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POSITIVE SOLUTIONS OF A SECOND-ORDER NEUMANN BOUNDARY VALUE PROBLEM WITH A PARAMETER

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Abstract

In this paper, we consider the Neumann boundary value problem with a parameter $\lambda \in (0, \infty)$:

$$\begin{cases} -(p(t)x'(t))' + q(t)x(t) = \lambda g(t)f(x(t)), & 0 \le t \le 1, \\ x'(0) = x'(1) = 0. \end{cases}$$

By using fixed point theorems in a cone, we obtain some existence, multiplicity and nonexistence results for positive solutions in terms of different values of λ. We also prove an existence and uniqueness theorem and show the continuous dependence of solutions on the parameter λ.

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1. Introduction

In this paper, we consider the following Neumann boundary value problem (NBVP) with a parameter $\lambda \in (0, \infty)$:

$$\begin{cases} -(p(t)x'(t))' + q(t)x(t) = \lambda g(t)f(x(t)), & 0 \le t \le 1, \\ x'(0) = x'(1) = 0, \end{cases} \tag{1.1}$$

where $p(t) \in C^1[0, 1]$, $p(t) > 0$; $q(t) \in C[0, 1]$, $q(t) \ge 0$ and $q(t) \not\equiv 0$; $g : [0, \infty) \to [0, \infty)$ is continuous and $\int_0^1 g(s)\, ds > 0$; $f : [0, \infty) \to [0, \infty)$ is continuous and $f \not\equiv 0$. We assume that these conditions on p, q, f, g are satisfied throughout the paper unless otherwise specified.

A function $x \in C^2[0, 1]$ is said to be a nontrivial solution of (1.1) if and only if x satisfies (1.1) and $x(t) \not\equiv 0$. Moreover, if $x(t) \ge 0$ for $t \in [0, 1]$, then x is said to be a positive solution of (1.1).

By using fixed point theorems in a cone, we give some existence, multiplicity and nonexistence results for positive solutions of (1.1) (see Theorem 3.1), and we also investigate the existence and uniqueness of solutions of (1.1) and their continuous

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dependence on the parameter λ (see Theorem 4.1). Similar results for the periodic boundary value problem are obtained in Graef *et al.* [6] for the case where $p(t) = 1$ and $q(t) = \rho^2$ for some $\rho > 0$. So our results can be regarded as extensions of the results in [6]. We note that, in D'Agui [4], some results on the existence of three solutions for the NBVP in a more general form than (1.1) are proved by using a three critical points theorem. Studies of the boundary value problem with a parameter can also be found in [15, 16]. For more work on (1.1) with $\lambda = 1$, we refer readers to [3, 7–9, 12–14, 17–20] and the references therein.

This paper is organised as follows. Some notation and preliminary lemmas are given in Section 2. Then existence, multiplicity and nonexistence results for positive solutions are derived in terms of different values of λ in Section 3. An existence and uniqueness theorem as well as the result of continuous dependence of solutions on λ are presented in Section 4.

2. Preliminaries

Let $X = C[0, 1]$ with norm $\|x\| = \max_{0\le t\le 1} |x(t)|$, and $P = \{x \in C[0, 1]; x(t) \ge 0\}$. Then P is a normal cone in $C[0, 1]$, and $P^\circ \ne \emptyset$. Let $x_1, x_2 \in X$. We write $x_1 \le x_2$ if $x_2 - x_1 \in P$; $x_1 < x_2$ if $x_1 \le x_2$ and $x_1 \ne x_2$; $x_1 \ll x_2$ if $x_2 - x_1 \in P^\circ$. We call the set $[x_1, x_2] = \{x \in X : x_1 \le x_2\}$ an order interval in X. An operator $T : [x_1, x_2] \to X$ is called increasing (or nondecreasing) if $Tx \le Ty$ for any $x, y \in [x_1, x_2]$ and $x \le y$, and T is called strongly increasing if $Tx \ll Ty$ for any $x, y \in [x_1, x_2]$ and $x < y$.

From the results in [5], we obtain the following two lemmas, which will be useful for the proofs of our main results.

Lemma 2.1. *Let X be a Banach space, $P \subset X$ a normal cone with $P^\circ \ne \emptyset$. Let $\psi_1, \psi_2, \psi_3, \psi_4 \in X$ with $\psi_1 < \psi_2 < \psi_3 < \psi_4$ and suppose that the strongly increasing completely continuous map $G : [\psi_1, \psi_4] \to X$ satisfies*

$$\psi_1 \le G(\psi_1), G(\psi_2) < \psi_2, \psi_3 < G(\psi_3), G(\psi_4) \le \psi_4.$$

Then G has at least three fixed points x_1, x_2, x_3 such that $x_1 \ll x_2 \ll x_3$.

Lemma 2.2. *Let X be a Banach space and P be a cone in X. Assume that Q_1, Q_2 are bounded open subsets of X with $0 \in Q_1 \subset \bar{Q}_1 \subset Q_2$, and let $A : P \cap (\bar{Q}_2 \setminus Q_1) \to P$ be a completely continuous operator such that $\|Ax\| \ge \|x\|$ for any $x \in P \cap \partial Q_1$ and $\|Ax\| \le \|x\|$ for any $x \in P \cap \partial Q_2$. Then A has a fixed point in $P \cap (\bar{Q}_2 \setminus Q_1)$.*

We write

$$F(x) = \begin{cases} f(x)/x, & x > 0, \\ \limsup\limits_{t\to 0} f(t)/t, & x = 0, \end{cases}$$

and $f_0 = F(0)$, $f_\infty = \lim_{x\to\infty} F(x)$. We also need the functions

$$f^*(x) = \max_{0\le t\le x}\{f(t)\} \quad \text{and} \quad f_*(x) = \min_{0\le t\le x}\{f(t)\},$$

and we write $f^*_\infty = \lim_{x\to\infty} f^*(x)/x$ and $f^*_0 = \lim_{x\to 0} f^*(x)/x$.

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LEMMA 2.3 [15]. *Assume that* $f : [0, \infty) \to [0, \infty)$ *is continuous and* $f(x) > 0$ *for* $x > 0$. *Then* $f_\infty^* = f_\infty$ *and* $f_0^* = f_0$.

The following results are due to Li [8]. Let $L = \max_{0\le t\le 1}\{p(t)q(t)\}$. Then, by [8, Lemma 1], for each $h \in C[0, 1]$, the NBVP

$$\begin{cases} -(p(t)x'(t))' + Lx(t)/p(t) = h(t), & 0 \le t \le 1, \\ x'(0) = x'(1) = 0, \end{cases}$$

has the unique solution

$$x(t) = (Th)(t) = \int_0^1 G(t, s)h(s)\, ds,$$

where

$$G(t, s) = \begin{cases} \dfrac{(e^{\sqrt{L}\int_0^t \frac{ds}{p(s)}} + e^{-\sqrt{L}\int_0^t \frac{ds}{p(s)}})(e^{\sqrt{L}\int_s^1 \frac{dt}{p(t)}} + e^{-\sqrt{L}\int_s^1 \frac{dt}{p(t)}})}{2\sqrt{L}(e^{\sqrt{L}\int_0^1 \frac{ds}{p(s)}} + e^{-\sqrt{L}\int_0^1 \frac{ds}{p(s)}})}, & 0 \le t \le s \le 1, \\ \dfrac{(e^{\sqrt{L}\int_0^s \frac{dt}{p(t)}} + e^{-\sqrt{L}\int_0^s \frac{dt}{p(t)}})(e^{\sqrt{L}\int_t^1 \frac{ds}{p(s)}} + e^{-\sqrt{L}\int_t^1 \frac{ds}{p(s)}})}{2\sqrt{L}(e^{\sqrt{L}\int_0^1 \frac{ds}{p(s)}} + e^{-\sqrt{L}\int_0^1 \frac{ds}{p(s)}})}, & 0 \le s \le t \le 1. \end{cases}$$

Let $k = \min_{0\le s,t\le 1} G(t, s)$ and $K = \max_{0\le s,t\le 1} G(t, s)$. Then it is clear that $K > k > 0$. Moreover, we can see easily that $T : P \to P$ is a linear completely continuous operator since $G(t, s)$ is continuous. Let

$$(Bx)(t) = \frac{L - p(t)q(t)}{p(t)} x(t), \quad x \in P, t \in [0, 1].$$

Then $TB : P \to P$ is a linear completely continuous operator and $\|TB\| < 1$ (see [8, Lemma 2]). Moreover, for each $h \in C[0, 1]$, the NBVP

$$\begin{cases} -(p(t)x'(t))' + q(t)x(t) = h(t), & 0 \le t \le 1, \\ x'(0) = x'(1) = 0, \end{cases}$$

has a solution $x(t) = (I - TB)^{-1}Th(t)$ (see [8, Lemma 3]).

Now we define the map $T_\lambda : P \to P$ by

$$T_\lambda x(t) = \lambda(I - TB)^{-1}T(gf(x))(t), \quad 0 \le t \le 1.$$

As in the proof of [9, Lemmas 3–5], we can prove that T_λ is completely continuous. Then $x \in P \setminus \{0\}$ is a fixed point of T_λ if and only if x is a positive solution of (1.1).

3. Existence, multiplicity and nonexistence of positive solutions

In this section we give the existence, multiplicity and nonexistence results of positive solutions of (1.1).

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Theorem 3.1.

(i) *Assume that* $f(t) > 0$ *for* $t \geq 0$. *Then, given* $R > 0$, *there exist* $0 < \lambda_1 < \lambda_0$ *such that* (1.1) *has at least a positive solution* $x(t)$ *with* $\|x\| \leq R$ *for* $\lambda_1 \leq \lambda \leq \lambda_0$. *Moreover, if* $f_\infty = 0$, (1.1) *has at least a positive solution for all* $\lambda > 0$.

(ii) *Assume that* f *is strictly increasing and* $f_\infty = f_0 = 0$. *Then there exists* $\lambda_0 > 0$ *such that* (1.1) *has at least two positive solutions* x_1, x_2 *with* $0 \ll x_1 \ll x_2$ *for* $\lambda \in (\lambda_0, \infty)$.

(iii) *Assume that* $F(x)$ *is bounded in* $[0, \infty)$. *Then there exists* $\lambda_1 > 0$ *such that* (1.1) *has no positive solution for* $\lambda \in (0, \lambda_1)$.

Proof. (i) For $r > 0$, we write $\Omega_r = \{x \in X : \|x\| < r\}$, $\bar{\Omega}_r = \{x \in X : \|x\| \leq r\}$ and $\partial\Omega_r = \{x \in X : \|x\| = r\}$.

For $x \in \partial\Omega_R \cap P$,

$$\begin{aligned} \|T_\lambda x\| &= \lambda \left\| (I - TB)^{-1} \int_0^1 G(t, s) g(s) f(x(s))\, ds \right\| \\ &\leq \lambda K f^*(R) \|(I - TB)^{-1}\| \int_0^1 g(s)\, ds. \end{aligned}$$

Let

$$\lambda_0 = \frac{R}{K f^*(R) \|(I - TB)^{-1}\| \int_0^1 g(s)\, ds}. \tag{3.1}$$

Then, for each $0 < \lambda \leq \lambda_0$,

$$\|T_\lambda x\| \leq R = \|x\| \quad \text{for } x \in \partial\Omega_R \cap P.$$

Let $R_1 > 0$ be such that

$$R_1 < \frac{k f_*(R)}{K f^*(R) \|I - TB\|\, \|(I - TB)^{-1}\|} R. \tag{3.2}$$

Clearly, $R_1 < R$ and then, for $x \in \partial\Omega_{R_1} \cap P$,

$$\begin{aligned} \|T_\lambda x\| &= \lambda \left\| (I - TB)^{-1} \int_0^1 G(t, s) g(s) f(x(s))\, ds \right\| \\ &\geq \lambda \frac{\| \int_0^1 G(t, s) g(s) f(x(s))\, ds \|}{\|I - TB\|} \\ &\geq \lambda \frac{k f_*(R_1) \int_0^1 g(s)\, ds}{\|I - TB\|} \\ &\geq \lambda \frac{k f_*(R) \int_0^1 g(s)\, ds}{\|I - TB\|}. \end{aligned}$$

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Set

$$\lambda_1 = \frac{R_1 \|(I - TB)\|}{k f_*(R) \int_0^1 g(s)\, ds}.$$

Then $\lambda_1 < \lambda_0$ by (3.1) and (3.2), and, for each $\lambda \geq \lambda_1$,

$$\|T_\lambda x\| \geq R_1 = \|x\| \quad \text{for } x \in \partial\Omega_{R_1} \cap P.$$

Now it follows from Lemma 2.2 that T_λ has a fixed point in $(\bar{\Omega}_R \setminus \Omega_{R_1}) \cap P$ for each $\lambda_1 \leq \lambda \leq \lambda_0$. Consequently, (1.1) has a positive solution $x(t)$ with $\|x\| \leq R$ for each $\lambda_1 \leq \lambda \leq \lambda_0$.

Given $\lambda > 0$, since $f(t) > 0$ for $t \in [0, \infty)$, we have $f_*(r)/r \to \infty$ as $r \to 0$. So we can choose r_1 sufficiently small such that

$$0 < r_1 \leq \lambda \frac{k f_*(r_1) \int_0^1 g(s)\, ds}{\|I - TB\|}.$$

Then, for $x \in \partial\Omega_{r_1} \cap P$,

$$\begin{aligned}
\|T_\lambda x\| &= \lambda \left\| (I - TB)^{-1} \int_0^1 G(t, s) g(s) f(x(s))\, ds \right\| \\
&\geq \lambda \frac{\| \int_0^1 G(t, s) g(s) f(x(s))\, ds\|}{\|I - TB\|} \\
&\geq \lambda \frac{k f_*(r_1) \int_0^1 g(s)\, ds}{\|I - TB\|} \\
&\geq r_1 = \|x\|.
\end{aligned}$$

Also, since $f_\infty = 0$, we have $f_\infty^* = 0$ by Lemma 2.3. Then there exists $r_2 \in (r_1, \infty)$ such that $f^*(r_2) \leq \varepsilon r_2$ for some small $\varepsilon > 0$ satisfying

$$\varepsilon \lambda K \|(I - TB)^{-1}\| \int_0^1 g(s)\, ds < 1.$$

Thus, for $x \in \partial\Omega_{r_2} \cap P$,

$$\begin{aligned}
\|T_\lambda x\| &= \lambda \left\| (I - TB)^{-1} \int_0^1 G(t, s) g(s) f(x(s))\, ds \right\| \\
&\leq \lambda f^*(r_2) K \|(I - TB)^{-1}\| \int_0^1 g(s)\, ds \\
&\leq \lambda \varepsilon r_2 K \|(I - TB)^{-1}\| \int_0^1 g(s)\, ds \\
&< r_2 = \|x\|.
\end{aligned}$$

Then T_λ has a fixed point in $(\bar{\Omega}_{r_2} \setminus \Omega_{r_1}) \cap P$ by Lemma 2.2, and consequently (1.1) has a positive solution for $\lambda > 0$.

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(ii) It is easy to see that there exists $b \in (0, \infty)$ such that $F(b) = \max\{F(x) : x \in [0, \infty)\} > 0$ since $f_0 = f_\infty = 0$ and $f \not\equiv 0$. We let

$$m = \min_{0 \le t \le 1} (I - TB)^{-1} Tg(t), \quad M = \|(I - TB)^{-1} Tg\|. \tag{3.3}$$

Then $0 < m \le M$. Let $\lambda_0 = 1/mF(b)$. Noticing that $F(t) \to 0$ as $t \to 0$ or $t \to \infty$, given $\lambda > \lambda_0$, there exist $a \in (0, b)$ and $c \in (b, \infty)$ such that $\lambda MF(a) < 1$ and $\lambda MF(c) < 1$. That is,

$$\lambda Mf(a) < a \quad \text{and} \quad \lambda Mf(c) < c. \tag{3.4}$$

Since f is strictly increasing, we can verify easily that T_λ is strongly increasing in any order interval in P. It is clear that

$$T_\lambda 0 = 0,$$

and by (3.4),

$$\begin{aligned} T_\lambda a &= \lambda(I - TB)^{-1} T(f(a)g) = \lambda f(a)(I - TB)^{-1} Tg \le \lambda f(a) M < a, \\ T_\lambda b &= \lambda(I - TB)^{-1} T(f(b)g) = \lambda f(b)(I - TB)^{-1} Tg \ge \lambda f(b) m > b, \\ T_\lambda c &= \lambda(I - TB)^{-1} T(f(c)g) = \lambda f(c)(I - TB)^{-1} Tg \le \lambda f(c) M < c. \end{aligned} \tag{3.5}$$

Then it follows from Lemma 2.1 that T_λ has three fixed points x_0, x_1, x_2 in $[0, c]$ such that $x_0 \ll x_1 \ll x_2$. So x_1, x_2 are two fixed points of T_λ such that $0 \le x_0 \ll x_1 \ll x_2$. This means that (1.1) has two positive solutions x_1, x_2 with $0 \ll x_1 \ll x_2$ for each $\lambda \in (\lambda_0, \infty)$.

(iii) Since $F(x)$ is bounded, we may let $\mathcal{F} = \sup_{x \in [0,\infty)} F(x)$ and $\lambda_1 = 1/M\mathcal{F}$, where M is given in (3.3). Suppose that (1.1) has a positive solution x_λ for $\lambda \in (0, \lambda_1)$. Then

$$\begin{aligned} \|x_\lambda\| &= \|\lambda(I - TB)^{-1} T(gf(x_\lambda))\| \\ &\le \|\lambda(I - TB)^{-1} T(\mathcal{F}\|x_\lambda\| g)\| \\ &= \lambda\mathcal{F}\|x_\lambda\| \cdot \|(I - TB)^{-1} T(g)\| \\ &= \lambda M\mathcal{F}\|x_\lambda\| < \|x_\lambda\|, \end{aligned}$$

which is a contradiction. So (1.1) has no positive solution for $\lambda \in (0, \lambda_1)$.

This completes the proof. □

Remark 3.2.

(a) Theorem 3.1 extends [6, Theorem 2.1]. In fact, results similar to Theorem 3.1 were established in [6] for the special case of (1.1) when $p(t) = 1$ and $q(t) = \rho^2$ for some $\rho > 0$ (see [6, Theorem 2.1(a), (c), (e)]).

(b) In the proof of Theorem 3.1(ii), we get three fixed points of T_λ in $[0, c]$. However, there are possibly only two fixed points in $P \setminus \{0\}$ since x_1 may be 0. For example, let $f(x) = \min\{x^\sigma, x^\varsigma\}$ with $\sigma > 1, \varsigma < 1$. Then the conditions of Theorem (ii) hold. We can choose $a < 1$ so small that

$$\lambda M a^{\sigma - 1} < 1. \tag{3.6}$$

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From the proof of Lemma 2.1 (see [5, Section 20]), $0 \le x_1 < a$. Then by (3.5), (3.6) and the monotonicity of T_λ,

$$x_1 = T_\lambda^n x_1 \le T_\lambda^n a \le (\lambda M)^{(\sigma^n-1)/(\sigma-1)} a^{\sigma^n} = (\lambda M a^{\sigma-1})^{(\sigma^n-1)/(\sigma-1)} a \to 0 \quad \text{as } n \to \infty.$$

4. Existence and uniqueness of positive solutions

In this section, we will use the following assumption:

(H) $f : [0, \infty) \to (0, \infty)$ is nondecreasing, and there exists $\theta \in (0, 1)$ such that $f(\alpha x) \ge \alpha^\theta f(x)$ for $\alpha \in (0, 1)$ and $x \in [0, \infty)$.

The main result of this section is the following theorem.

Theorem 4.1. *Assume that (H) holds and $g(t) > 0$ for $t \ge 0$. Then (1.1) has a unique positive solution $x_\lambda(t)$ with $x_\lambda(t) > 0$, $t \in [0, 1]$, for $\lambda \in (0, \infty)$. Furthermore, such a solution $x_\lambda(t)$ satisfies the following properties:*

(i) *$x_\lambda(t)$ is nondecreasing in λ;*
(ii) *$\lim_{\lambda\to 0^+} \|x_\lambda\| = 0$ and $\lim_{\lambda\to\infty} \|x_\lambda\| = \infty$;*
(iii) *x_λ is continuous in λ, that is, if $\lambda \to \lambda_0$, then $\|x_\lambda - x_{\lambda_0}\| \to 0$.*

Proof. We first show that (1.1) has a solution for any $\lambda \in (0, \infty)$. By (H), for $x \in P$, $\alpha \in (0, 1)$,

$$T_\lambda(\alpha x) = \lambda(I - TB)^{-1} T(g f(\alpha x)) \ge \lambda(I - TB)^{-1} T(\alpha^\theta g f(x)) = \alpha^\theta T_\lambda x. \tag{4.1}$$

Similarly,

$$T_\lambda(\beta x) \le \beta^\theta T_\lambda x \quad \text{for } \beta > 1. \tag{4.2}$$

Let $\Phi = \lambda(I - TB)^{-1} \int_0^1 g(s)\,ds > 0$. Then it is easy to see that

$$0 < k f(\Phi)\Phi \le T_\lambda(\Phi) \le K f(\Phi)\Phi.$$

Define $\bar{C}$ and $\bar{D}$ by

$$\bar{C} = \sup\{\mu : \mu\Phi \le T_\lambda(\Phi)\} \quad \text{and} \quad \bar{D} = \inf\{\mu : \mu\Phi \ge T_\lambda(\Phi)\}. \tag{4.3}$$

Clearly, $k f(\Phi) \le \bar{C} \le \bar{D} \le K f(\Phi)$. Choose C and D such that

$$0 < C < \min\{1, \bar{C}^{1/(1-\theta)}\} \quad \text{and} \quad \max\{1, \bar{D}^{1/(1-\theta)}\} < D < \infty.$$

Define two sequences $\{x_k(t)\}$ and $\{y_k(t)\}$ by

$$\begin{cases} x_1 = C\Phi,\ x_{k+1} = T_\lambda x_k, & k = 1, 2, \ldots, \\ y_1 = D\Phi,\ y_{k+1} = T_\lambda y_k, & k = 1, 2, \ldots. \end{cases} \tag{4.4}$$

Then, by (4.1) and (4.3),

$$x_2 = T_\lambda x_1 = T_\lambda(C\Phi) \ge C^\theta T_\lambda(\Phi) \ge C^\theta \bar{C}\Phi \ge C^\theta C^{1-\theta}\Phi = x_1, \tag{4.5}$$

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and similarly, by (4.2) and (4.3),

$$y_2 \leq y_1. \tag{4.6}$$

Since f is nondecreasing, it is easy to verify that T_λ is nondecreasing in any order interval in P. Noticing that $x_1 < y_1$, by (4.4)–(4.6) it then follows that

$$C\Phi = x_1 \leq x_2 \leq \cdots \leq x_k \leq \cdots \leq y_k \leq \cdots \leq y_2 \leq y_1 = D\Phi. \tag{4.7}$$

Let $d = C/D$, so that $d \in (0, 1)$. We claim that

$$x_k \geq d^{\theta^{k-1}} y_k \quad \text{for } k = 1, 2, \ldots. \tag{4.8}$$

In fact, it is obvious that $x_1 = dy_1$, so that (4.8) is true for $k = 1$. Assume that (4.8) holds for $k = n$. Then it follows from (4.1) and the monotonicity of T_λ that

$$x_{n+1} = T_\lambda x_n \geq T_\lambda(d^{\theta^{n-1}} y_n) \geq (d^{\theta^{n-1}})^\theta T_\lambda y_n = d^{\theta^n} y_{n+1},$$

which means that (4.8) holds for $k = n + 1$, and then (4.8) holds for all $k = 1, 2, \ldots$. By (4.7) and (4.8),

$$\|x_k - y_k\| \leq (1 - d^{\theta^{k-1}})\|y_k\| \leq (1 - d^{\theta^{k-1}})D\Phi.$$

Thus there exists a function $x_\lambda \in P$ with $x_\lambda \geq C\Phi$ and

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x_\lambda,$$

and x_λ is a fixed point of T_λ. Therefore, $x_\lambda(t)$ is a positive solution of (1.1) with $x_\lambda(t) > 0$ for $t \in [0, 1]$.

We now show the uniqueness of the positive solution $x_\lambda(t)$ of (1.1) with $x_\lambda(t) > 0$, $t \in [0, 1]$, for each $\lambda \in (0, \infty)$. Assume, to the contrary, that there exists another positive solution $\bar{x}_\lambda(t)$ of (1.1) such that $\bar{x}_\lambda(t) > 0$ for $t \in [0, 1]$. Then $T_\lambda \bar{x}_\lambda = \bar{x}_\lambda$. Let

$$\alpha_0 = \sup\{\alpha > 0 : x_\lambda \geq \alpha \bar{x}_\lambda\}.$$

It is easy to see that $\alpha_0 \in (0, \infty)$ is well defined. We now show that $\alpha_0 \geq 1$. In fact, if $\alpha_0 < 1$, by (4.1) and the monotonicity of T_λ,

$$x_\lambda = T_\lambda x_\lambda \geq T_\lambda(\alpha_0 \bar{x}_\lambda) \geq \alpha_0^\theta T_\lambda \bar{x}_\lambda = \alpha_0^\theta \bar{x}_\lambda.$$

This contradicts the definition of α_0 since $\alpha_0^\theta > \alpha_0$. Hence, $x_\lambda \geq \bar{x}_\lambda$. Similarly, we can show that $\bar{x}_\lambda \geq x_\lambda$. Therefore, $x_\lambda = \bar{x}_\lambda$, and (1.1) has a unique positive solution $x_\lambda(t)$ with $x_\lambda(t) > 0$, $t \in [0, 1]$, for each $\lambda \in (0, \infty)$.

Finally, we prove properties (i)–(iii) of the solution x_λ of (1.1).

(i) Let $0 < \lambda_1 \leq \lambda_2$. Then $T_{\lambda_i} x_{\lambda_i} = x_{\lambda_i}$, $i = 1, 2$. Let

$$\bar{\eta} = \sup\{\eta : x_{\lambda_2} \geq \eta x_{\lambda_1}\}.$$

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Clearly, $\bar{\eta} \in (0, \infty)$ is well defined. We assert that $\bar{\eta} \geq 1$. Indeed, if $\bar{\eta} < 1$, by (4.1) and the monotonicity of T_λ,

$$x_{\lambda_2} = T_{\lambda_2} x_{\lambda_2} = \frac{\lambda_2}{\lambda_1} T_{\lambda_1} x_{\lambda_2} \geq \frac{\lambda_2}{\lambda_1} T_{\lambda_1}(\bar{\eta} x_{\lambda_1}) \geq \frac{\lambda_2}{\lambda_1} \bar{\eta}^\theta T_{\lambda_1} x_{\lambda_1} = \frac{\lambda_2}{\lambda_1} \bar{\eta}^\theta x_{\lambda_1}.$$

This contradicts the definition of $\bar{\eta}$ since $(\lambda_2/\lambda_1)\bar{\eta}^\theta > \bar{\eta}$. Therefore, $\bar{\eta} \geq 1$ and $x_{\lambda_2} \geq \bar{\eta} x_{\lambda_1} \geq x_{\lambda_1}$ and (i) is true.

(ii) For $0 < \lambda_1 \leq \lambda_2$, we have $x_{\lambda_1} \leq x_{\lambda_2}$ by (i). Then by the monotonicity of T_λ,

$$x_{\lambda_1} = T_{\lambda_1} x_{\lambda_1} = \frac{\lambda_1}{\lambda_2} T_{\lambda_2} x_{\lambda_1} \leq \frac{\lambda_1}{\lambda_2} T_{\lambda_2} x_{\lambda_2} = \frac{\lambda_1}{\lambda_2} x_{\lambda_2}. \tag{4.9}$$

Now fix λ_2 and let $\lambda_1 \to 0+$; we obtain $\|x_{\lambda_1}\| \to 0$. On the other hand, fix λ_1 and let $\lambda_2 \to \infty$; we obtain $\|x_{\lambda_2}\| \to \infty$.

(iii) Suppose that $\lambda_0 > 0$. Let $\lambda > \lambda_0$. As in (4.9) we can show that $x_{\lambda_0} \leq (\lambda_0/\lambda) x_\lambda$. Let

$$l_\lambda = \sup\{l > 0 : x_{\lambda_0} \geq l x_\lambda\}.$$

Then $0 < l_\lambda \leq \lambda_0/\lambda_1 < 1$. From (4.1) and the monotonicity of T_λ,

$$x_{\lambda_0} = T_{\lambda_0} x_{\lambda_0} \geq T_{\lambda_0}(l_\lambda x_\lambda) \geq l_\lambda^\theta T_{\lambda_0} x_\lambda = l_\lambda^\theta \frac{\lambda_0}{\lambda} T_\lambda x_\lambda = l_\lambda^\theta \frac{\lambda_0}{\lambda} x_\lambda.$$

By the definition of l_λ, $l_\lambda \geq l_\lambda^\theta \lambda_0/\lambda$; that is, $l_\lambda \geq (\lambda_0/\lambda)^{1/(1-\theta)}$. So we obtain

$$x_{\lambda_0} \geq l_\lambda x_\lambda \geq (\lambda_0/\lambda)^{1/(1-\theta)} x_\lambda,$$

and then

$$\|x_{\lambda_0} - x_\lambda\| \leq (1 - (\lambda_0/\lambda)^{1/(1-\theta)}) \|x_{\lambda_0}\| \to 0 \quad \text{as } \lambda \to \lambda_0 + 0.$$

That is, T_λ is right-continuous at λ_0. Similarly, we can prove that T_λ is left-continuous at λ_0. This completes the proof. □

REMARK 4.2. We note that results similar to Theorem 4.1 have been established in [6, 10, 11] for other types of boundary value problem, and some ideas of the proof of Theorem 4.1 are also from [6, 10, 11]. For some more work in this area, we refer readers to [1, 2].

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