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On Deformations of Pairs (Manifold, Coherent Sheaf)

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Abstract. We analyse infinitesimal deformations of pairs $(X, \mathcal{F})$ with $\mathcal{F}$ a coherent sheaf on a smooth projective variety X over an algebraically closed field of characteristic 0. We describe a differential graded Lie algebra controlling the deformation problem, and we prove an analog of a Mukai–Artamkin theorem about the trace map.

1 Introduction

Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X over an algebraically closed field of characteristic 0; in particular $\mathcal{F}$ admits a finite locally free resolution. The *sheaf trace morphism* is defined to be

$$\mathrm{Tr}\colon \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow \mathcal{O}_X, \tag{1.1}$$

and the *trace maps* $\mathrm{Tr}^i\colon \mathrm{Ext}^i_X(\mathcal{F},\mathcal{F}) \to H^i(X,\mathcal{O}_X)$ are the morphisms induced in hypercohomology by Tr [1,16,27].

The maps Tr^1 and Tr^2 have a clear interpretation in the setting of deformation theory of $\mathcal{F}$ and of the determinant bundle $\det\mathcal{F}$ of $\mathcal{F}$. Every deformation of $\mathcal{F}$ induces naturally a deformation of the determinant line bundle $\det\mathcal{F}$; the vector spaces $\mathrm{Ext}^i_X(\mathcal{F},\mathcal{F})$, $i = 1,2$, are the tangent and obstruction spaces of the functor $\mathrm{Def}_{\mathcal{F}}$ of deformations of $\mathcal{F}$; the spaces $H^i(X,\mathcal{O}_X)$, $i = 1,2$, are the tangent and obstruction spaces of the functor $\mathrm{Def}_{\det\mathcal{F}}$ of deformations of $\det\mathcal{F}$. Then the maps Tr^i, $i = 1,2$, are induced by the natural transformation $\mathrm{Def}_{\mathcal{F}} \to \mathrm{Def}_{\det\mathcal{F}}$.

Moreover, since we are in characteristic 0, the Picard functor is unobstructed and this allows proving that Tr^2 annihilates all the obstructions to deformations of $\mathcal{F}$. As a consequence, we have the following well-known result.

Theorem 1.1 (Mukai–Artamkin [1,22]) *In the above situation, if* Tr^2 *is injective, then the deformation functor* $\mathrm{Def}_{\mathcal{F}}$ *is unobstructed. If* Tr^1 *is surjective and* Tr^2 *is injective, then the natural transformation* $\mathrm{Def}_{\mathcal{F}} \to \mathrm{Def}_{\det\mathcal{F}}$ *is smooth.*

Mukai and Artamkin assumed that $\mathcal{F}$ is simple. However, it is immediately clear that this assumption is used only for the pro-representability of the functor $\mathrm{Def}_{\mathcal{F}}$

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and hence for the existence of the universal deformation of $\mathcal{F}$, while the proof of the smoothness of the semi-universal deformation works also without the simpleness assumption.

One of the main goals of this paper is to extend the previous result to the case of infinitesimal deformations of pairs $(X, \mathcal{F})$, consisting of deformations of both the variety X and the coherent sheaf $\mathcal{F}$.

As a first result, we describe a well-defined homotopy class of differential graded Lie algebras associated with the deformations of the pair $(X, \mathcal{F})$.

For every quasi-coherent sheaf of DG-Lie algebras $\mathcal{L}^*$ on an algebraic variety X, $R\Gamma(X, \mathcal{L}^*)$ is defined, up to homotopy equivalence, to be the DG-Lie algebras of derived sections and we shall say that a deformation problem is controlled by $\mathcal{L}^*$ if it is controlled in the usual way by the DG-Lie algebra $R\Gamma(X, \mathcal{L}^*)$. A canonical representative for $R\Gamma(X, \mathcal{L}^*)$ is given by the totalization, in the simplicially enriched model category of DG-Lie algebras, of the cosimplicial space of Čech cochains of $\mathcal{L}^*$ with respect to any open affine cover of X; see Section 6.

For a coherent sheaf $\mathcal{F}$ on a projective smooth variety X, it is known that the deformations of $\mathcal{F}$ are controlled by the sheaf of the DG-Lie algebra of endomorphisms of any finite locally free resolution of $\mathcal{F}$ [6].

It is also known that if $\mathcal{E}$ is a locally free sheaf on a smooth variety X, then the deformations of the pair $(X, \mathcal{E})$ are controlled by the DG-Lie algebra associated with the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol [13, 20, 25]. For a general coherent sheaf $\mathcal{F}$, the sheaf of differential operators on $\mathcal{F}$ is equally well defined [9, §16.8], but, in general, it is not the right object controlling the deformations of pairs.

Therefore, we first introduce the right algebraic object associated with the deformation problem; we shall call this object the *module of derivations of pairs*. It extends the sheaf of differential operators on $\mathcal{F}$, involving a finite locally free resolution of $\mathcal{F}$. This allows us to define a coherent sheaf of DG-Lie algebras over X controlling the deformations of the pair $(X, \mathcal{F})$ (Theorem 7.11).

In particular, the cohomology groups $T^i_{(X,\mathcal{F})}$ of the associated DG-Lie algebra fit into a long exact sequence:

$$\cdots \longrightarrow \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{F}) \longrightarrow T^i_{(X,\mathcal{F})} \longrightarrow H^i(X, \Theta_X) \longrightarrow \operatorname{Ext}^{i+1}_X(\mathcal{F}, \mathcal{F}) \longrightarrow \cdots,$$

where Θ_X denotes the tangent sheaf of X. In particular, we recover the well-known fact that if $\operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) = 0$, then the natural transformation $\operatorname{Def}_{(X,\mathcal{F})} \to \operatorname{Def}_X$ is smooth, since it is surjective on tangent spaces ($T^1_{(X,\mathcal{F})} \to H^1(X, \Theta_X)$) and injective on obstruction spaces ($T^2_{(X,\mathcal{F})} \to H^2(X, \Theta_X)$).

Then we devote our attention to the natural transformations

$$\operatorname{Def}_{(X,\mathcal{F})} \longrightarrow \operatorname{Def}_{(X,\det\mathcal{F})} \longrightarrow \operatorname{Def}_X .$$

In particular, we describe an extension of the trace maps to the DG-Lie algebra of differential operators with principal symbol (Theorem 5.6).

Finally, we are able to prove the following result (Theorem 7.16), which is the analog of Theorem 1.1, for deformations of pairs.

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Theorem 1.2 *Let $\mathcal{F}$ be a coherent sheaf on a projective smooth variety X defined over an algebraically closed field of characteristic* 0*. Consider the trace maps*

$$\mathrm{Tr}^i\colon \mathrm{Ext}^i_X(\mathcal{F},\mathcal{F}) \longrightarrow H^i(X,\mathcal{O}_X).$$

(i) *If the map* Tr^1 *is surjective and the map* Tr^2 *is injective, then the natural transformation* $\mathrm{Def}_{(X,\mathcal{F})} \to \mathrm{Def}_{(X,\det\mathcal{F})}$ *is smooth.*

(ii) *If* Tr^2 *is injective and* $\mathrm{Def}_{(X,\det\mathcal{F})}$ *is unobstructed, then* $\mathrm{Def}_{(X,\mathcal{F})}$ *is unobstructed.*

(iii) *If* $H^0(X,\Theta_X) = 0$ *and* $\mathrm{Ext}^0_X(\mathcal{F},\mathcal{F}) = \mathbb{K}$, e.g., *if $\mathcal{F}$ is simple, then* $\mathrm{Def}_{(X,\mathcal{F})}$ *is pro-representable.*

For instance, if $\mathcal{F}$ is a simple coherent sheaf [22, p. 101] of positive rank over a surface S with trivial canonical bundle, then Tr^1 is surjective, Tr^2 is injective, and therefore the natural transformation $\mathrm{Def}_{(X,\mathcal{F})} \to \mathrm{Def}_{(X,\det\mathcal{F})}$ is smooth. When $\mathcal{F}$ is locally free, Theorem 1.2 was already proved in [13] by using transcendental methods, and hence over the field of complex numbers.

Our proof of Theorem 1.2 is almost entirely algebraic and it relies on the the explicit description of a DG-Lie algebra controlling the deformations of the pair $(X,\mathcal{F})$ and the extension of the trace maps.

Another derived extension of the determinant map was introduced in [24] from the derived stack of perfect complex to the derived stack of line bundle.

The paper is organized as follows. The first sections of this paper are devoted to the introduction and the study of the first properties of the module of derivations of pairs; we prove that it behaves very well with respect to all the canonical constructions, injective and projective resolutions, de Rham complexes, and Fitting stratifications. In Section 5, we define a coherent sheaf of DG-Lie algebras associated with derivations of pairs and the extension of the trace map (Theorem 5.6) that generalises the classical one given in (1.1). Section 6 is included for the reader's convenience; there we review the relevant notions on deformation functors associated with differential graded Lie algebras and with semicosimplicial DG-Lie algebras. In Section 7, we analyse the infinitesimal deformations of pairs, describing the DG-Lie algebra that controls these deformations (Theorem 7.11), and we prove the main theorem (Theorem 1.2) about the trace map (Theorem 7.16).

2 Derivations and Automorphisms of Pairs

Let $A \to R$ be a morphism of unitary commutative rings and M, N two R-modules. We shall denote by $\mathrm{Hom}_A(M,N)$ (resp.: $\mathrm{Hom}_R(M,N)$) the R-module of A-linear (resp.: R-linear) maps $M \to N$: the R-module structure on $\mathrm{Hom}_A(M,N)$ is induced by the R-module structure on N. We shall denote by $\mathrm{Der}_A(R,N) = \mathrm{Hom}_R(\Omega_{R/A},N)$ the R-module of A-linear derivations $R \to N$. We shall refer to the R-module $R \oplus M$ as the trivial extension whenever $R \oplus M$ is considered as a commutative ring equipped with the product $(r,m)(s,n) = (rs, rn + sm)$ [25, p. 10].

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Definition 2.1 Let $A \to R$ be a morphism of unitary commutative rings and M an R-module. The R-module of A-derivations of the pair (R, M) is defined as

$$D_A(R,M) = \left\{ (h,u) \in \mathrm{Der}_A(R,R) \times \mathrm{Hom}_A(M,M) \;\middle|\; \begin{array}{l} u(rm) - ru(m) = h(r)m, \\ \text{for every } r \in R, m \in M \end{array} \right\}.$$

Remark 2.2 In the setup of the above definition we have the following.

- If $r_1, r_2, \ldots \in R$ generate R as an A-algebra and $m_1, m_2, \ldots \in M$ generate M as R-module, then every $(h,u) \in D_A(R,M)$ is uniquely determined by $h(r_1), h(r_2), \ldots$ and $u(m_1), u(m_2), \ldots$.
- If $R \oplus M$ is the trivial extension of R by M, then a pair

$$(h,u) \in \mathrm{Hom}_A(R,R) \times \mathrm{Hom}_A(M,M)$$

belongs to $D_A(R,M)$ if and only if the map $R \oplus M \to R \oplus M$, $(r,m) \mapsto (h(r), u(m))$ is an A-derivation.

Lemma 2.3 *There exists an exact sequence of R-modules*

$$0 \longrightarrow D_A(R,M) \longrightarrow \mathrm{Der}_A(R \oplus M, R \oplus M) \xrightarrow{\Phi} \mathrm{Hom}_R(M,R) \oplus \mathrm{Der}_A(R,M),$$

where the R-module structure on the derivations of the trivial extension $R \oplus M$ is induced by the inclusion $R \to R \oplus M$.

Proof Every element of $\mathrm{Hom}_A(R \oplus M, R \oplus M)$ is represented by a matrix of A-linear maps $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with

$$a \in \mathrm{Hom}_A(R,R), \quad b \in \mathrm{Hom}_A(M,R), \quad c \in \mathrm{Hom}_A(R,M), \quad d \in \mathrm{Hom}_A(M,M),$$

and then there exists a natural isomorphism of R-modules

$$\begin{aligned} &\mathrm{Hom}_A(R \oplus M, R \oplus M) \\ &\qquad = \mathrm{Hom}_A(R,R) \oplus \mathrm{Hom}_A(M,R) \oplus \mathrm{Hom}_A(R,M) \oplus \mathrm{Hom}_A(M,M). \end{aligned}$$

We have already noticed that $(h,u) \in D_A(R,M)$ if and only if

$$\left(\begin{smallmatrix} h & 0 \\ 0 & u \end{smallmatrix}\right) \in \mathrm{Der}_A(R \oplus M, R \oplus M).$$

Therefore, defining $\Phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (b,c)$, it is straightforward to see that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a derivation, then b is R-linear and c is a derivation. It also easy to prove that the image of Φ is $\{b \in \mathrm{Hom}_R(M,R) \mid b(m)n + mb(n) = 0 \text{ for all } m,n \in M\} \oplus \mathrm{Der}_A(R,M)$. ■

Lemma 2.4 *In the setup of Definition 2.1, let us denote by*

$$q\colon D_A(R,M) \longrightarrow \mathrm{Hom}_A(M,M) \quad \textit{and} \quad \alpha\colon D_A(R,M) \longrightarrow \mathrm{Der}_A(R,R),$$

the projection maps restricted to $D_A(R,M)$.

(i) *If M is a faithful module,* i.e., *if* $\mathrm{ann}(M) = 0$, *then q is injective and its image is the submodule of differential operators of first order with principal symbol.*

(ii) *There exists an exact sequence of R-modules*

$$0 \longrightarrow \mathrm{Hom}_R(M,M) \longrightarrow D_A(R,M) \xrightarrow{\alpha} \mathrm{Der}_A(R,R). \tag{2.1}$$

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Proof The only nontrivial statement is the one concerning the image of q. Recall that a morphism $v \in \mathrm{Hom}_A(M,M)$ is a differential operator of first order if for every $r \in R$ the map $[v,r]\colon M \to M$, where $[v,r](m) = v(rm) - rv(m)$, is a morphism of R-modules. In this case the symbol $\sigma(v)$ is defined as the map

$$\sigma(v)\colon R \to \mathrm{Hom}_R(M,M), \quad \sigma(v)(r) = [v,r],$$

and it is called *principal* if $\sigma(v)(r)$ is a scalar multiple of the identity for every $r \in R$. Notice that the symbol $\sigma(v)$ is an A-derivation, since for every $r,s \in R$, we have

$$\sigma(v)(rs) = [v,rs] = [v,r]s + r[v,s].$$

If $(h,u) \in D_A(R,M)$, then for every $r \in R$, $[u,r] = h(r)\,\mathrm{Id}_M \in \mathrm{Hom}_R(M,M)$. Conversely, let $v \in \mathrm{Hom}_A(M,M)$ be a first order differential operator such that $\sigma(v)(R) \subseteq R\,\mathrm{Id}_M$. Since M is faithful, for every $r \in R$ there exists a unique $h(r) \in R$ such that $\sigma(v)(r) = [v,r] = h(r)\,\mathrm{Id}_M$. Since $\sigma(v)$ is an A-derivation, $h\colon R \to R$ is also an A-derivation and therefore $(h,v) \in D_A(R,M)$. ∎

The restriction $\alpha\colon D_A(R,M) \to \mathrm{Der}_A(R,R)$ of the projection on the first factor is called the anchor map of the pair. It is clear that $D_A(R,M)$ is a Lie subalgebra of $\mathrm{Der}_A(R,R) \times \mathrm{Hom}_A(M,M)$ and we have the relation

$$[x,ry] = \alpha(x)(r)y + r[x,y], \quad x,y \in D_A(R,M), r \in R,$$

called Poisson identity. Note that the bracket on $\mathrm{Der}_A(R,R)$, and then on $D_A(R,M)$, is bilinear over A and not over R.

Example 2.5 (Lie derivative) Let $A \to R$ be a morphism of unitary commutative rings. Then every $h \in \mathrm{Der}_A(R,R)$ gives a canonical element $(h,L_h) \in D_A(R,\Omega_{R/A})$ uniquely determined by the equation $L_h(dx) = d(h(x))$, $x \in R$.

Recall that we may define the module of Kähler differentials as $\Omega_{R/A} = I/I^2$, where I is the kernel of the multiplication map $R \otimes_A R \to R$, the differential dx is the class of $x \otimes 1 - 1 \otimes x$ in I/I^2, and the R-module structure is induced by the morphism of A-algebras $R \to R \otimes_A R$, $r \mapsto r \otimes 1$ [21, §25]. For every $h \in \mathrm{Der}_A(R,R)$ we define $L_h\colon I/I^2 \to I/I^2$ as the factorization to the quotient of the derivation

$$k\colon R \otimes_A R \to R \otimes_A R, \quad k(x \otimes y) = h(x) \otimes y + x \otimes h(y).$$

The equation $L_h(dx) = d(h(x))$ is trivially satisfied. For every $r,s,x \in R$, we have

$$\begin{aligned} L_h(rsdx) &= k((r \otimes 1)(sx \otimes 1 - s \otimes x)) \\ &= (h(r) \otimes 1)(sx \otimes 1 - s \otimes x) + (r \otimes 1)k(sx \otimes 1 - s \otimes x) \\ &= h(r)sdx + rL_h(sdx), \end{aligned}$$

and then $(h,L_h) \in D_A(R,\Omega_{R/A})$.

The definition of $D_A(R,M)$ extends naturally to $D_A(R,M_\bullet)$, where $M_\bullet$ is a diagram of R-modules over a small category I. Here we are mainly interested in two cases. The former is when I is just a set, a diagram of R-modules is just a collection $\{M_i\}$,

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$i \in I$, and $D_A(R, M_\bullet)$ is the limit of the diagram of anchor maps $\alpha\colon D_A(R, M_i) \to \mathrm{Der}_A(R, R)$. For later use, it is notationally convenient to define

$$\begin{aligned}\prod_i{}^{\times} D_A(R, M_i) = D_A(R, M_\bullet) &= \lim_i (\alpha\colon D_A(R, M_i) \longrightarrow \mathrm{Der}_A(R, R)) \\ &= \left\{ (h, \dots, u_i, \dots) \mid (h, u_i) \in D_A(R, M_i), \text{for all } i \in I \right\}.\end{aligned}$$

The latter is when $M_\bullet = \{M_1 \xrightarrow{f} M_2\}$ is a morphism of two R-modules. Then $D_A(R, M_\bullet)$ is the set of triples (h, u_1, u_2), with

$$(h, u_1) \in D_A(R, M_1), \quad (h, u_2) \in D_A(R, M_2), \quad fu_1 = u_2 f.$$

Therefore, there exists an exact sequence of R-modules

$$0 \longrightarrow D_A(R, M_\bullet) \longrightarrow D_A(R, M_1) \times_{\mathrm{Der}_A(R,R)} D_A(R, M_2) \xrightarrow{\Phi} \mathrm{Hom}_R(M_1, M_2),$$

where $\Phi((h, u_1), (h, u_2)) = fu_1 - u_2 f$.

Lemma 2.6 *Let $A \to R$ be a morphism of unitary commutative rings and M, N two R-modules. Then we have two natural R-linear morphisms of Lie algebras:*

(2.2)

$$\begin{aligned}D_A(R, M \otimes_R N) \xleftarrow{\Phi} D_A(R, M) \times_{\mathrm{Der}_A(R,R)} D_A(R, N) &\xrightarrow{\Psi} D_A(R, \mathrm{Hom}_R(M, N)), \\ \Phi((h, u), (h, v)) &= (h, u \otimes_R \mathrm{Id}_N + \mathrm{Id}_M \otimes_R v), \\ \Psi((h, u), (h, v)) &= (h, f \mapsto vf - fu).\end{aligned}$$

Proof Straightforward. Notice that $u \otimes_R \mathrm{Id}_N + \mathrm{Id}_M \otimes_R v$ is well defined, although u, v are not R-linear and the two addends $u \otimes_R \mathrm{Id}_N$, $\mathrm{Id}_M \otimes_R v$ are not defined. Similarly $vf - fu$ is a morphism of R-modules, although vf and fu are only A-linear. ∎

As a particular case of Lemma 2.6 we obtain a natural R-linear morphisms of Lie algebras (the transpose):

$$\begin{aligned}(-)^T\colon D_A(R, M) &\longrightarrow D_A(R, \mathrm{Hom}_R(M, R)), \\ (h, u) &\longmapsto (h, u)^T = \Psi((h, u), (h, h)) = (h, f \mapsto hf - fu).\end{aligned} \tag{2.3}$$

Definition 2.7 The Leibniz extension of a derivation of the pair $(h, u) \in D_A(R, M)$ is the sequence $(h, u_n) \in D_A(R, \wedge_R^n M)$, $n \geq 0$, uniquely determined by the formulas

$$u_0 = h, \quad u_n(m_1 \wedge \cdots \wedge m_n) = \sum_{i=1}^{n} m_1 \wedge \cdots \wedge u(m_i) \wedge \cdots \wedge m_n. \tag{2.4}$$

The Leibniz extension is properly defined: by the universal property of wedge products, (2.4) defines a sequence of A-linear maps $\tilde{u}_n\colon \wedge_A^n M \to \wedge_A^n M$. Now $\wedge_R^n M$ is the quotient of $\wedge_A^n M$ by the A-submodule H generated by all the elements

$$m_1 \wedge \cdots \wedge rm_i \wedge m_{i+1} \wedge \cdots \wedge m_n - m_1 \wedge \cdots \wedge m_i \wedge rm_{i+1} \wedge \cdots \wedge m_n, \quad r \in R, m_j \in M; 0 < i < n,$$

and it is immediate to verify that $\tilde{u}_n(H) \subset H$ and then that $(h, \tilde{u}_n)$ factors to a derivation of the pair $(h, u_n) \in D_A(R, \wedge_R^n M)$. The Leibniz extension is functorial in the following sense: given a morphism of R-modules $f\colon M \to N$ and $(h, u) \in D_A(R, M)$,

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$(h, v) \in D_A(R, N)$ such that $vf = fu$, then $v_n f^{\wedge n} = f^{\wedge n} u_n$ for every n. Moreover, for every n, the map

$$D_A(R, M) \longrightarrow D_A(R, \wedge_R^n M), \quad (h, u) \longmapsto (h, u_n),$$

is a morphism of Lie algebras. This follows immediately from the fact that, for every $(h, u), (k, v) \in D_A(R, M)$, we have

$$\begin{aligned} u_n v_n(m_1 \wedge \cdots \wedge m_n) = &\sum_{i=1}^{n} m_1 \wedge \cdots \wedge uv(m_i) \wedge \cdots \wedge m_n \\ &+ \sum_{i<j} m_1 \wedge \ldots u(m_i) \wedge \cdots \wedge v(m_j) \cdots \wedge m_n \\ &+ \sum_{i<j} m_1 \wedge \ldots v(m_i) \wedge \cdots \wedge u(m_j) \cdots \wedge m_n. \end{aligned}$$

Definition 2.8 If M is a free module of rank n and $(h, u) \in D_A(R, M)$, we shall call $(h, u_n) \in D_A(R, \wedge_R^n M)$ the *trace* of (h, u).

The name trace is motivated by the fact that if $h = 0$, *i.e.*, if $u \in \mathrm{Hom}_R(M, M)$, then u_n is the multiplication by the trace of u.

2.1 Automorphisms of Pairs

Let $A \to R$ be a morphism of commutative unitary rings and let M be an R-module. We shall denote by $\mathrm{Aut}_A(R)$ the group of A-linear automorphisms of R, *i.e.*, the automorphism group of the A-algebra R, and by $\mathrm{Aut}_A(M)$ (resp. $\mathrm{Aut}_R(M)$) the group of A-linear (resp. R-linear) automorphisms of M.

Definition 2.9 The group of A-linear automorphisms of the pair (R, M) is defined as the subgroup $\mathrm{Aut}_A(R, M) \subset \mathrm{Aut}_A(R) \times \mathrm{Aut}_A(M)$ of pairs (θ, ϕ) such that $\phi(rm) = \theta(r)\phi(m)$ for every $r \in R$, $m \in M$.

Let $R \oplus M$ be the trivial extension of R by M. As in the proof of Lemma 2.3, there exists a natural inclusion

$$\mathrm{Hom}_A(R, R) \times \mathrm{Hom}_A(M, M) \subset \mathrm{Hom}_A(R \oplus M, R \oplus M),$$

and it is immediate to see that an element of $\mathrm{Hom}_A(R, R) \times \mathrm{Hom}_A(M, M)$ is an automorphism of the pair (R, M) if and only if it is an automorphism of the A-algebra $R \oplus M$.

The analog of the anchor map is the group homomorphism

$$\alpha \colon \mathrm{Aut}_A(R, M) \longrightarrow \mathrm{Aut}_A(R), \quad \alpha(\theta, \phi) = \theta,$$

whose kernel is $\mathrm{Aut}_R(M)$.

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There exists the analog of Lemma 2.6 and of the Leibniz extension for automorphisms of pairs. It is straightforward to verify that there exist two natural group homomorphisms

$$\begin{array}{ccc} & \mathrm{Aut}_A(R,M)\times_{\mathrm{Aut}_A(R)}\mathrm{Aut}_A(R,N) & \\ {\scriptstyle\Phi}\swarrow & & \searrow{\scriptstyle\Psi} \\ \mathrm{Aut}_A(R,M\otimes_R N) & & \mathrm{Aut}_A(R,\mathrm{Hom}_R(M,N)) \end{array} \tag{2.5}$$

$$\Phi((\theta,\phi),(\theta,\psi))=(\theta,\phi\otimes\psi),\quad \Psi((\theta,\phi),(\theta,\psi))=(\theta,f\mapsto\psi f\phi^{-1}).$$

Similarly, every $(\theta,\phi)\in\mathrm{Aut}_A(R,M)$ gives a sequence $(\theta,\phi_n)\in\mathrm{Aut}_A(R,\wedge^n_R M)$, $n\geq 0$, uniquely determined by the formulas

$$\phi_0=\theta,\quad \phi_n(m_1\wedge\cdots\wedge m_n)=\phi(m_1)\wedge\cdots\wedge\phi(m_n).$$

If M is free of rank n, we write $(\theta,\phi_n)=\det(\theta,\phi)$. When $\theta=\mathrm{Id}$, we recover the usual notion of the determinant of an R-linear endomorphism.

Assume now that A contains the field $\mathbb{Q}$ of rational numbers and let

$$(h,u)\in D_A(R,M)\subset\mathrm{Der}_A(R\oplus M,R\oplus M)$$

be a nilpotent derivation of pairs. Then its exponential

$$\exp(h,u)=\Big(\sum_{n=0}^{\infty}\frac{h^n}{n!},\sum_{n=0}^{\infty}\frac{u^n}{n!}\Big)$$

also preserves the direct sum decomposition $R\oplus M$ and then

$$\exp(h,u)\in\mathrm{Aut}_A(R,M)\subset\mathrm{Aut}_A(R\oplus M).$$

It is plain that the exponential commutes with the anchor maps and the usual properties of the exponential imply that exp commutes with the morphisms (2.2) and (2.5). The exponential also commutes with Leibniz extensions of derivations and automorphisms. This is immediate from the previous remark since the assumption $\mathbb{Q}\subset A$ implies that $\wedge^n_R M$ is a direct summand of $\otimes^n_R M$. In particular, the exponential of the trace is the determinant of the exponential.

Lemma 2.10 *Let R be a commutative unitary algebra over a field $\mathbb{K}$ of characteristic 0, and let M be an R-module. Then for every local Artin $\mathbb{K}$-algebra A with residue field $\mathbb{K}$, the group $\exp(D_{\mathbb{K}}(R,M)\otimes\mathfrak{m}_A)$ is naturally isomorphic to the group of A-linear automorphisms of the pair $(R\otimes A,M\otimes A)$ lifting the identity on (R,M).*

Proof We have already noticed that via the diagonal inclusion

$$\mathrm{Hom}_{\mathbb{K}}(R,R)\times\mathrm{Hom}_{\mathbb{K}}(M,M)\subset\mathrm{Hom}_{\mathbb{K}}(R\oplus M,R\oplus M),$$

a couple $(h,u)\in\mathrm{Hom}_{\mathbb{K}}(R,R)\times\mathrm{Hom}_{\mathbb{K}}(M,M)$ is a $\mathbb{K}$-linear derivation (resp. automorphism) of the pair (R,M) if and only if it is a $\mathbb{K}$-linear derivation (resp. automorphism) of the trivial extension $R\oplus M$. Thus, the lemma is an immediate consequence of the well-known fact [17, Proposition 5.44] that for every commutative unitary $\mathbb{K}$-algebra S, the group $\exp(\mathrm{Der}_{\mathbb{K}}(S,S)\otimes\mathfrak{m}_A)$ is naturally isomorphic to the group of A-linear automorphisms of $S\otimes A$ lifting the identity on S. ∎

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3 Lifting to Resolutions

Throughout this section, $A \to R$ is a fixed morphism of unitary commutative rings. For every $(h, u) \in D_A(R, M)$, the derivation h preserves the annihilator of M. In fact, if $r \in R$ and $rm = 0$ for every $m \in M$, then also $h(r)m = u(rm) - ru(m) = 0$, for every $m \in M$.

The above consideration shows that in general the anchor map

$$\alpha\colon D_A(R, M) \longrightarrow \mathrm{Der}_A(R, R)$$

is not surjective. For instance, if $M = R/I$, with I an ideal of R, then the image of α is the submodule of derivations preserving the ideal I. If $(h, u) \in D_A(R, M)$, we have seen that h preserves the annihilator of M, which is precisely the ideal I. Conversely, any derivation $h \in \mathrm{Der}_A(R, R)$ has a canonical lifting $(h, h) \in D_A(R, R)$, and if $h(I) \subset I$, then (h, h) factors to an element of $D_A(R, M)$.

Lemma 3.1 *In the above setup, if $f\colon P \to M$ is a surjective morphism of R-modules with P projective, then every $(h, u) \in D_A(R, M)$ lifts to an element $(h, v) \in D_A(R, P)$ such that $fv = uf$. In particular, we have the following.*

(i) *Every derivation of pairs lifts to any projective resolution.*
(ii) *If P is a projective R-module, then the anchor map $\alpha\colon D_A(R, P) \to \mathrm{Der}_A(R, R)$ is surjective.*

Proof Consider first the case when P is a free R-module with basis $\{e_i\}$. Choosing elements $v_i \in P$ such that $f(v_i) = u(f(e_i)) \in M$ for every i, then the A-linear map

$$v\colon P \longrightarrow P, \quad v\Big(\sum_i a_i e_i\Big) = \sum_i a_i v_i + h(a_i)e_i,$$

has the required properties. If P is not free, since every projective module is a direct summand of a free module, there exist a free module F together with two morphisms $i\colon P \to F$, $g\colon F \to P$ such that $gi = \mathrm{Id}_P$. Since $fg\colon F \to M$ is surjective, there exists $(h, w) \in D_A(R, F)$ lifting (h, u) and it is sufficient to take

$$(h, v) = (h, gwi) \in D_A(R, P).$$

If $M = 0$, then for every $h \in \mathrm{Der}_A(R, R)$ we have $(h, 0) \in D_A(R, 0)$ and the above computation gives the surjectivity of the anchor map $\alpha\colon D_A(R, P) \to \mathrm{Der}_A(R, R)$. ■

Lemma 3.2 *In the above setup, if $g\colon M \to J$ is an injective morphism of R-modules with J injective, then every $(h, u) \in D_A(R, M)$ extends to an element $(h, v) \in D_A(R, J)$ such that $vg = gu$. In particular, we have the following.*

(i) *Every derivation of pairs extends to any injective resolution.*
(ii) *If J is an injective R-module, then the anchor map $\alpha\colon D_A(R, J) \to \mathrm{Der}_A(R, R)$ is surjective.*

Proof Let $q\colon J \to I$ be an injective morphism with I injective as an A-module. Then the composition $qg\colon M \to J \to I$ is also injective. Consider the R-module $\mathrm{Hom}_A(R, I)$, where the R-module structure is given by $(t\psi)(r) = \psi(tr)$ for any $t, r \in R$ and any

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$\psi \in \mathrm{Hom}_A(R,I)$. Then the map

$$\beta\colon J \longrightarrow \mathrm{Hom}_A(R,I), \quad j \longmapsto \beta(j)(r) = q(rj) \in I, \quad \forall\, r \in R,$$

is an injective R-linear morphism. Since J is injective as an R-module, there exists a splitting $\gamma\colon \mathrm{Hom}_A(R,I) \to J$, such that $\gamma\beta = \mathrm{Id}_J$. Let $(h,u) \in D_A(R,M)$ be a fixed derivation of the pair; since I is injective as A-module and u is a morphism of A-modules, there exists a morphism $w \in \mathrm{Hom}_A(I,I)$ such that $wqg = qgu$:

$$\begin{array}{ccc} M & \xrightarrow{qg} & I \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ M & \xrightarrow{qg} & I. \end{array}$$

Consider now the map $w_h\colon \mathrm{Hom}_A(R,I) \to \mathrm{Hom}_A(R,I)$ defined by

$$w_h(\psi) = w \circ \psi - \psi \circ h,$$

for any $\psi \in \mathrm{Hom}_A(R,I)$. Then $(h,w_h) \in D_A(R,\mathrm{Hom}_A(R,I))$ since w_h is A-linear and $w_h(t\psi) = tw_h(\psi) + h(t)\psi$, for any $t,r \in R$. Indeed, we have

$$w_h(t\psi)(r) = (w \circ t\psi - t\psi \circ h)(r) = w(\psi(tr)) - \psi(th(r)),$$

while

$$\begin{aligned} tw_h(\psi)(r) &= t(w \circ \psi - \psi \circ h)(r) = w(\psi(tr)) - \psi(h(tr)) \\ &= w(\psi(tr)) - \psi(th(r)) - \psi(rh(t)). \end{aligned}$$

Then we prove that (h,w_h) extends (h,u), *i.e.*, $\beta gu = w_h\beta g$. For every $r \in R$ and every $m \in M$, we have

$$\begin{aligned} (w_h\beta g(m))(r) &= w(\beta g(m)(r)) - \beta g(m)(h(r)) = w(q(rg(m)) - q(h(r)g(m)) \\ &= w(q(g(rm)) - q(g(h(r)m)) = qgu(rm) - q(g(h(r)m)) \\ &= qg(ru(m)) + qg(h(r)m) - q(g(h(r)m)) \\ &= qg(ru(m)) = \beta g(u(m))(r)\,. \end{aligned}$$

Finally, it is sufficient to take $(h,v) = (h,\gamma w_h\beta) \in D_A(R,J)$. Item (i) is now clear and item (ii) follows by considering the injective morphism $0 \to J$. ∎

Corollary 3.3 *Let $0 \to K \xrightarrow{\alpha} P \xrightarrow{\beta} M \to 0$ be a short exact sequence of projective R-modules, and denote by*

$$L = D_A(R, K \xrightarrow{\alpha} P) = D_A(R, P \xrightarrow{\beta} M) = \left\{ (h,u) \in D_A(R,P) \mid u(\alpha(K)) \subset \alpha(K) \right\}.$$

Then the natural morphisms of Lie algebras

$$L = D_A(R, K \xrightarrow{\alpha} P) \to D_A(R,K), \quad L = D_A(R, K \xrightarrow{\alpha} P) \to D_A(R,P),$$

$$D_A(R, P \xrightarrow{\beta} M) \to D_A(R,M),$$

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fit in the following exact sequences of R-modules

$$0 \longrightarrow L \longrightarrow D_A(R,P) \xrightarrow{p} \mathrm{Hom}_R(K,M) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}_R(P,K) \xrightarrow{j} L \to D_A(R,M) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}_R(M,P) \xrightarrow{h} L \longrightarrow D_A(R,K) \longrightarrow 0,$$

where $p(h,u) = \beta u \alpha$, $j(v) = (0, \alpha v)$, $h(u) = (0, u\beta)$.

Proof Since $\beta\alpha = 0$, the morphism p is properly defined and it is surjective because M is projective, $P \simeq K \oplus M$, and therefore

$$\mathrm{Hom}_R(P,P) \longrightarrow \mathrm{Hom}_R(K,M), \quad u \longmapsto \beta u \alpha,$$

is surjective. The surjectivity of $L \to D_A(R,M)$ is given by Lemma 3.1 and therefore the anchor map $\alpha\colon L \to \mathrm{Der}_A(R,R)$ is also surjective. The third exact sequence follows by the snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & L & \longrightarrow & \mathrm{Der}_A(R,R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(K,K) & \longrightarrow & D_A(R,K) & \longrightarrow & \mathrm{Der}_A(R,R) & \longrightarrow & 0 \end{array}$$

where $S = \{f \in \mathrm{Hom}_R(P,P) \mid f(K) \subset K\}$. ∎

Consider now a morphism of commutative unitary rings $A \to R$ and a cochain complex $C = \{\cdots \to C^i \xrightarrow{d} C^{i+1} \to \cdots\}$ of R-modules. Then we can define $D_A(R,C)$ as in Definition 2.1, by replacing $\mathrm{Hom}_A(M,M)$ with the module of morphisms of cochain complexes of A-modules. Equivalently $D_A(R,C)$ is defined considering C as a diagram of R-modules over the ordered set $\mathbb{Z}$.

However, for the application we have in mind, it is more convenient to consider the DG-Lie subalgebra

$$D_A^*(R,C) = \left\{ (h,u) \in \mathrm{Der}_A^*(R,R) \times \mathrm{Hom}_A^*(C,C) \;\middle|\; \begin{array}{l} u(rx) - ru(x) = h(r)x \\ \text{for every } r \in R, x \in C \end{array} \right\}.$$

For the definition and the main properties of $\mathrm{Der}_A^*(R,R)$ and $\mathrm{Hom}_A^*(C,C)$, we refer the reader to [15, §1]. Notice that the differential is the internal derivation $\delta = [d,-]$ and $\mathrm{Der}_A^i(R,R) = 0$ for every $i \neq 0$: this implies that $D_A^i(R,C) = \mathrm{Hom}_R^i(C,C)$ for every $i \neq 0$, and $D_A^0(R,C) = \prod^{\times} D_A(R,C^i)$ is the limit of the diagram of anchor maps $D_A(R,C^i) \to \mathrm{Der}_A(R,R)$. Finally, note that $D_A(R,C) = Z^0(D_A^*(R,C))$.

In order to extend Lemma 3.1 and Corollary 3.3 to the differential graded case, it is useful to work in the projective model structure on the category of unbounded cochain complexes [12, Theorem 2.3.11]: with respect to this model structure, a morphism of cochain complexes $M \to N$ is a weak-equivalence if it is a quasi-isomorphism. It is a fibration if it is degreewise surjective. It is a cofibration if it has the left lifting property with respect to all the trivial fibrations. Moreover, if $M \to N$ is a cofibration, then for every i the map $M^i \to N^i$ is a split injective with projective cokernel; the converse holds whenever N is bounded above.

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In particular, for every cofibrant complex P, since P^i is projective for every i, there exists a short exact sequence

$$0 \longrightarrow \mathrm{Hom}^*_R(P,P) \longrightarrow D^*_A(R,P) \longrightarrow \mathrm{Der}_A(R,R) \longrightarrow 0. \tag{3.1}$$

Proposition 3.4 *Let $f\colon K \to P$ be a trivial cofibration between cofibrant complexes of R-modules. Then the two natural DG-Lie algebra morphisms*

$$D^*_A(R,K) \longleftarrow D^*_A(R,K \xrightarrow{f} P) \longrightarrow D^*_A(R,P)$$

are quasi-isomorphisms of complexes of R-modules. In particular, $D^*_A(R,K)$, $D^*_A(R,P)$ are quasi-isomorphic DG-Lie algebras.

Proof By assumption f is a cofibration and then we have a short exact sequence of cofibrant complexes

$$0 \to K \xrightarrow{f} P \to M \to 0.$$

Since K^i, P^i, M^i are projective modules for every i, Corollary 3.3 gives two short exact sequences of complexes

$$0 \longrightarrow D^*_A(R,K \xrightarrow{f} P) \longrightarrow D^*_A(R,P) \longrightarrow \mathrm{Hom}^*_R(K,M) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Hom}^*_R(M,P) \longrightarrow D^*_A(R,K \xrightarrow{f} P) \longrightarrow D^*_A(R,K) \longrightarrow 0.$$

Finally, since M is cofibrant acyclic, the natural map $M \to \mathrm{Cone}(\mathrm{Id}_M)$ is a trivial cofibration and then admits a left inverse; hence the complexes M, $\mathrm{Hom}^*_R(M,P)$, and $\mathrm{Hom}^*_R(K,M)$ are contractible. ∎

Corollary 3.5 *Let P and K be quasi-isomorphic cofibrant complexes of R-modules. Then $D^*_A(R,P)$ is quasi-isomorphic to $D^*_A(R,K)$ as a DG-Lie algebra.*

Proof By general facts of model category theory [12, Lemma 1.1.12], two cofibrant complexes P and K are quasi-isomorphic if and only if there exists a span of trivial cofibrations $P \to Q \leftarrow K$. Therefore, the conclusion follows from Proposition 3.4. ∎

Example 3.6 If $P \to M$ is a projective resolution of an R-module M, then Corollary 3.5 implies that the graded Lie algebra $H^*(D^*_A(R,P))$ depends only on M. Moreover, by (3.1) we have $H^i(D^*_A(R,P)) = \mathrm{Ext}^i_R(M,M)$ for every $i \neq 0,1$ and, by (2.1) of Lemma 2.4, there exists an exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_R(M,M) \longrightarrow H^0(D^*_A(R,P)) \longrightarrow \mathrm{Der}_A(R,R) \longrightarrow \mathrm{Ext}^1_R(M,M) \longrightarrow H^1(D^*_A(R,P)) \longrightarrow 0.$$

In fact, $\mathrm{Ext}^i_R(M,M) = H^i(\mathrm{Hom}^*_R(P,M)) = H^i(\mathrm{Hom}^*_R(P,P))$ for every i. By Lemma 3.1, the natural map $Z^0(D^*_A(R,P)) \to D_A(R,M)$ is surjective; its kernel is given by the R-linear morphisms of complexes $P \to P$ inducing the trivial map on M. Since P is a projective resolution, these morphisms are exactly the ones homotopic to 0, and then there exists a natural isomorphism $H^0(D^*_A(R,P)) = D_A(R,M)$. Since

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$\mathrm{Ext}^i_R(M,M) = 0$ for $i < 0$, the above equality is completely equivalent to the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}^{-2}_R(P,P) \xrightarrow{\delta} \mathrm{Hom}^{-1}_R(P,P) \xrightarrow{\delta} D_A(R,P) \longrightarrow D_A(R,M) \longrightarrow 0.$$

4 Anchor Invariance of Fitting Ideals

The content of this section is not relevant for the remaining part of the paper, and it is written as an application of the previous results that we consider of independent interest.

Let $A \to R$ be a morphism of unitary commutative rings and M an R-module. We have seen that, unless M is either projective or injective, the anchor map

$$D_A(R,M) \xrightarrow{\alpha} \mathrm{Der}_A(R,R)$$

is generally not surjective.

Theorem 4.1 *In the above setup, if M is finitely generated, then for every $(h,u) \in D_A(R,M)$, the derivation h preserves the Fitting ideals of M.*

Proof Recall [4, 26] that the Fitting ideals of M, $\mathrm{Fitt}_0(M) \subset \mathrm{Fitt}_1(M) \subset \cdots \subset R$, are defined by considering any free resolution $F \xrightarrow{f} R^m \xrightarrow{p} M \to 0$ and the exterior powers $\wedge^i_R F \xrightarrow{f^{\wedge i}} \wedge^i_R R^m$. Then, for every $i \geq 0$, the Fitting ideal $\mathrm{Fitt}_{m-i}(M) \subset R$ is the ideal generated by the coefficients of all elements in the image of $f^{\wedge i}$, with respect to the canonical basis of $\wedge^i_R R^m$. The definition is independent of the choice of the resolution and then $\mathrm{Fitt}_m(M) = R$ whenever M is generated by m elements.

By Lemma 3.1, the derivation of the pair (h,u) lifts to a couple $(h,v) \in D_A(R,R^m)$ and $(h,w) \in D_A(R,F)$ such that $up = pv$ and $fw = vf$. Let us prove first that the ideal $I = \mathrm{Fitt}_{m-1}(M)$ is preserved by h. Let $e_1,\dots,e_m$ be the canonical basis of R^m. For every $x \in F$, we have

$$f(x) = \sum a_i e_i, \quad f(w(x)) = \sum b_i e_i, \quad a_i, b_i \in I,$$

and then

$$\sum h(a_i)e_i = v(f(x)) - \sum a_i v(e_i) = \sum b_i e_i - \sum a_i v(e_i) \in I \cdot R^m,$$

proving that $h(a_i) \in I$ for every i. As regards the invariance of the Fitting ideals $\mathrm{Fitt}_{m-r}(M)$ for $r > 1$, it is sufficient to repeat the above argument to the maps

$$f^{\wedge r} \colon \wedge^r_R F \longrightarrow \wedge^r_R R^m$$

and to the Leibniz extensions of (h,v) and (h,w). ∎

Corollary 4.2 *Let $A \to R$ be a morphism of noetherian rings and let M and N be two finitely generated R-modules. If a derivation $h \in \mathrm{Der}_A(R,R)$ lifts to $D_A(R,M)$ and to $D_A(R,N)$, then h preserves the Fitting ideals of $\mathrm{Tor}^R_i(M,N)$ and $\mathrm{Ext}^i_R(M,N)$.*

Proof Let $P \to M$ be a projective resolution, then h lifts to an element of $D_A(R,P)$, and by Lemma 2.6 h lifts also to $D_A(R,\mathrm{Tor}^R_i(M,N))$ and $D_A(R,\mathrm{Ext}^i_R(M,N))$. ∎

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Joining Theorem 4.1 and Example 2.5 we get a new proof of the following classical result.

Corollary 4.3 (R. Hart [10]) *Let A be a commutative ring and R a finitely generated commutative A-algebra. Then every A-derivation of R preserves the Fitting ideals of $\Omega_{R/A}$.*

More generally, taking the Leibniz extension of the Lie derivative, we also obtain that every A-derivation of R preserves the Fitting ideals of $\wedge^n_R\Omega_{R/A}$ for every n. Notice also that the Leibniz extension of the Lie derivative commutes with the de Rham differential.

Example 4.4 Assume that R is a principal ideal domain and that M is a finitely generated R-module. Then a derivation $h \in \mathrm{Der}_A(R,R)$ lifts to $D_A(R,M)$ if and only if h preserves the Fitting ideals of M. In fact, we have a cyclic decomposition

$$M = \frac{R}{(a_1)}e_1 \oplus \cdots \oplus \frac{R}{(a_n)}e_n, \quad a_i | a_{i+1},$$

and therefore the Fitting ideals are $(a_1a_2\cdots a_n) \subset \cdots \subset (a_1a_2) \subset (a_1)$. Assume that h preserves all the Fitting ideals, then $h(a_i) \in (a_i)$ for every i; this is clear if either $i = 1$ or $a_i = 0$, while for $i > 1$ and $a_i \neq 0$, we have

$$h(a_1\cdots a_{i-1}) \in (a_1\cdots a_{i-1}), \quad h(a_1\cdots a_i) \in (a_1\cdots a_i),$$
$$h(a_1\cdots a_i) = h(a_1\cdots a_{i-1})a_i + a_1\cdots a_{i-1}h(a_i),$$

and then $a_1\cdots a_{i-1}h(a_i) \in (a_1\cdots a_{i-1}a_i)$. Now the map

$$u\colon M \longrightarrow M, \quad u\Big(\sum x_ie_i\Big) = \sum h(x_i)e_i$$

gives a derivation of the pair (h,u).

5 Coherent Sheaves of DG-Lie Algebras and Trace Maps

Assume now that $X \to \operatorname{Spec} A$ is a scheme over A, and denote by

$$\Theta_{X/A} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_{X/A}, \mathcal{O}_X) = \mathcal{D}\mathrm{er}_A(\mathcal{O}_X, \mathcal{O}_X)$$

the relative tangent sheaf. Given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we can define $\mathcal{D}_A(X,\mathcal{F})$ as the subsheaf of $\Theta_{X/A} \times \mathcal{H}\mathrm{om}_A(\mathcal{F},\mathcal{F})$ of all the elements satisfying the same condition of Definition 2.1. Similarly, for every morphism $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules, we can define the sheaf $\mathcal{D}_A(X, \mathcal{F} \xrightarrow{f} \mathcal{G})$, for instance, by the exact sequence

$$0 \longrightarrow \mathcal{D}_A(X, \mathcal{F} \xrightarrow{f} \mathcal{G}) \longrightarrow \mathcal{D}_A(X,\mathcal{F}) \times_{\Theta_{X/A}} \mathcal{D}_A(X,\mathcal{G}) \xrightarrow{\Phi} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}),$$

where $\Phi(h,u,v) = fu - vf$.

If $\mathcal{F}$ is coherent and $p\colon Y \to X$ is the affine morphism of schemes such that $p_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}$ [11, Exercise II.5.17], then by Lemma 2.3 we have an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{D}_A(X,\mathcal{F}) \longrightarrow p_*\Theta_{Y/A} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X) \oplus \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_{X/A},\mathcal{F}). \tag{5.1}$$

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Proposition 5.1 *In the above situation, if A is noetherian, X is of finite type over A, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves, then also $\mathcal{D}_A(X,\mathcal{F})$, $\mathcal{D}_A(X,\mathcal{G})$, and $\mathcal{D}_A(X,\mathcal{F}\xrightarrow{f}\mathcal{G})$ are coherent. For every open affine subset $U\subset X$, we have*

$$\mathcal{D}_A(X,\mathcal{F})(U)\cong D_A(\mathcal{O}_X(U),\mathcal{F}(U)),$$

$$\mathcal{D}_A(X,\mathcal{F}\xrightarrow{f}\mathcal{G})(U)\cong D_A(\mathcal{O}_X(U),\mathcal{F}(U)\xrightarrow{f}\mathcal{G}(U)).$$

Proof According to the above assumptions, the scheme Y is also of finite type over A and the morphism p is finite. In particular, both $\Omega_{X/A}$ and $p_*\Theta_{Y/A}$ are coherent sheaves and so $\mathcal{D}_A(X,\mathcal{F})$ is coherent, being the kernel of a morphism of coherent sheaves. Given an open affine subset $U\subset X$, the proof that the natural map $\mathcal{D}_A(X,\mathcal{F})(U)\to D_A(\mathcal{O}_X(U),\mathcal{F}(U))$ is an isomorphism follows by comparing the exact sequences obtained by applying the functor $\Gamma(U,\underline{\ })$ to (5.1) and the exact sequence of Lemma 2.3. The statement about $\mathcal{D}_A(X,\mathcal{F}\xrightarrow{f}\mathcal{G})$ is proved in the same way. ∎

Example 5.2 Let $\mathcal{F}$ be a locally free sheaf on X, then the sheaf $\mathcal{D}_A(X,\mathcal{F})$ is isomorphic to the sheaf of A-linear first order differential operators on $\mathcal{F}$ with scalar symbol. Via this isomorphism, the exact sequence

$$0\longrightarrow\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})\longrightarrow\mathcal{D}_A(X,\mathcal{F})\longrightarrow\Theta_{X/A}\longrightarrow 0$$

corresponds, up to isomorphism, to the Atiyah extension of $\mathcal{F}$ [2], [20, Example 2.3], [25, p. 145].

Definition 5.3 Let $X\to\operatorname{Spec}A$ be a noetherian scheme over a noetherian ring A. By a (quasi)coherent sheaf of Lie algebras over X/A we mean a (quasi)coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$, together an A-bilinear bracket $\mathcal{L}\times\mathcal{L}\xrightarrow{[\cdot,\cdot]}\mathcal{L}$ inducing a structure of a Lie algebra over A on every stalk of $\mathcal{L}$. Note that the bracket is not $\mathcal{O}_X$-linear, so this is not the same structure as a Lie algebra in the monoidal category of quasi coherent modules. A morphism of (quasi)coherent sheaves of Lie algebras is a morphism of sheaves of $\mathcal{O}_X$-modules commuting with the brackets.

For instance, if $\mathcal{F}$ is a coherent sheaf on X, then the anchor map

$$\alpha\colon\mathcal{D}_A(X,\mathcal{F})\to\Theta_{X/A}$$

is a morphism of coherent sheaves of Lie algebras over X/A. If $\mathcal{F}\to\mathcal{G}$ is a morphism of coherent sheaves, then the natural maps

$$\mathcal{D}_A(X,\mathcal{F}\longrightarrow\mathcal{G})\longrightarrow\mathcal{D}_A(X,\mathcal{F}),\quad\mathcal{D}_A(X,\mathcal{F}\longrightarrow\mathcal{G})\longrightarrow\mathcal{D}_A(X,\mathcal{G}),$$

are morphisms of coherent sheaves of Lie algebras over X/A.

Recall that a differential graded (DG) Lie algebra over a commutative ring A is the data of a cochain complex (L,d) of A-modules, together with an A-bilinear map $[\cdot,\cdot]\colon L\times L\to L$ (called a bracket) of degree 0, such that the following conditions are satisfied.

- (graded skew symmetry) $[x,y]=-(-1)^{ij}[y,x]\in L^{i+j}$, for every $x\in L^i$ and $y\in L^j$.

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- $[x,x]=0$ for every $x\in L^{2i}$, and $[x,[x,x]]=0$ for every $x\in L^{2i+1}$, with $i\in\mathbb{Z}$.
- (graded Jacobi identity) $[x,[y,z]]=[[x,y],z]+(-1)^{ij}[y,[x,z]]$, for every $x\in L^i$, $y\in L^j$, and $z\in L$.
- (graded Leibniz rule) $d[x,y]=[dx,y]+(-1)^i[x,dy]$, for every $x\in L^i$ and $y\in L^j$.

A morphism of differential graded Lie algebras $\chi\colon L\to M$ is a morphism of cochain complexes that commutes with brackets.

Definition 5.4 Let $X\to\operatorname{Spec}A$ be a noetherian scheme over a noetherian ring A. By a quasi-coherent sheaf of DG-Lie algebras over X/A, we mean a complex $\mathcal{L}^*$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules, together with an A-bilinear bracket

$$\mathcal{L}^*\times\mathcal{L}^*\xrightarrow{[\cdot,\cdot]}\mathcal{L}^*$$

inducing a structure of DG-Lie algebra over A on every stalk of $\mathcal{L}$. A quasi-coherent sheaf of DG-Lie algebras $\mathcal{L}^*$ is coherent if $\oplus\mathcal{L}^i$ is a coherent sheaf. A morphism of quasi-coherent sheaves of DG-Lie algebras is a morphism of complexes of $\mathcal{O}_X$-modules commuting with the brackets.

Example 5.5 Let $X\to\operatorname{Spec}A$ be a noetherian scheme over a noetherian ring A, and let $\mathcal{E}^*$ be a bounded complex of coherent sheaves of $\mathcal{O}_X$-modules. For every index i, the sheaf $\mathcal{H}\mathrm{om}^i_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)=\prod_j\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}^j,\mathcal{E}^{j+i})$ is coherent, and then $\mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)$ is a coherent sheaf of DG-Lie algebras.

As in Section 3, we can define the complex of $\mathcal{O}_X$-modules $\mathcal{D}^*_A(X,\mathcal{E}^*)$ as the subsheaf of $\mathcal{D}\mathrm{er}^*_A(\mathcal{O}_X,\mathcal{O}_X)\times\mathcal{H}\mathrm{om}^*_A(\mathcal{E}^*,\mathcal{E}^*)$, whose elements are the pairs (h,u) such that $u(rm)-ru(m)=h(r)m$, for every $r\in\mathcal{O}_X$, $m\in\mathcal{E}^*$. For every $i\neq 0$, we have $\mathcal{D}^i_A(X,\mathcal{E}^*)=\mathcal{H}\mathrm{om}^i_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)$, while $\mathcal{D}^0_A(X,\mathcal{E}^*)$ is the limit of the set of anchor maps $\mathcal{D}^0_A(X,\mathcal{E}^j)\xrightarrow{\alpha}\mathcal{D}\mathrm{er}_A(\mathcal{O}_X,\mathcal{O}_X)=\Theta_{X/A}$. Thus $\mathcal{D}^*_A(X,\mathcal{E}^*)$ is a coherent sheaf of DG-Lie algebras and there exists a short exact sequence of coherent sheaves of DG-Lie algebras over X/A:

$$0\longrightarrow\mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)\longrightarrow\mathcal{D}^*_A(X,\mathcal{E}^*)\xrightarrow{\alpha}\Theta_{X/A}. \tag{5.2}$$

According to Lemma 3.1, the anchor map $\mathcal{D}^*_A(X,\mathcal{E}^*)\xrightarrow{\alpha}\Theta_{X/A}$ is surjective whenever every sheaf $\mathcal{E}^j$ is locally free. Analogously to Remark 5.1, for every open affine subset $U\subset X$, we have $\mathcal{D}_A(X,\mathcal{E}^*)(U)\cong D_A(\mathcal{O}_X(U),\mathcal{E}^*(U))$.

It is well known that, for every bounded complex $\mathcal{E}^*$ of locally free sheaves, we can define the trace map

$$\operatorname{Tr}\colon\mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)\longrightarrow\mathcal{O}_X, \tag{5.3}$$

which is a surjective morphism of complexes of coherent sheaves: given

$$f\in\mathcal{H}\mathrm{om}^i_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*),$$

$\operatorname{Tr}(f)=0$ for $i\neq 0$. If $i=0$, then f is the datum of a finite sequence of morphisms of locally free sheaves $f_j\colon\mathcal{E}^j\to\mathcal{E}^j$, and $\operatorname{Tr}(f)=\sum_j(-1)^j\operatorname{Tr}(f_j)$, where

$$\operatorname{Tr}\colon\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}^j,\mathcal{E}^j)\to\mathcal{O}_X$$

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is the usual trace map.

It is easy to verify that Tr is also a morphism of sheaves of DG-Lie algebras where $\mathcal{O}_X$ is equipped with the trivial bracket. Since the differential on $\mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)$ is an adjoint operator, it is sufficient to prove that $\mathrm{Tr}([f, g]) = 0$ for every

$$f \in \mathcal{H}\mathrm{om}^i_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) \quad \text{and} \quad g \in \mathcal{H}\mathrm{om}^j_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*).$$

This is clear if $i + j \neq 0$, and so it is not restrictive to assume $i = -j$. Since the trace is $\mathcal{O}_X$-linear, we can reduce this fact to the following statement from linear algebra: let V_i, $i \in \mathbb{Z}$, be a sequence of finite-dimensional vector spaces, with $V_i \neq 0$ for at most finitely many indices i. Then for every integer p and any two sequences of linear maps,

$$f_i \colon V_i \longrightarrow V_{i+p}, \qquad g_i \colon V_i \longrightarrow V_{i-p},$$

we have $\sum_i (-1)^i \mathrm{Tr}(f_{i-p}g_i - (-1)^{p^2} g_{i+p}f_i) = 0$. In fact, by a basic fact of linear algebra we have $\mathrm{Tr}(f_{i-p}g_i) = \mathrm{Tr}(g_i f_{i-p})$ for every i, and then

$$\begin{aligned}\sum_i (-1)^i \mathrm{Tr}(f_{i-p}g_i - (-1)^{p^2} g_{i+p}f_i) &= \sum_i (-1)^i \mathrm{Tr}(f_{i-p}g_i) - (-1)^{i+p} \mathrm{Tr}(g_{i+p}f_i) \\ &= \sum_i (-1)^i \mathrm{Tr}(f_{i-p}g_i) - (-1)^i \mathrm{Tr}(g_i f_{i-p}) = 0.\end{aligned}$$

Denoting by $\mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)_0$ the sheaf of traceless endomorphisms of $\mathcal{E}^*$, *i.e.*, the kernel of (5.3), and by $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{F})_0$ its hypercohomology groups, there exists a long exact sequence

$$\begin{aligned}(5.4)\quad \cdots \longrightarrow \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathrm{Tr}} H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F})_0 \\ \longrightarrow \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathrm{Tr}} H^2(X, \mathcal{O}_X) \longrightarrow \cdots.\end{aligned}$$

By the results of Section 2, we can extend the trace map $\mathrm{Tr}\colon \mathcal{H}\mathrm{om}^0_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) \to \mathcal{O}_X$ to a Lie algebra morphism

$$(5.5)\qquad \mathrm{Tr}\colon \mathcal{D}^0_A(X, \mathcal{E}^*) \longrightarrow \mathcal{D}_A(X, \det \mathcal{E}^*).$$

Assume that the finite complex $\mathcal{E}^*$ is nonzero only for degrees between n and m; then by definition, an element of $\mathcal{D}^0_A(X, \mathcal{E}^*)$ is given by a sequence $(h, u_n, \dots, u_m)$, where $h \in \mathcal{D}\mathrm{er}_A(\mathcal{O}_X, \mathcal{O}_X)$ and $(h, u_i) \in \mathcal{D}_A(X, \mathcal{E}^i)$ for every i. Taking the trace of every pair (h, u_i) (Definition 2.8) we obtain a finite sequence of derivations of pairs

$$(h, v_i) \in \mathcal{D}_A(X, \det \mathcal{E}^i), \quad v_i = \mathrm{Tr}(u_i), i = n, \dots, m.$$

Then for every odd index i, we consider the transpose (2.3):

$$(h, v_i)^T \in \mathcal{D}_A(X, (\det \mathcal{E}^i)^{-1}).$$

Since

$$[(h, v_0), (h, v_1)^T] \in \mathcal{D}_A(X, \det \mathcal{E}^0) \times_{\mathcal{D}\mathrm{er}_A(\mathcal{O}_X, \mathcal{O}_X)} \mathcal{D}_A(X, (\det \mathcal{E}^1)^{-1}),$$

we can apply the Lie morphism Φ of Lemma 2.6 in order to obtain an element

$$\mathrm{Tr}(h, u_0, u_1) \in \mathcal{D}_A(X, (\det \mathcal{E}^0) \otimes (\det \mathcal{E}^1)^{-1}).$$

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It is now clear that, with a finite number of constructions as above, we have a well-defined element

$$\mathrm{Tr}(h, u_n, \dots, u_m) \in \mathcal{D}_A(X, \det \mathcal{E}^*) = \mathcal{D}_A(X, \bigotimes_{i=n}^{m} (\det \mathcal{E}^i)^{(-1)^i}).$$

Moreover, we have also proved that all the maps considered above,

$$\mathcal{D}_A(X, \mathcal{E}^i) \longrightarrow \mathcal{D}_A(X, \det \mathcal{E}^i), \quad \prod_i{}^{\times} \mathcal{D}_A(X, \det \mathcal{E}^i) \longrightarrow \mathcal{D}_A(X, \det \mathcal{E}^*),$$

are morphisms of sheaves of Lie algebras. Keeping in mind that

$$\mathcal{D}^0_A(X, \mathcal{E}^*) = \prod_i{}^{\times} \mathcal{D}_A(X, \mathcal{E}^i), \quad \mathcal{D}^j_A(X, \mathcal{E}^*) = \mathcal{H}\mathrm{om}^j_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*), \quad j \neq 0,$$

we can glue together (5.5) and (5.3) and obtain a morphism of sheaves of DG-Lie algebras

$$\mathrm{Tr}\colon \mathcal{D}^*_A(X, \mathcal{E}^*) \longrightarrow \mathcal{D}_A(X, \det \mathcal{E}^*)\,. \tag{5.6}$$

The following theorem is now clear.

Theorem 5.6 *For every bounded complex of locally free sheaves $\mathcal{E}^*$ on a scheme X of finite type over a noetherian ring A, there exists a commutative diagram of morphisms of coherent sheaves of DG-Lie algebras*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) & \longrightarrow & \mathcal{D}^*_A(X, \mathcal{E}^*) & \xrightarrow{\alpha} & \Theta_{X/A} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{D}_A(X, \det \mathcal{E}^*) & \xrightarrow{\alpha} & \Theta_{X/A} & \longrightarrow & 0
\end{array}$$

6 A Short Review of Deformation Theory via DG-Lie Algebras

The main references for this section are [6,8,17–19]. From this section, and throughout the rest of the paper, we work over a fixed algebraically closed field $\mathbb{K}$ of characteristic zero. We denote by **Set** the category of sets and by $\mathbf{Art}_{\mathbb{K}}$ the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$, and by $\mathbf{DGLA} = \mathbf{DGLA}_{\mathbb{K}}$ the category of DG-Lie algebras over $\mathbb{K}$. Unless otherwise specified, for every local algebra $A \in \mathbf{Art}_{\mathbb{K}}$, we denote by $\mathfrak{m}_A$ its maximal ideal.

Given a DG-Lie algebra L over $\mathbb{K}$, we can define two functors of Artin rings. The Maurer–Cartan functor $\mathrm{MC}_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is defined by

$$\mathrm{MC}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \tfrac{1}{2}[x,x] = 0 \right\},$$

where the DG-Lie structure on $L \otimes \mathfrak{m}_A$ is obtained by scalar extension from the DG-Lie structure on L. The deformation functor $\mathrm{Def}_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\text{gauge}} = \frac{\left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \tfrac{1}{2}[x,x] = 0 \right\}}{\exp(L^0 \otimes \mathfrak{m}_A)},$$

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where the gauge action $*\colon \exp(L^0 \otimes \mathfrak{m}_A) \times \mathrm{MC}_L(A) \to \mathrm{MC}_L(A)$ may be defined by the explicit formula

$$e^a * x := x + \sum_{n\geq 0} \frac{[a,-]^n}{(n+1)!}([a,x] - da).$$

A quasi-isomorphism of DG-Lie algebras is a morphism that induces an isomorphism in cohomology. Two DG-Lie algebras L and M are said to be quasi-isomorphic or homotopy equivalent if they are equivalent under the equivalence relation generated by quasi-isomorphisms.

By standard facts about deformation functors [18,19], it is known that the tangent space to Def_L is isomorphic to $H^1(L)$ and that $H^2(L)$ is an obstruction space.

Remark 6.1 Every morphism $L \to M$ of DG-Lie algebras induces a natural transformation of the associated deformation functors $\mathrm{Def}_L \to \mathrm{Def}_M$. A basic result [19, Theorema 1.5] asserts that if $H^0(L) \to H^0(M)$ is surjective, $H^1(L) \to H^1(M)$ is bijective, and $H^2(L) \to H^2(M)$ is injective, then $\mathrm{Def}_L \to \mathrm{Def}_M$ is an isomorphism.

Lemma 6.2 *Let L be a DG-Lie algebra over a field of characteristic 0 with $H^1(L)$ finite-dimensional. If the natural map*

$$N^0 = \{x \in L^0 \mid dx = 0, [x, L^1] = 0\} \longrightarrow H^0(L)$$

is surjective, then Def_L is pro-representable.

Proof Notice first that N^0 is a Lie subalgebra of L^0. Then define $N^i = L^i$ for every $i \geq 2$ and choose a vector subspace $N^1 \subset L^1$ such that $L^1 = d(L^0) \oplus N^1$. The inclusion of DG-Lie algebras $N = \bigoplus_{i\geq 0} N^i \hookrightarrow L$ satisfies the condition of Remark 6.1 and then $\mathrm{Def}_L = \mathrm{Def}_N$. On the other side, the gauge action on MC_N is trivial and then $\mathrm{Def}_N = \mathrm{MC}_N$ is pro-representable by Schlessinger's theorem [23, Theorem 2.11]. ∎

Let Δ_{mon} be the category whose objects are finite ordinal sets and whose morphisms are order-preserving injective maps between them. A semicosimplicial differential graded Lie algebra is a covariant functor $\Delta_{\mathrm{mon}} \to \mathbf{DGLA}$. Equivalently, a semicosimplicial DG-Lie algebra $\mathfrak{g}^\Delta$ is a diagram

$$\mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \mathfrak{g}_2 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots,$$

where each $\mathfrak{g}_i$ is a DG-Lie algebra, and for each $i > 0$, there are $i+1$ morphisms of DG-Lie algebras $\partial_{k,i}\colon \mathfrak{g}_{i-1} \to \mathfrak{g}_i$, $k = 0, \ldots, i$, such that $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \geq l$. Here we use the non-standard notation of lower indexes for a semicosimplicial object, since the upper indexes are already used to denote degrees.

We denote by $\mathrm{Tot}(\mathfrak{g}^\Delta)$ the image of a semicosimplicial differential graded Lie algebra $\mathfrak{g}^\Delta$ via the Thom–Whitney totalization functor $\mathrm{Tot}\colon \mathbf{DGLA}^{\Delta_{\mathrm{mon}}} \to \mathbf{DGLA}$. We refer to [3, §], [6, §3.2], [7, §3] and [15, §3] for an explicit description of the DG-Lie algebra $\mathrm{Tot}(\mathfrak{g}^\Delta)$ and its properties. Here, we remind only that there exists a quasi-isomorphism of complexes of vector spaces between $\mathrm{Tot}(\mathfrak{g}^\Delta)$ and the total complex associated with the cochain complex $C(\mathfrak{g}^\Delta)$ (considering any $\mathfrak{g}^\Delta$ as a semicosimplicial object in the abelian category of DG-vector spaces). As a consequence of this fact, if $f\colon \mathfrak{g}^\Delta \to \mathfrak{h}^\Delta$ is a morphism of semicosimplicial differential graded Lie algebras such

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that $f\colon \mathfrak{g}_i \to \mathfrak{h}_i$ is a quasi-isomorphism of DG-Lie algebras for every i, then the image $\mathrm{Tot}(f)\colon \mathrm{Tot}(\mathfrak{g}^\Delta) \to \mathrm{Tot}(\mathfrak{h}^\Delta)$ is a quasi-isomorphism.

Considering the DG-Lie algebra $\mathrm{Tot}(\mathfrak{g}^\Delta)$, we can naturally associate with any semicosimplicial DG-Lie algebra $\mathfrak{g}^\Delta$ the functor of Artin rings $\mathrm{Def}_{\mathrm{Tot}(\mathfrak{g}^\Delta)}\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$. According to [6, Definition 3.1, Definition 3.3], we can also associate with $\mathfrak{g}^\Delta$ two other functors of Artin rings. The former $Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is defined, for any $A \in \mathbf{Art}_{\mathbb{K}}$, by

$$Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)(A) = \left\{ (l,m) \in (\mathfrak{g}_0^1 \times \mathfrak{g}_1^0) \otimes \mathfrak{m}_A \;\middle|\; \begin{array}{l} dl + \frac{1}{2}[l,l] = 0,\ \partial_{1,1} l = e^m * \partial_{0,1} l, \\ \partial_{0,2} m \bullet -\partial_{1,2} m \bullet \partial_{2,2} m \\ \qquad = dn + [\partial_{2,2}\partial_{0,1} l, n] \\ \text{for some } n \in \mathfrak{g}_2^{-1} \otimes \mathfrak{m}_A \end{array} \right\}.$$

The latter $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is defined, for any $A \in \mathbf{Art}_{\mathbb{K}}$, by

$$H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)(A) = Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)(A)/\sim,$$

where two elements (l_0, m_0) and $(l_1, m_1) \in Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)(A)$ are equivalent under the relation $\sim$ if and only if there exist elements $a \in \mathfrak{g}^0{}_0 \otimes \mathfrak{m}_A$ and $b \in \mathfrak{g}_1^{-1} \otimes \mathfrak{m}_A$ such that

$$e^a * l_0 = l_1, \quad -m_0 \bullet -\partial_{1,1} a \bullet m_1 \bullet \partial_{0,1} a = db + [\partial_{0,1} l_0, b].$$

Theorem 6.3 *Let $\mathfrak{g}^\Delta$ be a semicosimplicial DG-Lie algebra such that $H^j(\mathfrak{g}_i) = 0$ for all $i \geq 0$ and $j < 0$. Then there exists an isomorphism of functors of Artin rings*

$$\mathrm{Def}_{\mathrm{Tot}(\mathfrak{g}^\Delta)} \longrightarrow H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta). \tag{6.1}$$

Proof See [6, Theorem 4.10]. ∎

Remark 6.4 If each $\mathfrak{g}_i$ is concentrated in degree zero, *i.e.*, $\mathfrak{g}^\Delta$ is a semicosimplicial Lie algebra, then the functor $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ has an easier explicit description [7, §1]:

$$H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

$$H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)(A) = \frac{\{x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A \mid e^{\partial_0 x} e^{-\partial_1 x} e^{\partial_2 x} = 1\}}{\sim},$$

where $x \sim y$ if and only if there exists $a \in \mathfrak{g}_0 \otimes \mathfrak{m}_A$, such that $e^{-\partial_1 a} e^x e^{\partial_0 a} = e^y$.

Let $\mathcal{L}$ be a coherent sheaf of DG-Lie algebras over an algebraic variety X and $\mathcal{U} = \{U_i\}$ an affine open cover of X. In this case, the Čech (double) complex $C(\mathcal{U}, \mathcal{L})$ of $\mathcal{L}$ is exactly the total cochain complex associated with the semicosimplicial DG-Lie algebra

$$\mathcal{L}(\mathcal{U})\colon \quad \prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i,j} \mathcal{L}(U_{ij}) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \prod_{i,j,k} \mathcal{L}(U_{ijk}) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots,$$

where the face operators $\partial_h\colon \prod_{i_0,\dots,i_{k-1}} \mathcal{L}(U_{i_0\cdots i_{k-1}}) \to \prod_{i_0,\dots,i_k} \mathcal{L}(U_{i_0\cdots i_k})$ are given by $\partial_h(x)_{i_0\dots i_k} = x_{i_0\dots\widehat{i_h}\cdots i_k|U_{i_0\cdots i_k}}$, for $h = 0,\dots,k$.

For notational convenience, we denote by $\mathrm{Tot}(\mathcal{U}, \mathcal{L})$ the Thom–Whitney totalization of the semicosimplicial DG-Lie algebra $\mathcal{L}(\mathcal{U})$. Note that in this case

$$H^i(\mathrm{Tot}(\mathcal{U}, \mathcal{L})) = H^i(C(\mathcal{U}, \mathcal{L})) = \check{\mathbb{H}}^i(\mathcal{U}, \mathcal{L}) = \mathbb{H}^i(X, \mathcal{L}). \tag{6.2}$$

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Remark 6.5 For any coherent sheaf of DG-Lie algebras $\mathcal{L}$, the quasi-isomorphism class of $\mathrm{Tot}(\mathcal{U},\mathcal{L})$ does not depend on the choice of the affine open cover [6]. If $\mathcal{L}\to\mathcal{G}$ is a morphism of coherent sheaves of DG-Lie algebras that is a quasi-isomorphism at every stalk, then for any affine open cover $\mathcal{U}$, the induced morphism of DG-Lie algebras $\mathrm{Tot}(\mathcal{U},\mathcal{L})\to\mathrm{Tot}(\mathcal{U},\mathcal{G})$ is a quasi-isomorphism of DG-Lie algebras.

This motivates the following definition.

Definition 6.6 Let $\mathcal{L}$ be a (quasi) coherent sheaf of DG-Lie algebras over an algebraic variety X. Suppose that, for an open affine cover $\mathcal{U}$ of X (and so for all), the DG-Lie algebra $\mathrm{Tot}(\mathcal{U},\mathcal{L})$ controls a given deformation problem. Then we also say that $\mathcal{L}$ controls the deformation problem. Note that in this case $\mathbb{H}^1(X,\mathcal{L})$ is the tangent space and $\mathbb{H}^2(X,\mathcal{L})$ is an obstruction space for the controlled deformation problem.

7 Deformations of Pairs (Scheme, Coherent Sheaf)

Henceforth, unless otherwise specified, we denote by X a scheme of finite type over the field $\mathbb{K}$.

Definition 7.1 Let $\mathcal{F}$ be a coherent sheaf on X. An infinitesimal deformation of the pair $(X,\mathcal{F})$ over $A\in\mathbf{Art}_{\mathbb{K}}$ is the data $(\mathcal{X}_A,\mathcal{F}_A)$, where

- $\mathcal{X}_A$ is an infinitesimal deformation of X over A, *i.e.*, a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X}_A \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}\mathbb{K} & \longrightarrow & \operatorname{Spec} A, \end{array}$$

  where π is flat;
- $\mathcal{F}_A$ is a coherent sheaf of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{X}_A$, flat over A, and a morphism $\mathcal{F}_A\to\mathcal{F}$, inducing an isomorphism $\mathcal{F}_A\otimes_{\mathcal{O}_{\mathcal{X}_A}}\mathcal{O}_{\mathcal{X}}\to\mathcal{F}$.

Definition 7.2 Let $\mathcal{F}$ be a coherent sheaf on a scheme X. Two infinitesimal deformations $(\mathcal{X}_A,\mathcal{F}_A)$ and $(\mathcal{X}'_A,\mathcal{F}'_A)$ of the pair $(X,\mathcal{F})$ over A are isomorphic if there exist an isomorphism of deformations $f\colon\mathcal{X}_A\to\mathcal{X}'_A$ and an isomorphism $\mathcal{F}_A\to f^*\mathcal{F}'_A$ of coherent sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules.

We recall that the trivial infinitesimal deformation of $(X,\mathcal{F})$ over $A\in\mathbf{Art}_{\mathbb{K}}$ is given by the pair $(X\times\operatorname{Spec}A,\mathcal{F}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}\times\operatorname{Spec}\mathcal{A}}=\mathcal{F}\otimes_{\mathbb{K}}A)$.

An infinitesimal deformation of $(X,\mathcal{F})$ is locally trivial if it is locally in X isomorphic to the trivial infinitesimal deformation.

Definition 7.3 Let $\mathcal{F}$ be a coherent sheaf on a scheme X. The functor of infinitesimal deformations of the pair $(X,\mathcal{F})$ is the functor $\mathrm{Def}_{(X,\mathcal{F})}\colon\mathbf{Art}_{\mathbb{K}}\to\mathbf{Set}$, where $\mathrm{Def}_{(X,\mathcal{F})}(A)$ is defined to be the set of isomorphism classes of deformations of the pair $(X,\mathcal{F})$ over A. The functor of locally trivial infinitesimal deformations of the

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pair $(X,\mathcal{F})$ is the functor $\mathrm{Def}^{\mathrm{lt}}_{(X,\mathcal{F})}\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, where $\mathrm{Def}^{\mathrm{lt}}_{(X,\mathcal{F})}(A)$ is defined to be the set of isomorphism classes of locally trivial deformations of the pair $(X,\mathcal{F})$ over A.

Remark 7.4 If X is a smooth variety, then any infinitesimal deformation of X is locally trivial. If $\mathcal{F}$ is a locally free sheaf over a variety X, then any infinitesimal deformation of the sheaf is locally trivial. Then there exists an isomorphism of functors $\mathrm{Def}^{\mathrm{lt}}_{(X,\mathcal{F})} \cong \mathrm{Def}_{(X,\mathcal{F})}$, for a locally free sheaf $\mathcal{F}$ on a smooth variety X.

7.1 Locally Trivial Infinitesimal Deformations

Let us analyse the data that define a locally trivial infinitesimal deformation $(\mathcal{X}_A,\mathcal{F}_A)$ of $(X,\mathcal{F})$ over A. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an affine open cover of X. The deformation $\mathcal{X}_A$ of X is trivial over every affine open subset; thus for any $i\in I$, there exists an isomorphism $\alpha_i\colon \mathcal{O}_X(U_i)\otimes A\to \mathcal{O}_{\mathcal{X}_A}(U_i)$. On every $U_i\times \operatorname{Spec} A$, we have the sheaf $\mathcal{F}_{|U_i}\otimes A$ of $\mathcal{O}_X(U_i)\otimes A$-modules. Since α_i is an isomorphism, we can view $\mathcal{F}_{|U_i}\otimes A$ as a sheaf of $\mathcal{O}_{\mathcal{X}_A}(U_i)$-modules; the $\mathcal{O}_{\mathcal{X}_A}(U_i)$-modules structure on $\mathcal{F}_{|U_i}\otimes A$ is given by $s\cdot x=\alpha_i{}^{-1}(s)x$, for any $s\in\mathcal{O}_{\mathcal{X}_A}(U_i)$ and $x\in\mathcal{F}_{|U_i}\otimes A$.

To give a locally trivial deformation $\mathcal{F}_A$ of the sheaf $\mathcal{F}$ over X_A, we need to glue the sheaves $\mathcal{F}_{|U_i}\otimes A$ of $\mathcal{O}_{\mathcal{X}_A}(U_i)$-modules over double intersections. The isomorphisms $\alpha_{i|U_{ij}}\colon \mathcal{O}_X(U_{ij})\otimes A\to \mathcal{O}_{\mathcal{X}_A}(U_{ij})$ and $\alpha_{j_{|U_{ij}}}\colon \mathcal{O}_X(U_{ij})\otimes A\to \mathcal{O}_{\mathcal{X}_A}(U_{ij})$ induce a structure of sheaf of $\mathcal{O}_{\mathcal{X}_A}(U_{ij})$-module on the sheaves $\mathcal{F}_{|U_i}\otimes A_{|U_{ij}}$ and $\mathcal{F}_{|U_i}\otimes A_{|U_{ij}}$, respectively. To glue these sheaves, we need an isomorphism

$$\psi_{ij}\colon \mathcal{F}_{|U_i}\otimes A_{|U_{ij}}\longrightarrow \mathcal{F}_{|U_i}\otimes A_{|U_{ij}}$$

of $\mathcal{O}_{\mathcal{X}_A}(U_{ij})$-modules, *i.e.*, for all $s\in\mathcal{O}_{\mathcal{X}_A}(U_{ij})$, $\psi_{ij}(s\cdot x)=s\cdot\psi(x)$. By the explicit definition of $\mathcal{O}_{\mathcal{X}_A}(U_{ij})$-modules structures, we have

$$\psi_{ij}(s\cdot x)=\psi_{ij}(\alpha^{-1}_{i|U_{ij}}(s)x)\quad\text{ and }\quad s\cdot\psi_{ij}(x)=\alpha^{-1}_{j|U_{ij}}(s)\psi_{ij}(x).$$

Since $\alpha_{i|U_{ij}}$ is an isomorphism, there exists $t\in\mathcal{O}_X(U_{ij})\otimes A$ such that $s=\alpha_{i|U_{ij}}(t)$. Therefore, the linearity reads as follows: $\psi_{ij}(tx)=\alpha^{-1}_{j|U_{ij}}\alpha_{i|U_{ij}}(t)\psi_{ij}(x)$, for any

$$t\in\mathcal{O}_X(U_{ij})\otimes A\quad\text{ and }\quad x\in\mathcal{F}_{|U_{ij}}\otimes A.$$

In conclusion, in order to define a locally trivial infinitesimal deformation of $(X,\mathcal{F})$ over A, it is enough to give, for any U_{ij}, an automorphism $\theta_{ij}=\alpha^{-1}_{j|U_{ij}}\alpha_{i|U_{ij}}$ of

$$\mathcal{O}_X(U_{ij})\otimes A$$

satisfying the cocycle condition on the triple intersections, an automorphism $\psi_{ij}\in \mathrm{Hom}_{\mathbb{K}}(\mathcal{F}_{|U_{ij}}\otimes A,\mathcal{F}_{|U_{ij}}\otimes A)$ satisfying the cocycle condition on the triple intersections and such that $\psi_{ij}(tx)=\theta_{ij}(t)(x)$, for any $t\in\mathcal{O}_X(U_{ij})\otimes A$ and $x\in\mathcal{F}_{|U_{ij}}\otimes A$.

Since we are in characteristic zero, according to Lemma 2.10, we can take the logarithm and conclude that $(\theta_{ij},\psi_{ij})=e^{x_{ij}}$, where $x_{ij}\in\mathcal{D}_{\mathbb{K}}(X,\mathcal{F})(U_{ij})\otimes\mathfrak{m}_A$; the condition of gluing on triple intersections is equivalent to

$$e^{x_{jk}}e^{-x_{ik}}e^{x_{ij}}=1\in\exp(\mathcal{D}_{\mathbb{K}}(X,\mathcal{F})(U_{ijk})\otimes\mathfrak{m}_A),\quad\forall\ i,j,k\in I.$$

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As regards the isomorphisms, suppose that $(\mathcal{X}_A, \mathcal{F}_A)$ and $(\mathcal{X}'_A, \mathcal{F}'_A)$ are locally trivial isomorphic infinitesimal deformations of the pair $(X, \mathcal{F})$ over A. Then there exist an isomorphism of deformations $f: \mathcal{X}_A \to \mathcal{X}'_A$ and an isomorphism $\phi: \mathcal{F}_A \to f^*\mathcal{F}'_A$ of coherent sheaves of $\mathcal{O}_{\mathcal{X}_A}$-modules.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an affine open cover of X. Then for each $i \in I$, there exists an induced automorphism of $\mathcal{O}_X(U_i) \otimes A$:

$$b_i = \alpha_i'^{-1} f_{|U_i}^{-1} \alpha_i : \mathcal{O}_X(U_i) \otimes A \xrightarrow{\alpha_i} \mathcal{O}_{\mathcal{X}_A}(U_i) \xrightarrow{f_{|U_i}^{-1}} \mathcal{O}_{\mathcal{X}'_A}(U_i) \xrightarrow{\alpha_i'^{-1}} \mathcal{O}_X(U_i) \otimes A.$$

Therefore, $\alpha'_i b_i = f_{|U_i}^{-1} \alpha_i$, and so on U_{ij} we have

$$(\alpha'_i b_i)^{-1} (\alpha'_j b_j)_{|U_{ij}} = {\alpha_i}^{-1} f_{|U_{ij}} f_{|U_{ij}}^{-1} \alpha_j = {\alpha_i}^{-1} \alpha_{j|U_{ij}}.$$

This implies ${b_i}^{-1} \theta'_{ji} b_j = \theta_{ji}$ *i.e.*, $\theta_{ij} = {b_j}^{-1} \theta'_{ij} b_i$, where $\theta_{ij} = {\alpha_j}^{-1} \alpha_i$ and $\theta'_{ij} = {\alpha'_j}^{-1} \alpha'_i$ are the automorphisms of $\mathcal{O}_X(U_{ij}) \otimes A$ corresponding to the deformations $\mathcal{X}_A$ and $\mathcal{X}'_A$, respectively.

As regards the sheaves, $\phi_{U_i}: \mathcal{F}_{A|U_i} \to f^*\mathcal{F}'_{A|U_i}$ is an isomorphism of $\mathcal{O}_{\mathcal{X}_A}(U_i)$-modules for any i. The structure of $\mathcal{O}_{\mathcal{X}_A}$-modules on $\mathcal{F}_{A|U_i}$ is given by α_i, while the structure on $f^*\mathcal{F}'_{A|U_i}$ is given by $s \cdot x = \alpha_i'^{-1} f^{-1}(s) x$, for every $s \in \mathcal{O}_{\mathcal{X}_A}(U_i)$. Therefore, ϕ_{U_i} must satisfy $\phi_{U_i}(sx) = s\phi_{U_i}(x)$, for all $s \in \mathcal{O}_{\mathcal{X}_A}(U_i)$, where

$$\phi_{U_i}(sx) = \phi_{U_i}({\alpha_i}^{-1}(s)x) \quad \text{and} \quad s\phi_{U_i}(x) = \alpha_i'^{-1} f^{-1}(s) \phi_{U_i}(x).$$

Since α_i is an isomorphism, there exists $t \in \mathcal{O}_X(U_i) \otimes A$ such that $s = \alpha_i(t)$. Therefore, we have $\phi_{U_i}(tx) = \alpha_i'^{-1}(f^{-1}(\alpha_i(t)))\phi_{U_i}(x) = b_i(t)\phi_{U_i}(x)$, for any $t \in \mathcal{O}_X(U_i) \otimes A$ and $x \in \mathcal{F}_{|U_{ij}} \otimes A$. Moreover, let

$$\psi_{ij} \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{F}_{|U_{ij}} \otimes A, \mathcal{F}_{|U_{ij}} \otimes A) \quad \text{and} \quad \psi'_{ij} \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{F}_{|U_{ij}} \otimes A, \mathcal{F}_{|U_{ij}} \otimes A),$$

the automorphism of $\mathcal{O}_{\mathcal{X}_A}(U_{ij})$-modules corresponding to the deformations $\mathcal{F}_A$ and $\mathcal{F}'_A$, respectively. Then the following diagram must commute:

$$\begin{array}{ccc} \mathcal{F}_{|U_i} \otimes A_{|U_{ij}} & \xrightarrow{\phi_{i|U_{ij}}} & \mathcal{F}_{|U_i} \otimes A_{|U_{ij}} \\ \downarrow{\scriptstyle \psi_{ij}} & & \downarrow{\scriptstyle \psi'_{ij}} \\ \mathcal{F}_{|U_j} \otimes A_{|U_{ij}} & \xrightarrow{\phi_{j|U_{ij}}} & \mathcal{F}_{|U_j} \otimes A_{|U_{ij}} \end{array}$$

i.e., $\psi'_{ij}\phi_{i|U_{ij}} = \phi_{j|U_{ij}}\psi_{ij}$, that is $\phi_{j|U_{ij}}^{-1} \psi'_{ij} \phi_{i|U_{ij}} = \psi_{ij}$.

In conclusion, let $(\mathcal{X}_A, \mathcal{F}_A)$ and $(\mathcal{X}'_A, \mathcal{F}'_A)$ be two locally trivial isomorphic deformations of the pair $(X, \mathcal{F})$ over A, corresponding to the elements $\{(\theta_{ij}, \psi_{ij})\}$ and $\{(\theta'_{ij}, \psi'_{ij})\}$. The isomorphism given by $f: \mathcal{X}_A \to \mathcal{X}'_A$ and $\phi: \mathcal{F}_A \to f^*\mathcal{F}'_A$ corresponds to the elements $(b_i, \phi_i) \in \mathcal{D}_A\big(\mathcal{O}_X(U_i) \otimes A, \mathcal{F}(U_i) \otimes A\big)$, for any i, such that ${b_i}^{-1} \theta'_{ij} b_j = \theta_{ij}$ and ${\phi_i}^{-1} \psi'_{ij} \phi_j = \psi_{ij}$, for any i and j.

Since we are in characteristic zero according to Lemma 2.10, we can take the logarithms and write $(b_i, \phi_i) = e^{a_i}$, where $a_i \in \mathcal{D}_{\mathbb{K}}(X, \mathcal{F})(U_i) \otimes \mathfrak{m}_A$. The condition of gluing is equivalent to $e^{-a_i} e^{x_{ij}} e^{a_j} = e^{x'_{ij}}$, where $\{e^{x_{ij}} = (\theta_{ij}, \psi_{ij})\}$ and $\{e^{x'_{ij}} = (\theta'_{ij}, \psi'_{ij})\}$.

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Theorem 7.5 *Let $\mathcal{F}$ be a coherent sheaf on a scheme X of finite type over the field $\mathbb{K}$. Then the coherent sheaf of Lie algebras $\mathcal{D}_{\mathbb{K}}(X,\mathcal{F})$ controls the problem of locally trivial infinitesimal deformations of the pair $(X,\mathcal{F})$.*

Proof According to Definition 6.6, we must prove that for any affine open cover $\mathcal{U}=\{U_i\}$ of X, there exists an isomorphism of functors of Artin rings

$$\mathrm{Def}_{\mathrm{Tot}(\mathcal{U},\mathcal{D}_{\mathbb{K}}(X,\mathcal{F}))} \longrightarrow \mathrm{Def}^{\mathrm{lt}}_{(X,\mathcal{F})}.$$

This follows from the previous consideration and from the explicit description of the functor $\mathrm{Def}_{\mathrm{Tot}(\mathcal{U},\mathcal{D}_{\mathbb{K}}(X,\mathcal{F}))}$ given in Remark 6.4: for every $A\in\mathbf{Art}_{\mathbb{K}}$, we have

$$\mathrm{Def}_{\mathrm{Tot}(\mathcal{U},\mathcal{D}_{\mathbb{K}}(X,\mathcal{F}))}(A)=\frac{\left\{\{x_{ij}\}\in\prod_{i,j}\mathcal{D}_{\mathbb{K}}(X,\mathcal{F})(U_{ij})\otimes\mathfrak{m}_A \mid e^{x_{jk}}e^{-x_{ik}}e^{x_{ij}}=1\right\}}{\sim},$$

where $x\sim y$ if and only if there exists $\{a_i\}\in\prod_i\mathcal{D}_{\mathbb{K}}(X,\mathcal{F})(U_i)\otimes\mathfrak{m}_A$ such that $e^{-a_i|_{U_{ij}}}e^{x_{ij}}e^{a_j|_{U_{ij}}}=e^{y_{ij}}$. ∎

Example 7.6 Let Z be a closed subscheme of an algebraic scheme X and denote by $\Theta_X(-\log Z)\subset\Theta_X$ the subsheaf of tangent vector fields that are tangent to Z. It is known that the sheaf of Lie algebras $\Theta_X(-\log Z)$ controls the locally trivial infinitesimal deformations of the pair (X,Z) [14]. By the same argument used at the beginning of Section 3, we have two morphisms of sheaves of Lie algebras

$$\Theta_X(-\log Z)\xrightarrow{\Delta}\mathcal{D}_{\mathbb{K}}(X,\mathcal{O}_Z)\xrightarrow{\alpha}\Theta_X(-\log Z),\qquad \alpha\circ\Delta=\mathrm{Id}.$$

Indeed, the image of the anchor map α preserves the annihilator of the sheaf $\mathcal{O}_Z$ and, conversely, every derivation h preserving the ideal sheaf of Z induces a derivation of pair (h,h).

The geometrical interpretation of these morphisms is clear: the morphism Δ controls the natural transformation that associates with a locally trivial deformation of (X,Z) the corresponding locally trivial deformation of $(X,\mathcal{O}_Z)$. The anchor map α associates a locally trivial deformation of the pair $(X,\mathcal{O}_Z)$ with the corresponding locally trivial deformation of $(X,\mathrm{Supp}\,\mathcal{O}_Z)$.

7.2 Infinitesimal Deformations

Next we analyse all the infinitesimal deformations of a pair $(X,\mathcal{F})$. We assume that X is smooth and projective, so that every infinitesimal deformation of X is locally trivial and every coherent sheaf on X has a finite locally free resolution.

Let $\mathcal{E}^*\to\mathcal{F}$ be a finite locally free resolution of $\mathcal{F}$, *i.e.*, an exact sequence

$$0\longrightarrow\mathcal{E}^{-m}\xrightarrow{d}\mathcal{E}^{-m+1}\xrightarrow{d}\cdots\xrightarrow{d}\mathcal{E}^{-1}\xrightarrow{d}\mathcal{E}^{0}\longrightarrow\mathcal{F}\longrightarrow 0,$$

where $\mathcal{E}^j$ is a locally free sheaf for any j. According to Example 5.5, we can consider the sheaf of DG-Lie algebras $\mathcal{D}^*_{\mathbb{K}}(X,\mathcal{E}^*)$ associated with the complex $\mathcal{E}^*$.

Lemma 7.7 *Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X and $f\colon\mathcal{E}^*\to\mathcal{F}$ and $g\colon\mathcal{G}^*\to\mathcal{F}$ two finite locally free resolutions of $\mathcal{F}$. Then for any open affine cover $\mathcal{U}=\{U_i\}$ of X, the DG-Lie algebras* $\mathrm{Tot}(\mathcal{U},\mathcal{D}^*_{\mathbb{K}}(X,\mathcal{E}^*))$ *and* $\mathrm{Tot}(\mathcal{U},\mathcal{D}^*_{\mathbb{K}}(X,\mathcal{G}^*))$ *are quasi-isomorphic.*

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Proof Since the variety is smooth and projective, by the Hilbert Syzygy Theorem, there exist two bounded complexes $\mathcal{H}^*$ and $\mathcal{N}^*$ of locally free sheaves and a commutative diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^* \oplus \mathcal{G}^* & \longrightarrow & \mathcal{H}^* & \longrightarrow & \mathcal{N}^* & \longrightarrow & 0 \\ & & & \searrow^{f+g} & \downarrow{\scriptstyle h} & & & & \\ & & & & \mathcal{F} & & & & \end{array}$$

such that $h\colon \mathcal{H}^* \to \mathcal{F}$ is a quasi-isomorphism; in particular, the induced morphisms $\mathcal{E}^* \to \mathcal{H}^*$ and $\mathcal{G}^* \to \mathcal{H}^*$ are quasi-isomorphisms.

Let U be an affine open subset of X. Since the sections of a locally free sheaf of an $\mathcal{O}_X$-module over U is a projective $\mathcal{O}_X(U)$-module, the injection

$$\mathcal{E}^*(U) \oplus \mathcal{G}^*(U) \longrightarrow \mathcal{H}^*(U)$$

is a cofibration, since $\mathcal{H}^*, \mathcal{N}^*$ are bounded complexes and therefore $\mathcal{H}^*(U), \mathcal{N}^*(U)$ are cofibrant complexes of $\mathcal{O}_X(U)$-modules. Since U is affine, the maps

$$\mathcal{E}^*(U) \oplus \mathcal{G}^*(U) \longrightarrow \mathcal{F}(U) \quad \text{and} \quad \mathcal{H}^*(U) \longrightarrow \mathcal{F}(U)$$

are quasi-isomorphisms and then the induced maps

$$\mathcal{E}^*(U) \longrightarrow \mathcal{H}^*(U) \quad \text{and} \quad \mathcal{G}^*(U) \longrightarrow \mathcal{H}^*(U)$$

are trivial cofibrations.

Then we have the following diagram of morphisms of coherent sheaves of DG-Lie algebras:

$$\begin{aligned} \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*) \longleftarrow \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^* \xrightarrow{f} \mathcal{H}^*) \longrightarrow \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{H}^*) \\ \longleftarrow \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{G}^* \xrightarrow{g} \mathcal{H}^*) \longrightarrow \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{G}^*). \end{aligned}$$

According to Propositions 5.1 and 3.4, for every open $U_i \in \mathcal{U}$, we have a diagram of quasi-isomorphisms of DG-Lie algebras

$$\begin{array}{ccccc} \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^* \xrightarrow{f} \mathcal{H}^*)(U_i) & & & & \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{G}^* \xrightarrow{g} \mathcal{H}^*)(U_i) \\ \downarrow & \searrow & & \swarrow & \downarrow \\ \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(U_i) & & \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{H}^*)(U_i) & & \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{G}^*)(U_i) \end{array}$$

Finally, applying the Tot functor, we conclude that

$$\mathrm{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)) \quad \text{and} \quad \mathrm{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{G}^*))$$

are quasi-isomorphic DG-Lie algebras. ∎

Definition 7.8 Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X. For any $i \in \mathbb{Z}$, we define the coherent sheaves $\mathcal{T}^i_{(X,\mathcal{F})}$ to be the cohomology sheaves of $\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)$ for a finite locally free resolution $\mathcal{E}^* \to \mathcal{F}$ of $\mathcal{F}$: $\mathcal{T}^i_{(X,\mathcal{F})} := \mathcal{H}^i(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*))$. Similarly, we define the hyper-cohomology groups $T^i_{(X,\mathcal{F})} =$

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$\mathbb{H}^i(X, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*))$. Lemma 7.7 implies that the sheaves $\mathcal{T}^i_{(X,\mathcal{F})}$ and the groups $T^i_{(X,\mathcal{F})}$ are well defined, since they do not depend on the choice of the resolution.

The short exact sequence (5.2) of complexes of coherent sheaves

$$0 \longrightarrow \mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) \longrightarrow \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*) \longrightarrow \Theta_X \longrightarrow 0$$

gives a hypercohomology long exact sequence

(7.1)

$$0 \longrightarrow \mathrm{Ext}^0_X(\mathcal{F}, \mathcal{F}) \longrightarrow T^0_{(X,\mathcal{F})} \longrightarrow H^0(X, \Theta_X) \longrightarrow \mathrm{Ext}^0_X(\mathcal{F}, \mathcal{F}) \longrightarrow T^1_{(X,\mathcal{F})} \longrightarrow \cdots.$$

By the hypercohomology spectral sequence, we have

$$E_2^{p,q} = H^p(X, \mathcal{T}^q_{(X,\mathcal{F})}) \Rightarrow T^{p+q}_{(X,\mathcal{F})}.$$

According to Example 3.6, we have that $\mathcal{T}^i_{(X,\mathcal{F})} = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$, for $i \neq 0, 1$, and we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{T}^0_{(X,\mathcal{F})} \longrightarrow \Theta_X \longrightarrow \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{T}^1_{(X,\mathcal{F})} \longrightarrow 0.$$

In particular, the sheaf $\mathcal{T}^i_{(X,\mathcal{F})}$ vanishes in the locus where $\mathcal{F}$ has projective dimension $< i$.

7.3 Local Case

First, we analyse the infinitesimal deformations of the pair $(X, \mathcal{F})$ in the local case. Let $X = \mathrm{Spec}\, R$ be smooth and affine over $\mathbb{K}$ and $\mathcal{F} = \widetilde{M}$, where M is a finitely generated R-module. Let (E^*, d) be a finite complex of projective R-modules such that the sequence

$$0 \longrightarrow E^{-m} \xrightarrow{d} E^{-m+1} \xrightarrow{d} \cdots \xrightarrow{d} E^{-1} \xrightarrow{d} E^0 \longrightarrow M \longrightarrow 0$$

is exact. Finally, consider the DG-Lie algebra $D^*_{\mathbb{K}}(R, E^*)$.

Proposition 7.9 *In the notation above, the DG-Lie algebra $D^*_{\mathbb{K}}(R, E^*)$ controls the infinitesimal deformations of the pair $(X, \mathcal{F})$,* i.e.*, there exists an isomorphism of deformation functors* $\mathrm{Def}_{D^*_{\mathbb{K}}(R,E^*)} \cong \mathrm{Def}_{(X,\mathcal{F})}$.

Proof The variety $X = \mathrm{Spec}\, R$ is smooth and affine and so it has only trivial infinitesimal deformations [25, Theorem 1.2.4]. Therefore, for any $A \in \mathbf{Art}_{\mathbb{K}}$, any infinitesimal deformation of the pair $(X, \mathcal{F} = \widetilde{M})$ over A is of the form $(X \times \mathrm{Spec}\, A, F_A = \widetilde{M}_A)$, where M_A is an $R \otimes A$-module, that is, A-flat, together with a morphism $\pi\colon M_A \to M$, inducing an isomorphism $M_A \otimes_A \mathbb{K} \to M$ [6, §1].

The flatness condition allows us to lift the relations; therefore, any deformation M_A of M over A corresponds to an exact sequence $E^* \otimes A \to M_A$ that reduces to

$$E^* \longrightarrow M$$

when tensored with $\mathbb{K}$, *i.e.*, any deformation M_A of M over A corresponds to a deformed complex $(E^* \otimes A, d_A)$ and any of these complexes is of the form $(E^* \otimes A, d + x)$, for $x \in \mathrm{MC}_{D^*_{\mathbb{K}}(R,E^*)}(A) = \mathrm{MC}_{\mathrm{Hom}^*_{\mathbb{K}}(E^*,E^*)}(A)$.

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Two deformations $(X \times \operatorname{Spec} A, \mathcal{F}_A = \widetilde{M}_A)$ and $(X \times \operatorname{Spec} A, \mathcal{F}'_A = \widetilde{M}'_A)$ of $(X, \widetilde{M})$ over A are isomorphic if there exist an isomorphism of deformations

$$f\colon X \times \operatorname{Spec} A \to X \times \operatorname{Spec} A$$

and an isomorphism $\mathcal{F}_A \to f^*\mathcal{F}'_A$ of $R \otimes A$-modules.

In particular, the isomorphism of the modules lifts to an isomorphism of the deformed complexes and vice versa. Then an isomorphism is given by a pair (θ, ψ) where $\theta\colon R \otimes A \to R \otimes A$ is an isomorphism that lifts the identity of R and

$$\psi\colon (E^* \otimes A, d + x) \longrightarrow (E^* \otimes A, d + x')$$

is an isomorphism lifting the identity, such that $\psi(rm) = \theta(r)\psi(m)$, for any $r \in R \otimes A$ and any $m \in E^* \otimes A$.

As in Lemma 2.10, for any $A \in \mathbf{Art}_{\mathbb{K}}$ the group $\exp(D^0_{\mathbb{K}}(R, E^*) \otimes \mathfrak{m}_A)$ is naturally isomorphic to the group of A-automorphisms of the pair $(R \otimes A, E^* \otimes A)$ lifting the identity on (R, E^*); therefore, it corresponds to the group of the isomorphisms of deformations. ∎

Example 7.10 (*cf.* [5, Proposition A3]) Assume that X is a smooth surface, $D \subset X$ a reduced divisor, and $\mathcal{F}$ a sheaf of $\mathcal{O}_D$-modules. Assume that for every point $p \in D$, the stalk $\mathcal{F}_p$ is a torsion-free $\mathcal{O}_{D,p}$-module. Then by the Auslander–Buchsbaum Theorem [21, Theorem 19.1], the projective dimension of every stalk of the sheaf $\mathcal{F}$ is at most 1. In particular, $\mathcal{T}^i_{(X,\mathcal{F})} = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 0$, for every $i \geq 2$.

If X is affine, we have $\operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) = H^0(\mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) = 0$, and so the pair $(X, \mathcal{F})$ has unobstructed deformations.

7.4 Global Case

Finally, we analyse all the infinitesimal deformations of a pair $(X, \mathcal{F})$. Essentially, we must glue together the computations about the locally trivial infinitesimal deformations (Theorem 7.5) and the deformations in the local case (Proposition 7.9).

Theorem 7.11 *Let $\mathcal{F}$ be a coherent sheaf on a projective smooth variety X and let $\mathcal{E}^* \to \mathcal{F}$ be a finite locally free resolution of $\mathcal{F}$. Then the coherent sheaf of DG-Lie algebras $\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)$ controls the problem of deformations of the pair $(X, \mathcal{F})$. In particular, $T^1_{(X,\mathcal{F})}$ is the tangent space and $T^2_{(X,\mathcal{F})}$ is the obstruction space for* $\operatorname{Def}_{(X,\mathcal{F})}$.

Proof The proof follows the general lines already used in [6]. According to Definition 6.6, we need to prove that for any affine open cover $\mathcal{U} = \{U_i\}$ of X, there exists an isomorphism of functors of Artin rings $\operatorname{Def}_{\operatorname{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X,\mathcal{E}^*))} \to \operatorname{Def}_{(X,\mathcal{F})}$.

According to the isomorphism given in (6.1), it is enough to show that there exists an isomorphism $H^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(\mathcal{U}))) \to \operatorname{Def}_{(X,\mathcal{F})}$. Therefore, for any $A \in \mathbf{Art}_{\mathbb{K}}$, we need to prove that every element in $Z^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(\mathcal{U})))(A)$ corresponds to an infinitesimal deformation of the pair $(X, \mathcal{F})$ over A and that two elements are equivalent if and only if the corresponding deformations are isomorphic. This follows from the local study analysed in Proposition 7.9 and a gluing procedure as in Theorem 7.5. Indeed, an element $(l, m) \in Z^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(\mathcal{U})))(A)$

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gives a deformation of the pair $(X,\mathcal{F})$ as gluing of deformations on each U_i. We only stress the fact that the gluing condition on the isomorphisms involves an element in $\prod_{i,j,k}\mathcal{H}\mathrm{om}^{-1}_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)(U_{ijk})\otimes\mathfrak{m}_A$; this is due to the fact that we do not need to glue the deformed complexes but rather their cohomology to get a sheaf (see [6, §2] for more details about this). As regards the equivalence relation $\sim$, the first condition is the isomorphism of the induced deformation on each open set, the second condition gives the gluing of the local isomorphism to obtain a global isomorphism of the induced deformations.

More explicitly, an element $(l,m)\in Z^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X,\mathcal{E}^*)(\mathcal{U})))(A)$ is given by the sequences

$$l=\{l_i\}\in\prod_i\mathcal{H}\mathrm{om}^1_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)(U_i)\otimes\mathfrak{m}_A,$$
$$m=\{(h_{ij},u_{ij})\}\in\prod_{i,j}\mathcal{D}^0_{\mathbb{K}}(X,\mathcal{E}^*)(U_{ij})\otimes\mathfrak{m}_A$$

such that

(1) for any i, the element l_i satisfies the Maurer–Cartan equation, *i.e.*, $dl_i+\frac{1}{2}[l_i,l_i]=0$,
(2) for any i and j, the restriction of l_i and l_j to U_{ij} are gauge equivalent under m_{ij}, *i.e.*, $l_i|_{U_{ij}}=e^{m_{ij}}*l_j|_{U_{ij}}$,
(3) for any i, j, and k, we have the condition:

$$h_{jk}|_{U_{ijk}}\bullet -h_{ik}|_{U_{ijk}}\bullet h_{ij}|_{U_{ijk}}=0,$$
$$u_{jk}|_{U_{ijk}}\bullet -u_{ik}|_{U_{ijk}}\bullet u_{ij}|_{U_{ijk}}=[d+l_j|_{U_{ijk}},n_{ijk}],$$

for some $n=\{n_{ijk}\}\in\prod_{i,j,k}\mathcal{H}\mathrm{om}^{-1}_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)(U_{ijk})\otimes\mathfrak{m}_A$.

According to the local study, conditions (1) and (2) imply that every l_i defines a deformation of the pair $(X,\mathcal{F})$ on U_i and that the deformations induced by l_i and l_j on U_{ij} are isomorphic. Finally, the third condition implies that we can glue together the local deformations to get a global deformation of the pair $(X,\mathcal{F})$ over A. Indeed, the former condition is the gluing of the locally trivial deformation of U_i over the double intersections to obtain a deformation of X over A; the latter regards the gluing of the local deformations of the sheaf $\mathcal{F}$. We only stress the fact that the isomorphisms that glue the restrictions of the deformed complexes $(\mathcal{E}^*_{U_i}\otimes A,d+l_i)$ and $(\mathcal{E}^*_{U_j}\otimes A,d+l_j)$, for any i and j, satisfy the cocycle condition only up to homotopy. Indeed, we do not need to glue the restriction of the deformed complexes of sheaves together but rather their cohomology. Therefore, the gluing isomorphism must satisfy the cocycle condition only up to homotopy.

As regards the equivalence relation, suppose that

$$(l_0=\{l_{0,i}\},m_0=\{(h_{0,ij},u_{0,ij})\}),$$

and

$$(l_1=\{l_{1,i}\},m_1=\{(h_{1,ij},u_{1,ij})\})\in Z^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X,\mathcal{E}^*)(\mathcal{U})))(A)$$

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are equivalent under the relation $\sim$. Then we need to prove that the induced deformations are isomorphic. The equivalence implies the existence of

$$a = \{g_i, v_i\} \in \prod_i \mathcal{D}^0_{\mathbb{K}}(X, \mathcal{E}^*)(U_i) \otimes \mathfrak{m}_A$$
$$b = \{b_{ij}\} \in \prod_{i,j} \mathcal{H}\mathrm{om}^{-1}_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)(U_{ij}) \otimes \mathfrak{m}_A$$

such that

$$e^a * l_0 = l_1, \quad -m_0 \bullet -a_{i|U_{ij}} \bullet m_1 \bullet a_{j|U_{ij}} = db + [l_{0,j|U_{ij}}, b].$$

The first condition implies that the deformations of the pair induced by l_0 and l_1 are isomorphic locally on U_i; the second condition provides the gluing of the isomorphisms along double intersections. (We have a strict cocycle condition for the isomorphisms of the deformations of X and, as before, we have a homotopy cocycle condition for the gluing of the deformed complexes.)

As regards the tangent and obstruction spaces for $\mathrm{Def}_{(X,\mathcal{F})}$, we have that

$$H^1(\mathrm{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)))$$

is the tangent space and $H^2(\mathrm{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)))$ is an obstruction space. According to Definition 7.8 and to equality (6.2), we have that

$$H^i(\mathrm{Tot}(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(\mathcal{U}))) \cong \mathbb{H}^i(X, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)) = T^i_{(X,\mathcal{F})}. \qquad \blacksquare$$

Remark 7.12 In the situation of Theorem 7.11, consider the sheaf of truncated DG-Lie algebras

$$\sigma_{\le 0}\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*) = \mathcal{D}_{\mathbb{K}}(X, \mathcal{E}^*) \bigoplus_{i<0} \mathcal{H}\mathrm{om}^i_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) \subset \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*).$$

We have seen in Example 3.6 that the natural map $\mathcal{D}_{\mathbb{K}}(X, \mathcal{E}^*) \to \mathcal{D}_{\mathbb{K}}(X, \mathcal{F})$ induces a surjective quasi-isomorphism of coherent sheaves of DG-Lie algebras

$$\sigma_{\le 0}\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*) \to \mathcal{D}_{\mathbb{K}}(X, \mathcal{F}).$$

Therefore, the sheaf of truncated DG-Lie algebras $\sigma_{\le 0}\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)$ also controls the locally trivial deformations of the pair, and the obvious natural transformation

$$\mathrm{Def}^{\mathrm{lt}}_{(X,\mathcal{F})} \longrightarrow \mathrm{Def}_{(X,\mathcal{F})}$$

is induced by the inclusion $\sigma_{\le 0}\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*) \subset \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)$.

We are now ready to prove the last item of Theorem 1.2.

Proposition 7.13 *Let $\mathcal{F}$ be a coherent sheaf on a projective smooth variety X. Assume that $H^0(X, \Theta_X) = 0$ and $\mathrm{Ext}^0_X(\mathcal{F}, \mathcal{F}) = \mathbb{K}$. Then the functor $\mathrm{Def}_{(X,\mathcal{F})}$ is pro-representable.*

Proof Let $\mathcal{E}^* \to \mathcal{F}$ be a finite locally free resolution of $\mathcal{F}$. According to (7.1) the vector space $T^0_{(X,\mathcal{F})}$ is one-dimensional generated by the identity on the complex $\mathcal{E}^*$. Therefore, for every open affine cover $\mathcal{U}$ the vector space $H^0(\mathrm{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)))$ is one-dimensional and generated by the identity on $\mathcal{E}^*_{|U_{i_0\cdots i_k}}$ for every $k \ge 0$ and every

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$i_0, \dots, i_k$. In particular the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*))$ satisfies the condition of Lemma 6.2 and the conclusion follows by Theorem 7.11. ∎

Let X be a projective smooth variety, $\mathcal{F}$ a coherent sheaf on X, and $\mathcal{E}^* \to \mathcal{F}$ a finite locally free resolution. By definition $\det \mathcal{F} = \det \mathcal{E}^*$, and we have defined a morphism of sheaves of DG-Lie algebras (see equation (5.6)):

$$\mathrm{Tr}\colon \mathcal{D}_{\mathbb{K}}{}^*(X, \mathcal{E}^*) \longrightarrow \mathcal{D}_{\mathbb{K}}(X, \det \mathcal{F}).$$

By Theorem 7.11, the sheaf $\mathcal{D}_{\mathbb{K}}{}^*(X, \mathcal{E}^*)$ controls the deformations of the pair $(X, \mathcal{F})$, while the sheaf $\mathcal{D}_{\mathbb{K}}(X, \det \mathcal{F})$ controls the deformations of the pair $(X, \det \mathcal{F})$.

We can interpret this morphism as a natural transformation of deformation functors. Some deformations of the pair $(X, \mathcal{F})$ correspond to deformations of X and of the complex $\mathcal{E}^*$; then with these deformations we can easily associate a deformation of the pair $(X, \det \mathcal{F} = \det \mathcal{E}^*)$, considering the determinant of the deformed complex. In general, not all the deformations of $\mathcal{F}$ correspond to deformations of the complex, but this is true only locally on X and the deformations of the complex glue only in cohomology. Anyway, we can still define a deformation of the pair $(X, \det \mathcal{F})$, as we see in the following lemma.

Lemma 7.14 *Let X be a projective smooth variety and $\mathcal{F}$ a coherent sheaf on X. Then there exists a commutative diagram of deformation functors*

$$\begin{array}{ccc} \mathrm{Def}_{(X,\mathcal{F})} & \xrightarrow{\mathrm{Tr}} & \mathrm{Def}_{(X,\det\mathcal{F})} \\ & \searrow \quad \swarrow & \\ & \mathrm{Def}_X & \end{array}$$

where the diagonal arrows are the forgetful natural transformations, and the natural transformation Tr *is induced by the trace map* $\mathrm{Tr}\colon \mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*) \to \mathcal{D}^*_{\mathbb{K}}(X, \det \mathcal{F})$, *for any finite locally free resolution $\mathcal{E}^* \to \mathcal{F}$.*

Proof Let $\mathcal{U} = \{U_i\}$ be an open affine cover for X such that every sheaf $\mathcal{E}^k$ is free on every U_i. Fix $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a deformation $(X_A, \mathcal{F}_A) \in \mathrm{Def}_{(X,\mathcal{F})}(A)$. According to Theorem 7.11, this deformation corresponds to an element

$$(l, m) \in Z^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(\mathcal{U})))(A),$$

where

$$l = \{l_i\} \in \prod_i \mathcal{H}\mathrm{om}^1_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)(U_i) \otimes \mathfrak{m}_A,$$

$$m = \{(h_{ij}, u_{ij})\} \in \prod_{i,j} \mathcal{D}^0_{\mathbb{K}}(X, \mathcal{E}^*)(U_{ij}) \otimes \mathfrak{m}_A.$$

In particular, for any i, j, and k, we have the following conditions:

$$u_{jk}|_{U_{ijk}} \bullet -u_{ik}|_{U_{ijk}} \bullet u_{ij}|_{U_{ijk}} = [d + l_j|_{U_{ijk}}, n_{ijk}],$$

for some $n = \{n_{ijk}\} \in \prod_{i,j,k} \mathcal{H}\mathrm{om}^{-1}_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)(U_{ijk}) \otimes \mathfrak{m}_A$. Applying the trace morphisms to m, for any i and j, we have

$$\mathrm{Tr}(h_{ij}, u_{ij}) = (h_{ij}, \mathrm{Tr}(u_{ij})) \in \mathcal{D}^0_{\mathbb{K}}(X, \det \mathcal{F})(U_{ij}) \otimes \mathfrak{m}_A$$

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such that

$$\mathrm{Tr}(u_{jk}|_{U_{ijk}}) \bullet -\mathrm{Tr}(u_{ik}|_{U_{ijk}}) \bullet \mathrm{Tr}(u_{ij}|_{U_{ijk}}) = \mathrm{Tr}([d + l_j|_{U_{ijk}}, n_{ijk}]) = 0.$$

This implies that the element

$$\mathrm{Tr}(m) = \{(h_{ij}, \mathrm{Tr}(u_{ij})\} \in \prod_{i,j} \mathcal{D}^0_{\mathbb{K}}(X, \det \mathcal{F})(U_{ij}) \otimes \mathfrak{m}_A$$

satisfies the cocycle condition on triple intersections and so, by Theorem 7.5, it defines a deformation of the pair $(X, \det \mathcal{F})$ over A.

An analogous computation shows that the isomorphism class of the induced deformation of $(X, \det \mathcal{F})$ does not depend on the isomorphism class of the deformation of $(X, \mathcal{F})$. Indeed, let $(X_A, \mathcal{F}_A)$ and $(X'_A, \mathcal{F}'_A)$ be isomorphic deformations corresponding to the elements $(l_0 = \{l_{0,i}\}, m_0 = \{(h_{0,ij}, u_{0,ij})\})$ and

$$(l_1 = \{l_{1,i}\}, m_1 = \{(h_{1,ij}, u_{1,ij})\}) \in Z^1_{\mathrm{sc}}(\exp(\mathcal{D}^*_{\mathbb{K}}(X, \mathcal{E}^*)(\mathcal{U})))(A).$$

According to Theorem 7.11, the isomorphism corresponds to the existence of

$$a = \{g_i, v_i\} \in \prod_i \mathcal{D}^0_{\mathbb{K}}(X, \mathcal{E}^*)(U_i) \otimes \mathfrak{m}_A$$
$$b = \{b_{ij}\} \in tsl \prod_{i,j} \mathcal{H}\mathrm{om}^{-1}_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)(U_{ij}) \otimes \mathfrak{m}_A$$

such that

$$e^a * l_0 = l_1, \quad -m_0 \bullet -a_{i|U_{ij}} \bullet m_1 \bullet a_{j_{|U_{ij}}} = db + [l_{0,j_{|U_{ij}}}, b].$$

In particular, applying the trace morphisms for any i and j, we have

$$-\mathrm{Tr}(m_0) \bullet -\mathrm{Tr}(a_{i|U_{ij}}) \bullet \mathrm{Tr}(m_1) \bullet \mathrm{Tr}(a_{j_{|U_{ij}}}) = 0 \in \mathcal{D}^0_{\mathbb{K}}(X, \det \mathcal{F})(U_{ij}) \otimes \mathfrak{m}_A.$$

By Theorem 7.5, this implies that the induced deformations of $(X, \det \mathcal{F})$ are isomorphic. ∎

Remark 7.15 As a particular case, if we only consider the deformations of a coherent sheaf $\mathcal{F}$ on a fixed projective variety X, then trace induces a morphism of functors

$$\mathrm{Def}_{\mathcal{F}} \to \mathrm{Def}_{\det \mathcal{F}}.$$

In this case, the induced morphisms in cohomology are the ones already analysed in [27, Theorem 3.23].

Theorem 7.16 *Let X be a projective smooth variety and $\mathcal{E}^*$ a locally free finite resolution of a coherent sheaf $\mathcal{F}$. Consider the linear maps $\mathrm{Tr}^i\colon \mathrm{Ext}^i_X(\mathcal{F}, \mathcal{F}) \to H^i(X, \mathcal{O}_X)$ induced by the morphism of sheaves of DG-Lie algebras $\mathrm{Tr}\colon \mathcal{H}\mathrm{om}^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) \to \mathcal{O}_X$.*

If the map Tr^1 is surjective and the map Tr^2 is injective, then the induced natural transformation $\mathrm{Def}_{(X,\mathcal{F})} \to \mathrm{Def}_{(X,\det \mathcal{F})}$ is smooth. If Tr^2 is injective and $\mathrm{Def}_{(X,\det \mathcal{F})}$ is unobstructed, then $\mathrm{Def}_{(X,\mathcal{F})}$ is unobstructed.

By the exact sequence (5.4), the above assumptions on Tr^1 and Tr^2 are equivalent to assuming $\mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F})_0 = 0$.

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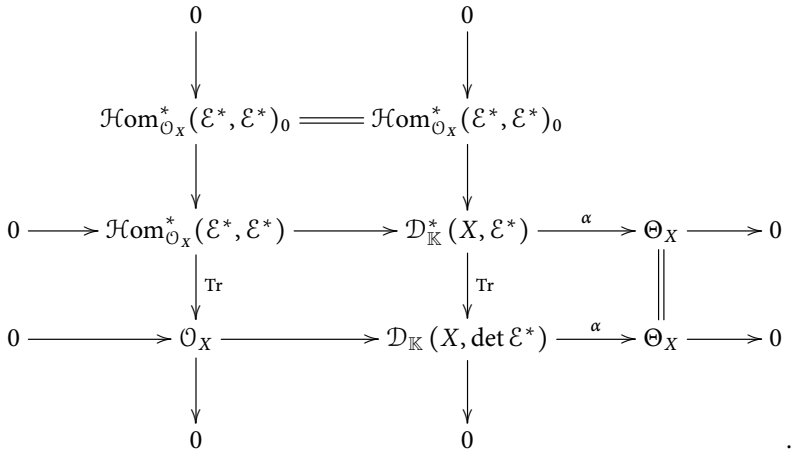
Proof By Theorem 5.6, there exists a commutative diagram with exact rows and columns of morphisms of coherent sheaves of DG-Lie algebras



The hypothesis on the morphisms $\mathrm{Tr}\colon \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*) \to \mathcal{O}_X$ implies that the same holds for the morphisms induced by $\mathrm{Tr}\colon \mathcal{D}_{\mathbb{K}}{}^*(X,\mathcal{E}^*) \to \mathcal{D}_{\mathbb{K}}(X,\det\mathcal{F})$. Then it is enough to apply the Standard Smoothness Criterion [19, Theorem 4.11] to the corresponding morphism of functors $\mathrm{Tr}\colon \mathrm{Def}_{(X,\mathcal{F})} \to \mathrm{Def}_{(X,\det\mathcal{F})}$.

If $\mathrm{Def}_{(X,\det\mathcal{F})}$ is unobstructed, then $\mathrm{Tr}^2\colon T^2_{(X,\mathcal{F})} \to T^2_{(X,\det\mathcal{F})}$ annihilates all the obstructions and its injectivity implies that also $\mathrm{Def}_{(X,\mathcal{F})}$ is unobstructed. ∎

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