BULL. AUSTRAL. MATH. SOC. 20M99, 06A50
VOL. 19 (1978), 59-65.

# Inverse semigroups and their natural order

## H. Mitsch

The natural order of an inverse semigroup defined by $a \le b \Longleftrightarrow a'b = a'a$ has turned out to be of great importance in describing the structure of it. In this paper an order-theoretical point of view is adopted to characterise inverse semigroups. A complete description is given according to the type of partial order an arbitrary inverse semigroup $S$ can possibly admit: a least element of $(S, \le)$ is shown to be the zero of $(S, \cdot)$ ; the existence of a greatest element is equivalent to the fact, that $(S, \cdot)$ is a semilattice; $(S, \le)$ is directed downwards, if and only if $S$ admits only the trivial group-homomorphic image; $(S, \le)$ is totally ordered, if and only if for all $a, b \in S$ , either $ab = ba = a$ or $ab = ba = b$ ; a finite inverse semigroup is a lattice, if and only if it admits a greatest element. Finally formulas concerning the inverse of a supremum or an infimum, if it exists, are derived, and right-distributivity and left-distributivity of multiplication with respect to union and intersection are shown to be equivalent.

## Introduction

Let $(S, \cdot)$ be an inverse semigroup, $(E, \cdot)$ the lower semilattice of all idempotent elements of $S$ . Then a partial order "$\le$" - the so-called "*natural order*" - can be defined on $S$ by

(1) $$a \le b \Longleftrightarrow ab' = aa'$$

Received 15 June 1978.



https://doi.org/10.1017/S0004972700008443 Published online by Cambridge University Press

($a'$ denotes the unique inverse of $a$ in $S$); the order "$\leq$" on $S$ restricted to $E$ coincides with the partial order of the lower semilattice $E$ : $e \leq f \Leftrightarrow e = ef = fe$ $(e, f \in E)$ . Several equivalent formulations of this definition are known (see [2, 3], [6]):

(2) $$a \leq b \Leftrightarrow a'b = a'a ,$$

(3) $$a \leq b \Leftrightarrow a = eb , \quad e \in E .$$

With respect to this order, $(S, \cdot)$ forms a *partially ordered semigroup*, that is to say, $a \leq b \Rightarrow ac \leq bc$ and $ca \leq cb$ for all $c \in S$ ; hence also $a \leq b \Leftrightarrow a' \leq b'$ (see [2, 3], [5]). The natural order on $S$ is even a special "natural" ordering in the sense of partially ordered semi-groups (see [5]):

(4) $$a \leq b \Leftrightarrow a = (aa')b = b(a'a) .$$

In fact, by (2), $a \leq b$ implies $a'b = a'a$ , thus, multiplying by $a$ on the left, $(aa')b = a$ ; conversely $a = (aa')b = eb$ with $e \in E$ implies $a \leq b$ , by (3). Furthermore

$$a \leq b \Leftrightarrow a' \leq b' \Leftrightarrow a' = (a'a)b' \Leftrightarrow a = b(a'a)$$

by taking inverses.

Order-theoretic considerations have turned out to be of great importance in the theory of inverse semigroups. In the following a description of special types of partial orderings which the natural order on an inverse semigroup can possibly assume will be given.

## Special natural orders

In every inverse semigroup $S$ we have, with respect to its natural order,

(5) $$a \leq e , \quad e \in E , \quad a \in S \Rightarrow a \in E$$

(by (3), $a = fe$ , $f \in E$ , thus $a \in E$);

(6) $$e \leq a , \quad e \in E , \quad a \in S \Rightarrow ea = ae = e$$

(by (4), $e = (ee')a = ea$ , $e = a(e'e) = ae$ ).

LEMMA 1. *An element $a$ of an inverse semigroup $S$ is the least element of $S$, if and only if $a$ is the zero of $S$.*

Proof. If $c$ is the zero of $S$ , then $c = ca$ for all $a \in S$ and

$c \in E$ ; hence by (3), $c \leq a$ for all $a \in S$ . Conversely if $c \leq a$ for all $a \in S$ , then, by (4), $c = (cc')a = a(c'c)$ . If there is $e \in E$ (that is if $E \neq \emptyset$ ), then with $a = e$ we obtain $c = (cc')e \in E$ and $c = ca = ac$ for all $a \in S$ ; if $E = \emptyset$ , then $S$ can not be an inverse semigroup, since $a \in S$ implies $aa' \in E$ . //

LEMMA 2. *An inverse semigroup* $S$ *possesses a greatest element, if and only if* $S$ *is an idempotent, commutative semigroup with identity (that is a semilattice).*

Proof. If $S$ is a semilattice with identity $e$ , then $a = ae$ for all $a \in S$ ; since $S = E$ , we conclude from (3) that $a \leq e$ for all $a \in S$ .

Conversely assume $a \leq i$ for all $a \in S$ (where $i$ is the greatest element of $S$ ); then by (1), $ai' = aa'$ for all $a \in S$ . For $e \in E$ we have $ei' = e$ , thus $e(i'i) = ei$ ; but $e, i'i \in E$ implies $ei \in E$ . Now if $a \in S$ is an arbitrary element, then $a \leq i$ implies, by (4), $a = (aa')i$ with $aa' \in E$ ; consequently $a \in E$ for all $a \in S$ . This means that $S$ is an idempotent, and thus commutative, semigroup (since idempotents commute in an inverse semigroup). Furthermore $a' = a$ for all $a \in S$ and $a \leq i$ implies $ai = ia = a$ for all $a \in S$ , and $i$ is the identity of $S$ . //

A partial order "$\leq$" on a set $S$ is called *directed downwards*, if for all $x, y \in S$ there exists a $z \in S$ with $z \leq x$ and $z \leq y$ ; the dual concept is *directed upwards*.

REMARKS. (1) If an inverse semigroup $S$ is directed upwards with respect to its natural order, then it is also directed downwards. In fact: if for all $x, y \in S$ there is $z \in S$ such that $x \leq z$ and $y \leq z$ , then, by (3), $x = ez$ and $y = fz$ , with $e, f \in E$ ; hence $fx = fez = efz = ey = u \in S$ , which means, again by (3), $u \leq x$ and $u \leq y$ , and $S$ is directed downwards.

(2) If $S$ is directed downwards, then $S$ is a reversible semigroup ([2, 3]). In fact, if for all $x, y \in S$ there is $z \in S$ with $z \leq x$ and $z \leq y$ , then, by (4), $z = ex = fy$ with $e, f \in E \subseteq S$ , and $z = xg = yh$ with $g, h \in S$ , which means $(Sx) \cap (Sy) \neq \emptyset$ and $(xS) \cap (yS) \neq \emptyset$ .

LEMMA 3. *An inverse semigroup* $S$ *is directed downwards with respect*

https://doi.org/10.1017/S0004972700008443 Published online by Cambridge University Press

*to its natural order, if and only if the right-reversible equivalence* $P_E$ *of* Dubreil, *defined by* $a \equiv b(P_E) \Leftrightarrow ea = fb$ *for some* $e, f \in E$ , *is the universal relation.*

Proof. $S$ is directed downwards if and only if for all $x, y \in S$ , there exists a $z \in S$ with $z \leq x$ and $z \leq y$ , or equivalently $z = ex = fy$ with $e, f \in E$ , or equivalently again, for all $x, y \in S$ , there exist $e, f \in E$ with $ex = fy$ , or finally equivalently $P_E$ is the universal relation on $S$ (see [2, 3], [4]). //

THEOREM 4. *An inverse semigroup* $S$ *is directed downwards with respect to its natural order, if and only if the only group-homomorphic image of* $S$ *is the trivial one.*

Proof. By [7] the least group-congruence on an arbitrary inverse semigroup $S$ (that is to say, a congruence $\rho$ such that $S/\rho$ is a group) is defined by $a \equiv b(\rho)$ if and only if $ea = eb$ for some $e \in E$ . If $S$ is directed downwards, then this is equivalent to the existence of $z \in S$ , given $x, y \in S$ , such that $z = (zz')x = (zz')y$ (by (4)), which means that $ex = ey$ for $e = zz' \in E$ . But this is equivalent to the fact that the least group-congruence on $S$ is the universal relation, which means that the maximal group-homomorphic image of $S$ is the trivial group; in other words there is no other group homomorphic to $S$ except the trivial one. //

REMARK. As a consequence we note that inverse semigroups the natural order of which is an upper-semilattice-, a lower-semilattice-, or a lattice-order, necessarily have only the trivial group-homomorphic image.

One order-theoretical extreme is the case where the natural order of $S$ is a total order; we prove:

THEOREM 5. *An inverse semigroup* $S$ *is totally ordered with respect to its natural ordering, if and only if* $ab = ba = a$ *or* $ab = ba = b$ *for all* $a, b \in S$ .

Proof. If $ab = ba = a$ or $ab = ba = b$ for all $a, b \in S$ , then for $b = a$ we get $a^2 = a$ , $a' = a$ for all $a \in S$ and $E = S$ . Thus for arbitrary $a, b \in S$ we have: if $a = ab$ then $a \leq b$ , and if $b = ab = ba$ , then $b \leq a$ by (3), and $(S, \leq)$ is totally ordered.

https://doi.org/10.1017/S0004972700008443 Published online by Cambridge University Press

Conversely if the natural order of $S$ is total, then we have $a \leq a'$ or $a' \leq a$ for all $a \in S$. By taking inverses we obtain $a' \leq a$ or $a \leq a'$; thus $a' = a$ for all $a \in S$ and $a = aa'a = a^3$ for all $a \in S$. Since $a^2 \leq a$ or $a \leq a^2$ for all $a \in S$, we get, multiplying by $a$, $a = a^3 \leq a^2$ or $a^2 \leq a^3 = a$; hence $a^2 = a$ for all $a \in S$, and $E = S$. But idempotents of an inverse semigroup commute, so that $ab = ba$ for all $a, b \in S$. Furthermore $a \leq b$ or $b \leq a$ for all $a, b \in S$ implies, by (4), $a = (aa')b = a^2b = ab$ or $b = ba$ for all $a, b \in S$, and we conclude that $ab = ba = a$ or $ab = ba = b$ for all $a, b \in S$. //

REMARK. If an inverse semigroup $S$ possesses an identity $e$, then $e$ is maximal with respect to the natural order of $S$. In fact suppose $e < a$ for an $a \in S$; then, by (1), $ea' = ee' = e^2 = e$; thus $a' = e$ and $a = (a')' = e$, a contradiction. Furthermore if $e$ is the least element of $S$, then $S = \{e\}$.

Concerning semilattice-orders, Lemma 2 solves the problem on the assumption that there is a greatest element. Equivalently we state

**LEMMA 6.** *Let $S$ be an inverse semigroup with identity $e$. Then $S$ is a lower semilattice with respect to its natural order, if $e$ is the greatest element of $S$.*

Finally we give an answer to the question: which are the finite inverse semigroups whose natural order is a lattice-order?

**THEOREM 7.** *Let $S$ be a finite inverse semigroup. Then $S$ is a lattice with respect to its natural order, if and only if $S$ possesses a greatest element.*

Proof. If $S$ has a greatest element, then, by Lemma 2, $(S, \leq)$ is a lower semilattice with respect to the operation "$\wedge$" defined by $a \wedge b = ab$ for all $a, b \in S$ (see Theorem 1.12 of [2]). Since $(S, \leq)$ is bounded from above, and since every non-void subset $T$ of $S$ has an infimum, $\inf T = a_1 \ldots a_n$ with $a_i \in T$, by a well-known result of lattice-theory every non-void subset of $S$ has also a supremum; in particular $\sup\{a, b\} = a \vee b = c_1 \ldots c_k$ where $c_i$ runs through all elements of $S$, which are upper bounds for $\{a, b\}$. The converse is trivial, since a finite lattice has a greatest element. //

COROLLARY 8. *Let $S$ be a finite inverse semigroup with identity $e$. Then $S$ is a lattice with respect to its natural order, if and only if $e$ is the greatest element of $S$.*

Concerning possibly existing infima and suprema in inverse semigroups we show:

LEMMA 9. *Let $S$ be an inverse semigroup. If $a, b$ are elements of $S$ for which* $\sup\{a, b\} = a \vee b$ $(\inf\{a, b\} = a \wedge b)$ *exists, then $a' \vee b'$ exists, too, and $(a \vee b)' = a' \vee b'$ ($a' \wedge b'$ exists, too, and $(a \wedge b)' = a' \wedge b'$).*

Proof. If $a \vee b$ exists for $a, b \in S$, then $a, b \leq a \vee b$ and $a', b' \leq (a \vee b)'$; if $x \in S$ is any upper bound for $\{a', b'\}$, then $a', b' \leq x$ implies $a, b \leq x'$; thus $a \vee b \leq x'$ and $(a \vee b)' \leq x$, that is $(a \vee b)'$ is the least upper bound for $\{a', b'\}$, and $(a \vee b)' = a' \vee b'$. //

LEMMA 10. *Let $S$ be an inverse semigroup which is a lattice with respect to its natural order. Then multiplication is left-distributive with respect to union (intersection), if and only if it is right-distributive with respect to union (intersection).*

Proof. Suppose $a(b \vee c) = ab \vee ac$ for all $a, b, c \in S$. Since $a, b \leq a \vee b$ implies $ac, bc \leq (a \vee b)c$, we have that $ac \vee bc \leq (a \vee b)c$ for all $a, b, c \in S$. Consequently in order to prove right-distributivity we have to show the converse inequality

$$\begin{aligned}[(a \vee b)c](ac \vee bc)' &= [(a \vee b)c][(ac)' \vee (bc)'] && \text{by Lemma 9}\\ &= [(a \vee b)c](c'a' \vee c'b')\\ &= [(a \vee b)c][c'(a' \vee b')] && \text{by assumption}\\ &= [(a \vee b)c][(a \vee b)c]' .\end{aligned}$$

By (1) this means $(a \vee b)c \leq ac \vee bc$ for all $a, b, c \in S$. The proof of the converse statement is similar. //

## References

[1] Garrett Birkhoff, *Lattice theory*, Third edition (American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, Rhode Island, 1967).

https://doi.org/10.1017/S0004972700008443 Published online by Cambridge University Press

[2] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Volume I (Mathematical Surveys, 7. American Mathematical Society, Providence, Rhode Island, 1961; reprinted 1964).

[3] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Volume II (Mathematical Surveys, 7. American Mathematical Society, Providence, Rhode Island, 1967; reprinted 1971).

[4] Paul Dubreil, *Contribution à la théorie des semi-groupes* (Mém. Acad. Sci. Inst. France (2) 63, no. 3, 52 pp (1941)).

[5] L. Fuchs, *Partially ordered algebraic systems* (Pure and Applied Mathematics, 28. Akadémiai Kiadó, Budapest; Pergamon Press, Oxford, London, New York, Paris, 1963).

[6] J.M. Howie, *An introduction to semigroup theory* (London Mathematical Society Monographs, 7. Academic Press [Harcourt Brace Jovanovich], London, New York, San Francisco, 1976).

[7] W.D. Munn, "A class of irreducible matrix representations of an arbitrary inverse semigroup", *Proc. Glasgow Math. Assoc.* 5 (1961-62), 41-48.

[8] В.В. Вагнер [V.V. Vagner], "Обобщенные группы" [Generalized groups], *Dokl. Akad. Nauk SSSR (N.S.)* 84 (1952), 1119-1122.

Mathematisches Institut der Universität Wien,
Wien,
Austria.

https://doi.org/10.1017/S0004972700008443 Published online by Cambridge University Press