate S, we note that

$$S(\mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}]) \subseteq \mathfrak{N}[\mathbf{x}] \otimes \mathfrak{N}[\mathbf{x}],$$

so $S(a \otimes a) = \mu(a \otimes a)$. However, $\mathrm{rank}\, q[\mathbf{x}] = \mathrm{rank}\, q - 2 = 0$ so

$$0 = q[\mathbf{x}](a \otimes a, a \otimes a) = (B \otimes B)(1 \otimes \omega(\mathbf{y}) \otimes 1 \otimes \omega(\mathbf{y}))(1 \otimes S \otimes 1)(a \otimes a \otimes a \otimes a) = \mu.$$

Hence, $S(a \otimes a)$ and $S(\omega(\mathbf{y})a \otimes \omega(\mathbf{y})a)$ are zero. Finally, $S(a \otimes \omega(\mathbf{y})a) = 2\omega(\mathbf{y})a \otimes a - (1 \otimes \omega(\mathbf{y}))S(a \otimes a) = 2\omega(\mathbf{y})a \otimes a$. If $\mathfrak{V}$ is a two-dimensional vector space with split form and u_1, w_1 is a splitting basis then $a \rightarrow u$, $\omega(\mathbf{y})a \rightarrow uw_1$ is an isomorphism of $\langle \mathfrak{N}, S, B, -1\rangle$ onto a spin system for $\mathfrak{V}$.

If $l = 1$ and $\dim \mathfrak{N} \otimes \mathfrak{N}/\mathrm{rad}\, q = 3$ we can again assume that a has been chosen in $\mathfrak{N}[\mathbf{x}]$ such that $B(a, \omega(\mathbf{y})a) = 1$. Since $q[\mathbf{x}] \neq 0$, $q[\mathbf{x}](a \otimes a, a \otimes a) = \nu \neq 0$. If $S(a \otimes a) = \mu a \otimes a$, then

$$\nu = (B[\mathbf{x}] \otimes B[\mathbf{x}])(1 \otimes S[\mathbf{x}] \otimes 1)(a \otimes a \otimes a \otimes a) = \mu.$$

Let $b = \omega(\mathbf{y})a$. Then $S(a \otimes b) = 2b \otimes a - \nu a \otimes b$. Let $\mathfrak{V}$ be a three-dimensional vector space with split form and splitting basis u_1, w_1, t. Then $a \rightarrow u$, $b \rightarrow uw_1$ will be an isomorphism if we can show that $\nu = 1$. A calculation of both sides of the equation

$$S(1 \otimes \omega(a \otimes b))(a \otimes a) + (1 \otimes \omega(a \otimes b))S(a \otimes a) = 2(\omega(a \otimes b) \otimes 1)(a \otimes a)$$

will reveal that $\nu = 1$.

If $l > 1$, we may assume that φ is an isomorphism of

$$\langle \mathfrak{N}[\mathbf{x}], S[\mathbf{x}], B[\mathbf{x}], (-1)^{l-1}\rangle$$

with a spin system for a vector space $\mathfrak{V}_0$ with split form and $\dim \mathfrak{V}_0 = (\dim \mathfrak{N} \otimes \mathfrak{N}/\mathrm{rad}\, q) - 2$. Furthermore, from the proof of Lemma 3 we know that if n and m are in $\mathfrak{N}[\mathbf{x}]$, then

$$\begin{aligned} S(n \otimes m) &= S[\mathbf{x}](n \otimes m), \\ S(\omega(\mathbf{y})n \otimes \omega(\mathbf{y})m) &= (\omega(\mathbf{y}) \otimes \omega(\mathbf{y}))S[\mathbf{x}](n \otimes m), \\ S(n \otimes \omega(\mathbf{y})m) &= 2\omega(\mathbf{y})n \otimes m - (1 \otimes \omega(\mathbf{y}))S[\mathbf{x}](n \otimes m). \end{aligned} \tag{9}$$

Let $\mathfrak{V} = \mathfrak{V}_0 \perp \langle u_l, w_l\rangle$, an orthogonal sum of $\mathfrak{V}_0$ and a hyperbolic plane. We then view the Clifford algebra for $\mathfrak{V}_0$ in an obvious way as a sub-algebra of the Clifford algebra for $\mathfrak{V}$. Define a linear transformation Φ of $\mathfrak{N}$ onto a spin system for $\mathfrak{V}$ as follows. Motivated by (6), for all n in $\mathfrak{N}[\mathbf{x}]$ we define

$$\begin{aligned} \Phi(n) &= (-1)^{l-1}u_l\varphi(n), \\ \Phi(\omega(\mathbf{y})n) &= (-1)^{l-1}u_l\varphi(n)w_l = \Phi(n)w_l. \end{aligned}$$

Then by (7) and (9), Φ is an isomorphism.

https://doi.org/10.4153/CJM-1971-099-8 Published online by Cambridge University Press

5. Identification of the usual Clifford groups. In this section the Clifford groups associated with a vector space $\mathfrak{V}$ with split form, which were defined in section two, are compared with the automorphism groups of a spin system for $\mathfrak{V}$. Recall that every element c of the Clifford algebra acts on $\mathfrak{N}$ by right multiplication $R(c) : n \to nc$.

LEMMA 4. *For every* $\mathbf{z}$ *in* $\mathfrak{N} \otimes \mathfrak{N}$ *there exists* v *in* $\mathfrak{V}$ *such that* $\omega(\mathbf{z}) = R(v)$. *Furthermore* $\omega(\mathbf{z})$ *is non-singular if and only if* v *is in* $\Gamma \cap \mathfrak{V}$.

Proof. It is sufficient to prove the first part for $\mathbf{z} = a \otimes b$ for some a and b in $\mathfrak{N}$. For all n in $\mathfrak{N}$, $\omega(a \otimes b)n = R(v)n$ where $v = \sum (B(a, bu_i)w_i + B(a, bw_i)u_i)$. For the next part, $\omega(\mathbf{z})$ is non-singular if and only if $\omega(\mathbf{z})^2 = \kappa 1$ for $\kappa \neq 0$, that is, if and only if $R(v)^2 = R(v^2) = \kappa 1$ so $v^2 = \kappa 1 \neq 0$ and v is in $\Gamma \cap \mathfrak{V}$.

Let J be the main involution on $\mathfrak{C} : J(c) = c$ for all c in $\mathfrak{C}^+$ and $J(c) = -c$ for all c in $\mathfrak{C}^-$ [**2**, p. 37]. Here $\mathfrak{C}^-$ denotes the subspace of $\mathfrak{C}$ spanned by all products of an odd number of elements of $\mathfrak{V}$. If $k = 2l$ and l is odd, let B_+ denote the bilinear form $B(1 \otimes J)$. If $k = 2l$ with l even, let $B_+ = B$. Then for all n and m in $\mathfrak{N}$ and v in $\mathfrak{V}$, $B_+(nv, m) = B_+(n, mv)$. Properties (i) and (ii) of Lemma 2 hold for B_+.

THEOREM 3. *Let* $\mathfrak{V}$ *be a k-dimensional vector space with a split form over a field of characteristic not two. Let* $\langle \mathfrak{N}, S, B, (-1)^l \rangle$ *be a spin system for* $\mathfrak{V}$ *and* R *be the representation of the Clifford algebra on* $\mathfrak{N}$.

(i) *If* $k = 2l$,
$$R(\Gamma) = \mathrm{Aut}(S),$$
$$R(\Gamma^+) = \mathrm{Aut}(S, J),$$
$$R(\Gamma_0) = \mathrm{Aut}(S, B_+),$$
$$R(\Gamma_0{}^+) = \mathrm{Aut}(S, B_+, J).$$

(ii) *If* $k = 2l + 1$, $R(\Gamma) = R(\Gamma^+) = \mathrm{Aut}(S)$. *Further, if* l *is even*, $R(\Gamma_0) = R(\Gamma_0{}^+) = \mathrm{Aut}(S, B)$, *while if* l *is odd*, $R(\Gamma_0{}^+) = \mathrm{Aut}(S, B)$.

Proof. First we show that $R(\Gamma) = \mathrm{Aut}(S)$ in all cases. If c is in Γ and $v_1, \ldots, v_k$ is an orthogonal basis for $\mathfrak{V}$, then $cv_1c^{-1}, \ldots, cv_kc^{-1}$ is an orthogonal basis for $\mathfrak{V}$ and for all i, $(v_i, v_i) = (cv_ic^{-1}, cv_ic^{-1})$. Hence, for all n and m in $\mathfrak{N}$, $(R(c) \otimes R(c))S(R(c)^{-1} \otimes R(c)^{-1})(n \otimes m) = \sum (v_i, v_i)^{-1}nc^{-1}v_ic \otimes mc^{-1}v_ic = \sum (cv_ic^{-1}, cv_ic^{-1})^{-1}nc^{-1}v_ic \otimes mc^{-1}v_ic = S(n \otimes m)$ by Lemma 1. Hence, $R(\Gamma) \subseteq \mathrm{Aut}(S)$.

To show the reverse inclusion suppose first that $k = 2l$. If T is in $\mathrm{Aut}(S)$, then $T = R(c)$ for some invertible c in $\mathfrak{C}$ because $\mathfrak{N}$ is a minimal right ideal in $\mathfrak{C}$, which is isomorphic to a full matrix algebra. Hence, for all n and m in $\mathfrak{N}$,

$$\sum (v_i, v_i)^{-1}nv_i \otimes mv_i = \sum (v_i, v_i)^{-1}nc^{-1}v_ic \otimes mc^{-1}v_ic.$$

Since $\mathfrak{N} \otimes \mathfrak{N}$ is a minimal right ideal in $\mathfrak{C} \otimes \mathfrak{C}$, which is also isomorphic to a full matrix algebra, the preceding equation implies that

$$\sum (v_i, v_i)^{-1}v_i \otimes v_i = \sum (v_i, v_i)^{-1}c^{-1}v_ic \otimes c^{-1}v_ic = \mathbf{x},$$

https://doi.org/10.4153/CJM-1971-099-8 Published online by Cambridge University Press

say. In terms of a splitting basis this becomes

$$\mathbf{x} = \sum (u_i \otimes w_i + w_i \otimes u_i) = \sum (c^{-1}u_i c \otimes c^{-1}w_i c + c^{-1}w_i c \otimes c^{-1}u_i c).$$

Then

$$\begin{aligned}(1 \otimes u)\mathbf{x}(1 \otimes w_1 \ldots w_{i-1}w_{i+1} \ldots w_l) &= \pm u_i \otimes uw_1 \ldots w_l \\ &= \sum (c^{-1}u_i c \otimes a_i + c^{-1}w_i c \otimes b_i)\end{aligned}$$

for some a_i, b_i, in $\mathfrak{C}$. Hence, each u_i (and similarly w_i) is a linear combination of the elements $c^{-1}u_1 c, \ldots, c^{-1}w_l c$, which must consequently be a basis for $\mathfrak{B}$. Therefore, c is in Γ.

If $k = 2l + 1$, let $t, u_1, \ldots, w_l$ be a splitting basis. Then $\mathfrak{C} = \mathfrak{C}_0 \otimes \mathfrak{C}_1$, a sum of two simple ideals, where $\mathfrak{C}_0$ is generated by $1 + t$ and $\mathfrak{C}_1$ by $1 - t$ and $\mathfrak{N}$ is a minimal right ideal in $\mathfrak{C}_0$. Then $J(\mathfrak{C}_0) = \mathfrak{C}_1$. If T is in $\mathrm{Aut}(S)$, then $T = R(c_0)$ for some invertible c_0 in $\mathfrak{C}_0$. Define $d = c_0 + J(c_0)$. Restricted to $\mathfrak{N}$, $R(d) = R(c)$. For every n and m in $\mathfrak{N}$,

$$\begin{aligned}&nt \otimes mt + \sum (nu_i \otimes mw_i + nw_i \otimes mu_i) \\ &\quad = nc_0^{-1}tc_0 \otimes mc_0^{-1}tc_0 + \sum (nc_0^{-1}u_i c_0 \otimes mc_0^{-1}w_i c_0 + nc_0^{-1}w_i c_0 \otimes mc_0^{-1}u_i c_0).\end{aligned}$$

Hence, for every x and y in $\mathfrak{C}_0$,

$$\begin{aligned}&xt \otimes yt + \sum (xu_i \otimes yw_i + xw_i \otimes yu_i) \\ &\quad = xc_0^{-1}tc_0 \otimes yc_0^{-1}tc_0 + \sum (xc_0^{-1}u_i c_0 \otimes yc_0^{-1}w_i c_0 + xc_0^{-1}w_i c_0 \otimes yc_0^{-1}u_i c_0).\end{aligned}$$

To the last equation we can apply $J \otimes 1$, $1 \otimes J$, and $J \otimes J$. The four equations taken together imply that for all a and b in $\mathfrak{C}$,

$$\begin{aligned}&at \otimes bt + \sum (au_i \otimes bw_i + aw_i \otimes bu_i) \\ &\quad = ad^{-1}td \otimes bd^{-1}td + \sum (ad^{-1}u_i d \otimes bd^{-1}w_i d + ad^{-1}w_i d \otimes bd^{-1}u_i d).\end{aligned}$$

Hence,

$$\begin{aligned}&t \otimes t + \sum (u_i \otimes w_i + w_i \otimes u_i) \\ &\quad = d^{-1}td \otimes d^{-1}td + \sum (d^{-1}u_i d \otimes d^{-1}w_i d + d^{-1}w_i d \otimes d^{-1}u_i d).\end{aligned}$$

As before, this implies that t and every u_i and w_i are linear combinations of $d^{-1}td, d^{-1}u_1 d, \ldots, d^{-1}w_l d$ and that d is in Γ. However, $J(d) = d$ so d is in Γ^+. Hence, in all cases, $\mathrm{Aut}(S) = R(\Gamma)$, and if k is odd, $\mathrm{Aut}(S) = R(\Gamma^+)$.

Let α be the main anti-automorphism on $\mathfrak{C}$. It is induced by $\alpha(v_1 \ldots v_j) = v_j \ldots v_1$. For $k = 2l$, there exists $\lambda \neq 0$ such that for all n and m in $\mathfrak{N}$, $n\alpha(m) = \lambda B_+(n, m)u$. For c in Γ_0, $c\alpha(c) = 1$. Therefore, $B_+(nc, mc) = B_+(n, m)$ and $R(\Gamma_0) \subseteq \mathrm{Aut}(S, B_+)$. Conversely, if T is in $\mathrm{Aut}(S, B_+)$, then $T = R(c)$ for some c in Γ and for all n and m in $\mathfrak{N}$, $nc\alpha(c)\alpha(m) = n\alpha(m)$. But $\mathfrak{C}\mathfrak{N} = \mathfrak{C}$ and $\alpha(\mathfrak{N})\mathfrak{C} = \mathfrak{C}$. Hence $c\alpha(c) = 1$ and c is in Γ_0. If $k = 2l + 1$ with l even, then $n\alpha(m) = \lambda B(n, m)u$ for some $\lambda \neq 0$ so $R(\Gamma_0) \subseteq \mathrm{Aut}(S, B)$.

If $k = 2l + 1$ with l odd, the role of α is taken by the anti-automorphism $\alpha^\sim$ induced by $\alpha^\sim(v_1 \ldots v_j) = (-1)^j v_j \ldots v_1$. Then $n\alpha^\sim(m) = \lambda B(n, m)u$ with

$\lambda \neq 0$. If c is in Γ_0^+, then $c\alpha^\sim(c) = c\alpha(c) = 1$ and $R(\Gamma_0^+) \subseteq \mathrm{Aut}(S, B)$. Conversely, let T be in $\mathrm{Aut}(S, B)$. Then $T = R(c)$ for some c in Γ^+ and $c = c_0 + J(c_0)$ for c_0 in $\mathfrak{C}_0$. If l is even, $nc\alpha(c)\alpha(m) = n\alpha(m)$ for all n and m in $\mathfrak{N}$. But $nc\alpha(c)\alpha(m) = nc_0\alpha(c_0)\alpha(m)$ so $c_0\alpha(c_0)$ is the unit element of $\mathfrak{C}_0$. Then $J(c_0)J\alpha(c_0) = J(c_0)\alpha J(c_0)$ is the unit element of $\mathfrak{C}_1$ so $c\alpha(c) = c_0\alpha(c_0) + J(c_0)\alpha J(c_0) = 1$ and c is in Γ_0^+. If l is odd, $nc\alpha^\sim(c)\alpha^\sim(m) = nc\alpha(c)\alpha^\sim(m) = n\alpha^\sim(m)$ for all n and m in $\mathfrak{N}$ and the same reasoning shows that c is in Γ_0^+.

Finally, say $k = 2l$. Obviously, for c in Γ, $R(c)$ commutes with J precisely when c is in Γ^+. Hence $R(\Gamma^+) = \mathrm{Aut}(S, J)$ and $R(\Gamma_0^+) = \mathrm{Aut}(S, B_+, J)$. This concludes the proof of Theorem 3.

If k and l are both odd, then $R(\Gamma_0) \neq \mathrm{Aut}(S, B)$ because t is in Γ_0 and $B(nt, mt) = -B(n, m)$.

The existence of the bilinear forms B and, if k is even, $B(1 \otimes J)$ shows that $R(\Gamma_0^+)$ can be included in certain proper subgroups of the general linear group on $\mathfrak{N}$. For example, if $k = 2l + 1$ is odd, and $l \equiv 0, 3 \pmod 4$, then B is a symmetric bilinear form with an orthogonal group as isometry group and $R(\Gamma_0^+)$ is contained in this orthogonal group. If $k = 2l + 1$ and $l \equiv 1, 2 \pmod 4$, then B is skew-symmetric and $R(\Gamma_0^+)$ is contained in the symplectic group which is the isometry group of B. For $k = 2l$ the situation is more complicated. If $l \equiv 0 \pmod 4$, both B and $B(1 \otimes J)$ are symmetric and $R(\Gamma_0^+)$ is contained in the intersection of the orthogonal groups for B and for $B(1 \otimes J)$. If $l \equiv 2 \pmod 4$, both bilinear forms are skew-symmetric and $R(\Gamma_0^+)$ is contained in the intersection of two symplectic groups. In case $l = 1, 3 \pmod 4$, one form is symmetric while the other is skew and $R(\Gamma_0^+)$ is in the intersection of an orthogonal and a symplectic group.

References

1. E. Artin, *Geometric algebra* (Interscience, New York, 1957).
2. C. Chevalley, *The algebraic theory of spinors* (Columbia University Press, New York, 1954).
3. ——— *The construction and study of certain important algebras* (The Mathematical Society of Japan, Tokyo, 1955).
4. N. Jacobson, *Lie algebras* (Interscience, New York, 1962).
5. J. A. Schouten, *On the geometry of spin spaces.* III, Indag. Math. *11* (1949), 336–346.

University of California,
Berkeley, California;
University of Toronto,
Toronto, Canada