*J. Austral. Math. Soc. Ser. B* 40(1999), 289–300

# SOLVABILITY OF GENERALIZED NONLINEAR SYMMETRIC VARIATIONAL INEQUALITIES

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(Received 26 February 1997; revised 12 June 1997)

**Abstract**

This paper deals with the study of a general class of nonlinear variational inequalities. An existence result is given, and a perturbed iterative scheme is analyzed for solving such problems.

## 1. Introduction and preliminaries

This paper deals with the solvability of a class of generalized equations of the form

$$(g(\overline{u}), h(\overline{u})) \in M. \tag{GE}$$

Here $M$ is a maximal monotone subset of $H \times H$, $H$ a real Hilbert space, and $g, h\colon H \to H$ are given nonlinear mappings. This problem contains as special cases various forms of variational and quasi-variational inequalities. The study of variational inequalities started in the sixties with the pioneering works of G. Fichera [6], J. L. Lions and G. Stampacchia [8] and J. J. Moreau [9]. If $M$ is the graph of a sub-differential mapping $\partial\varphi\colon H \rightrightarrows H$, where $\varphi$ is a proper, convex, lower semi-continuous function on $H$, then problem (GE) assumes the variational form

$$h(\overline{u}) \in \partial\varphi(g(\overline{u})).$$

In fact, it is the latter problem we shall be mainly concerned with. The extension of our results to the general problem (GE) is not difficult, and will only be discussed in the last paragraph. We now make the data of our problem more precise.

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let $g, h\colon H \to H$ be given nonlinear mappings and let

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https://doi.org/10.1017/S0334270000010912 Published online by Cambridge University Press

$\varphi\colon H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function such that $(\operatorname{range} g) \cap (\operatorname{dom} \varphi) \neq \emptyset$. We consider the following generalized variational inequality. Find $\overline{u} \in H$ such that

$$\langle h(\overline{u}), v - g(\overline{u})\rangle \leq \varphi(v) - \varphi(g(\overline{u})) \quad \forall\, v \in H. \tag{1.1}$$

We can write this equivalently as

$$h(\overline{u}) \in \partial\varphi(g(\overline{u})), \tag{1.1a}$$

where $\partial\varphi$ is the convex sub-differential of $\varphi$, that is,

$$\partial\varphi(x) := \{\xi \in H \mid \langle \xi, y - x\rangle \leq \varphi(y) - \varphi(x)\ \forall\, y \in H\}.$$

We list a few examples of Problem (1.1).

Let $m(u) := u - g(u)$. Then it is easily verified that $\overline{u}$ is a solution of (1.1) if and only if

$$\langle h(\overline{u}), y - \overline{u}\rangle \leq \varphi(y - m(\overline{u})) - \varphi(\overline{u} - m(\overline{u})) \quad \forall\, y \in H. \tag{1.2}$$

Indeed, this rewriting corresponds to a change of variables $y = v + m(\overline{u})$. Problem (1.2) has the form of a quasi-equilibrium problem [12].

If $K$ is a non-empty, closed and convex subset of $H$ and $\varphi := \delta_K$ is the indicator function of $K$, then (1.1) takes the form

$$g(\overline{u}) \in K, \quad \langle h(\overline{u}), v - g(\overline{u})\rangle \leq 0 \quad \forall\, v \in K \tag{1.3}$$

and the equivalent problem (1.2) takes the form

$$\overline{u} \in K(\overline{u}), \quad \langle h(\overline{u}), y - \overline{u}\rangle \leq 0 \quad \forall\, y \in K(\overline{u}), \tag{1.4}$$

where $K(u) := K + m(u)$. Problem (1.4) is a quasi-variational inequality. If $h$ is a potential operator, that is, there exists a convex and Gâteaux differentiable function $F\colon H \to \mathbb{R}$ such that

$$h(u) = -\nabla F(u),$$

then (1.4) is equivalent to

$$\overline{u} \in K(\overline{u}), \quad F(y) - F(\overline{u}) \geq 0 \quad \forall\, y \in K(\overline{u}).$$

If $g = \mathbf{I}$ (the identity mapping in $H$) and thereby $m(u) \equiv 0$, then (1.1) and (1.2) coincide and become the problem

$$\langle h(\overline{u}), y - \overline{u}\rangle \leq \varphi(y) - \varphi(\overline{u}) \quad \forall\, y \in H. \tag{1.5}$$

Furthermore, if $g = \mathbf{I}$, problem (1.4) becomes the variational inequality

$$\overline{u} \in K, \quad \langle h(\overline{u}), y - \overline{u} \rangle \leq 0 \quad \forall y \in K. \tag{1.6}$$

If $K$ is a closed, convex cone and $K^* := \{\xi \in H \mid \langle \xi, x \rangle \geq 0 \; \forall x \in K\}$ is its polar cone, then (1.3) becomes the quasi-complementarity problem

$$g(\overline{u}) \in K, \quad h(\overline{u}) \in -K^* \text{ and } \langle h(\overline{u}), g(\overline{u}) \rangle = 0. \tag{1.7}$$

If $g = \mathbf{I}$, then (1.7) is called a complementarity problem.

Let $\varphi^* \colon H \to \mathbb{R} \cup \{+\infty\}$ denote the conjugate function of $\varphi$, that is,

$$\varphi^*(\xi) := \sup_{x \in H} \big( \langle \xi, x \rangle - \varphi(x) \big).$$

The function $\varphi^*$ is again proper, convex and lower semi-continuous. Since

$$\xi \in \partial\varphi(x) \iff \varphi^*(\xi) + \varphi(x) = \langle \xi, x \rangle \iff x \in \partial\varphi^*(\xi),$$

we see *via* (1.1a) that $\overline{u}$ solves (1.1) if and only if

$$\langle \xi - h(\overline{u}), g(\overline{u}) \rangle \leq \varphi^*(\xi) - \varphi^*(h(\overline{u})) \quad \forall \xi \in H. \tag{1.8}$$

We can write this equivalently as

$$g(\overline{u}) \in \partial\varphi^*(h(\overline{u})). \tag{1.8a}$$

Since the dual problem (1.8) has the same structure as the primal problem (1.1), we call (1.1) a symmetric quasi-variational inequality. This added symmetry is the main reason for introducing a general mapping $g$, which may be different from $\mathbf{I}$. If $K$ is a closed, convex cone and $\varphi = \delta_K$, then $\varphi^* = \delta_{-K^*}$. Therefore and from $K^{**} = K$, the quasi-complementarity problem (1.7) is identical with its dual (see for example [12]).

REMARK 1. We observe that the solution set of (1.1) remains unchanged if, for arbitrary $\sigma > 0$ and $\tau > 0$, we replace simultaneously

$$\begin{aligned} &h(\cdot) \text{ by } \widetilde{h}(\cdot) := \tau h(\cdot), \\ &g(\cdot) \text{ by } \widetilde{g}(\cdot) := \sigma g(\cdot), \\ &\varphi(\cdot) \text{ by } \widetilde{\varphi}(\cdot) := \tau\sigma\varphi(\tfrac{\cdot}{\sigma}) \end{aligned}$$

and $\varphi^*(\cdot)$ by $(\widetilde{\varphi})^*(\cdot) = \sigma\tau\varphi^*(\frac{\cdot}{\tau})$ in (1.8). This scaling procedure will be employed later on.

## 2. Auxiliary tools

With $\varphi$ as before, let $J^{\varphi}(\cdot)\colon H \to H$ denote the proximity mapping which assigns to each $x \in H$ the unique minimizer of the strictly convex function $\varphi(\cdot) + \frac{1}{2}\|\cdot - x\|^2$ over $H$ (see [5, p. 39]). The following characterization is well known (see for example [5, p. 39] or [4, p. 25]):

$$z = J^{\varphi}(x) \iff \big(0 \le \langle z - x, v - z\rangle + \varphi(v) - \varphi(z) \quad \forall v \in H\big). \tag{2.1}$$

We can write equivalently

$$z = J^{\varphi}(x) \iff x - z \in \partial\varphi(z). \tag{2.1a}$$

In other words, $J^{\varphi} = (\mathbf{I} + \partial\varphi)^{-1}$. From (2.1a) it follows in particular that

$$x = J^{\varphi}(x + \xi) \iff \xi \in \partial\varphi(x) \iff x \in \partial\varphi^*(\xi) \iff \xi = J^{\varphi^*}(\xi + x) \tag{2.2}$$

for arbitrary $x, \xi \in H$. From (2.2) we can read off that

$$y = J^{\varphi}(y) + J^{\varphi^*}(y) \quad \forall y \in H. \tag{2.3}$$

We recall that $\partial\varphi$ is monotone, that is,

$$\langle \xi - \eta, x - y\rangle \ge 0 \text{ whenever } \xi \in \partial\varphi(x),\ \eta \in \partial\varphi(y).$$

We collect some simple properties of $J^{\varphi}$, which will be used later.

LEMMA 1. (a) *The proximity mapping $J^{\varphi}$ is non-expansive, that is,*

$$\|J^{\varphi}(x) - J^{\varphi}(y)\| \le \|x - y\| \quad \forall x, y \in H.$$

(b) *For all $x, a, b$ in $H$,*

$$\left\| J^{\varphi(\cdot - a)}(x) - J^{\varphi(\cdot - b)}(x) \right\| \le \|a - b\|.$$

(c) *For all $x$ and $a$ in $H$,*

$$a + J^{\varphi(\cdot)}(x - a) = J^{\varphi(\cdot - a)}(x).$$

PROOF. (a) Let $\xi := J^{\varphi}(x)$ and $\eta := J^{\varphi}(y)$. Then from (2.1a) we have

$$x - \xi \in \partial\varphi(\xi) \text{ and } y - \eta \in \partial\varphi(\eta),$$

and from the monotonicity of $\partial\varphi$ follows

$$0 \le \langle (x - \xi) - (y - \eta), \xi - \eta\rangle \le -\|\xi - \eta\|^2 + \|x - y\| \cdot \|\xi - \eta\|.$$

https://doi.org/10.1017/S0334270000010912 Published online by Cambridge University Press

Division by $\|\xi - \eta\|$ gives $\|\xi - \eta\| \le \|x - y\|$, as required.

(b) Let $\xi := J^{\varphi(\cdot - a)}(x)$ and $\eta := J^{\varphi(\cdot - b)}(x)$. Then from (2.1a) we obtain

$$x - \xi \in \partial\varphi(\xi - a) \text{ and } x - \eta \in \partial\varphi(\eta - b)$$

and from the monotonicity of $\partial\varphi$ follows

$$0 \le \langle (x - \eta) - (x - \xi), (\eta - b) - (\xi - a)\rangle \le -\|\xi - \eta\|^2 + \|\xi - \eta\| \cdot \|a - b\|.$$

Division by $\|\xi - \eta\|$ gives $\|\xi - \eta\| \le \|a - b\|$, as required.

(c) This statement follows readily from (2.1a).

## 3. Results

We shall mainly employ the following characterization.

LEMMA 2. *$\overline{u} \in H$ is a solution of* (1.1) *if and only if*

$$g(\overline{u}) = J^{\varphi}(g(\overline{u}) + h(\overline{u})). \tag{3.1}$$

PROOF. From (2.2) we derive that

$$\xi \in \partial\varphi(x) \iff x = J^{\varphi}(x + \xi).$$

Hence (1.1a) is equivalent to (3.1).

Dually, $\overline{u} \in H$ is a solution of (1.8) and therefore also a solution of (1.1), if and only if

$$h(\overline{u}) = J^{\varphi^*}(g(\overline{u}) + h(\overline{u})). \tag{3.2}$$

To problem (1.1) we can associate the following *Wiener-Hopf* condition

$$g(\overline{u}) = J^{\varphi}(\overline{v}), \quad h(\overline{u}) = \overline{v} - J^{\varphi}(\overline{v}). \tag{3.3}$$

LEMMA 3. *$\overline{u} \in H$ is a solution of* (3.1) *if and only if, together with some $\overline{v} \in H$ it is a solution of* (3.3).

PROOF. From (3.3) it follows, by adding both equations, that $g(\overline{u}) + h(\overline{u}) = \overline{v}$. On substituting for $\overline{v}$ in the first equation of (3.3), we obtain (3.1).

Now let (3.1) hold. Set $\overline{v} := g(\overline{u}) + h(\overline{u})$. Then from (3.1) we obtain $g(\overline{u}) = J^{\varphi}(\overline{v})$ and therefore $\overline{v} = J^{\varphi}(\overline{v}) + h(\overline{u})$. So both equations in (3.3) are satisfied.

Thus the Wiener-Hopf condition (3.3) is also necessary and sufficient for $\overline{u} \in H$ being a solution of (1.1). From (2.3) we see that condition (3.3) can be expressed in more symmetric form as

$$g(\overline{u}) = J^{\varphi}(\overline{v}), \quad h(\overline{u}) = J^{\varphi^*}(\overline{v}). \tag{3.4}$$

If $g$ is invertible, then we can solve condition (3.3) for $\overline{u}$ and write it as

$$\overline{u} = g^{-1}(J^{\varphi}(\overline{v})), \quad \overline{v} = h(g^{-1}(J^{\varphi}(\overline{v}))) + J^{\varphi}(\overline{v}). \tag{3.5}$$

If $g = \mathbf{I}$ and $\varphi = \delta_K$, then (3.5) becomes

$$\overline{u} = P_K(\overline{v}), \quad \overline{v} = h(P_K(\overline{v})) + P_K(\overline{v}),$$

where $P_K$ is the metric projection onto $K$. This is the original form of the Wiener-Hopf condition, as introduced by [16] in connection with the variational inequality (1.6). See also [1, 11, 13].

Since we do not feel that condition (3.3) has definite computational advantages over condition (3.1), let us return to the latter. Introducing the mapping $\Phi: H \to H$ as

$$\Phi(u) := u - g(u) + J^{\varphi}(g(u) + h(u)), \tag{3.6}$$

we can write condition (3.1) in fixed-point form as

$$\overline{u} = \Phi(\overline{u}). \tag{3.7}$$

Note that, for a given $u \in H$, $v = \Phi(u)$ is equivalent to

$$g_u(v) = J^{\varphi}(g_u(v) + h_u(v)),$$

where $g_u(v) := g(u) - u + v$ and $h_u(v) := h(u) + u - v$.

Because of the possibility of scaling we do not have to distinguish between Picard iterates $u^{n+1} := \Phi(u^n)$ and Mann-Toeplitz iterates $u^{n+1} := \lambda\Phi(u^n) + (1 - \lambda)u^n$ with $\lambda > 0$; the latter are simply Picard iterates of the operator $\Phi^{\lambda}(\cdot)$, which is obtained from $\Phi$ by replacing in (3.6)

$$\begin{aligned} &g(\cdot) \text{ by } \lambda g(\cdot), \\ &h(\cdot) \text{ by } \lambda h(\cdot), \\ &\varphi(\cdot) \text{ by } \lambda^2\varphi(\tfrac{\cdot}{\lambda}) =: \varphi^{\lambda}(\cdot). \end{aligned}$$

Compare with Remark 1 and note that $\lambda J^{\varphi}(x) = J^{\varphi^{\lambda}}(\lambda x)$.

Defining

$$\Psi(u) := u + h(u) - J^{\varphi^*}(g(u) + h(u)),$$

we see from (3.2) that $\overline{u}$ solves (1.1) if and only if

$$\overline{u} = \Psi(\overline{u}).$$

This, however, is the same as (3.7), since from (2.3) it follows that $\Psi(u) \equiv \Phi(u)$.

THEOREM 1. *Assume that there exist real numbers $\sigma > 0$, $\tau > 0$ and $k_1 \geq 0$, $k_2 \geq 0$ with $k_1 + k_2 < 1$ such that the mappings $G(u) := u - \sigma g(u)$ and $H(u) := u + \tau h(u)$ are Lipschitz continuous with Lipschitz constants $k_1$ and $k_2$ respectively. Then problem* (1.1) *has a unique solution $\overline{u} \in H$.*

PROOF. Let $\widetilde{g}(u) := \sigma g(u)$, $\widetilde{h}(u) := \tau h(u)$, $\widetilde{\varphi}(u) := \tau\sigma\varphi(\frac{u}{\sigma})$ and set

$$\widetilde{\Phi}(u) := u - \widetilde{g}(u) + J^{\widetilde{\varphi}}(\widetilde{g}(u) + \widetilde{h}(u)) = G(u) + J^{\widetilde{\varphi}}(H(u) - G(u)).$$

Using Remark 1, we have

$$\widetilde{\Phi}(\overline{u}) = \overline{u} \iff \Phi(\overline{u}) = \overline{u}.$$

Under the stated assumptions, $\widetilde{\Phi}$ is a contraction. Indeed, by Lemma 1(c) we have

$$\widetilde{\Phi}(u) = G(u) + J^{\widetilde{\varphi}}(H(u) - G(u)) = J^{\widetilde{\varphi}(\cdot - G(u))}(H(u))$$

and from Lemma 1(a), (b) we obtain

$$\left\| J^{\widetilde{\varphi}(\cdot - G(v))}(H(u)) - J^{\widetilde{\varphi}(\cdot - G(v))}(H(v)) \right\| \leq \|H(u) - H(v)\|,$$
$$\left\| J^{\widetilde{\varphi}(\cdot - G(u))}(H(u)) - J^{\widetilde{\varphi}(\cdot - G(v))}(H(u)) \right\| \leq \|G(u) - G(v)\|.$$

Hence

$$\begin{aligned}\left\|\widetilde{\Phi}(u) - \widetilde{\Phi}(v)\right\| &= \left\| J^{\widetilde{\varphi}(\cdot - G(u))}(H(u)) - J^{\widetilde{\varphi}(\cdot - G(v))}(H(v)) \right\| \\ &\leq \|G(u) - G(v)\| + \|H(u) - H(v)\| \\ &\leq (k_1 + k_2)\,\|u - v\|.\end{aligned}$$

Thus $\widetilde{\Phi}$ is a contraction and has a unique fixed point $\overline{u} = \widetilde{\Phi}(\overline{u})$, which is at the same time the unique fixed point of $\Phi$ and therefore, by Lemma 2, the unique solution of (1.1).

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REMARK 2. Assume there exists $\alpha > 0$ such that, for all $u, v \in H$,

$$\langle h(u) - h(v), u - v\rangle \leq -\alpha \|h(u) - h(v)\|^2 .$$

Then, for all $\tau$ with $0 \leq \tau \leq 2\alpha$, $H := \mathbf{I} + \tau h$ is non-expansive, since

$$\begin{aligned}\|H(u) - H(v)\|^2 &\leq \|u - v\|^2 - \tau(2\alpha - \tau)\|h(u) - h(v)\|^2 \\ &\leq \|u - v\|^2 .\end{aligned}$$

If, in addition, there exists $\beta > 0$ such that, for all $u, v \in H$,

$$\beta\|u - v\| \leq \|h(u) - h(v)\| ,$$

then $\alpha\beta \leq 1$ and for all $\tau$ with $0 < \tau < 2\alpha$, $H := \mathbf{I} + \tau h$ is a contraction, since

$$\|H(u) - H(v)\| \leq k\|u - v\|$$

with $k := \sqrt{1 - \tau(2\alpha - \tau)\beta^2} < 1$. A similar remark applies to $\mathbf{I} - \sigma g$.

REMARK 3. Assume that $g$ is invertible on $H$ and set $\psi(u) := h(g^{-1}(u))$. With $P := (\psi + \mathbf{I}) \circ J^{\varphi}$, it follows from (3.5) that we have to find a fixed point $\overline{v} = P(\overline{v})$ and then $\overline{u} := g^{-1}(J^{\varphi}(\overline{v}))$ solves (1.1). If $P$ is a contraction, then the Picard iterates $v^{n+1} := P(v^n)$ converge to $\overline{v}$ and the problem is solved. Now assume that $-\psi$ is co-coercive in the sense that, for all $u, v \in H$,

$$\langle \psi(u) - \psi(v), u - v\rangle \leq -\alpha\|\psi(u) - \psi(v)\|^2 ,$$

with $\alpha \geq \frac{1}{2}$. Then $P$ is non-expansive – see Remark 2. Hence, if $P$ has fixed points, then the Mann-Toeplitz iterates

$$v^{n+1} := \lambda P(v^n) + (1 - \lambda)v^n \text{ with } 0 < \lambda < 1 \tag{3.8}$$

converge weakly to a fixed point $\overline{v}$ of $P$ and $\left\|v^{n+1} - v^n\right\| \to 0$ [15]. As an example for this approach we consider the finite-dimensional Linear Complementarity Problem (LCP): To find $\overline{u} \in \mathbb{R}^N$ such that

$$\overline{u} \geq 0, \quad A\overline{u} + b \geq 0, \quad \langle \overline{u}, A\overline{u} + b\rangle = 0,$$

where $A$ is an $N \times N$ matrix and $b \in \mathbb{R}^N$. We assume that $\langle u, Au\rangle \geq 0$ for all $u \geq 0$ and that

$$\big(u \geq 0,\ Au \geq 0,\ \langle u, Au\rangle = 0,\ \langle u, b\rangle \leq 0\big) \implies u = 0.$$

Then (LCP) is solvable (see [7]). In this case, $P(v) = (\mathbf{I} - A)v^+ - b$, where $v^+$ denotes the positive part of $v$. If $\langle u, Au\rangle \geq \frac{1}{2}\|Au\|^2$ for all $u \in \mathbb{R}^N$, then $P$ is non-expansive, hence the iterates $v^n$ given by (3.8) converge to a fixed point of $P$ and the iterates $u^n := (v^n)^+$ converge to a solution of (LCP).

## 4. Perturbations

We assume now that we are given a sequence $\{\varphi^n\}_{n\in\mathbb{N}}$ of convex, proper, lower semi-continuous functions $\varphi^n : H \to \mathbb{R} \cup \{+\infty\}$ which converge to $\varphi$ in the sense of Mosco, that is, for every $u \in H$

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi^n(u_n)$$

holds for every sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H$ which converges weakly to $u$ and there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H$ which converges strongly to $u$ and satisfies

$$\varphi(u) \geq \limsup_{n\to\infty} \varphi^n(u_n).$$

We know from [2, Theorem 3.26] that if $\varphi^n$ converges to $\varphi$ in the sense of Mosco, then

$$J^{\varphi^n}(u) \to J^{\varphi}(u) \quad \forall\, u \in H. \tag{4.1}$$

We illustrate the perturbation scheme by the following examples.

EXAMPLE 1. *Penalty Schemes.* Let $K \subseteq H$ be nonempty, closed, convex and let $p : H \to \mathbb{R}$ be a penalty function for $K$, that is, $p$ is a lower semi-continuous and convex function satisfying

$$p(u) \geq 0 \;\; \forall\, u \in H, \;\text{ and } p(u) = 0 \iff u \in K.$$

Consider the sequence of functions $\{\varphi^n\}_{n\in\mathbb{N}}$ defined by $\varphi^n(u) := r_n p(u)$ for all $u \in H$. If $0 < r_n < r_{n+1}$ and $r_n \to +\infty$, then $\varphi^n$ converges to $\varphi := \delta_K$ in the sense of Mosco.

EXAMPLE 2. *Galerkin Schemes.* Let $K \subseteq H$ be nonempty, closed, convex and let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of closed, convex subsets of $K$ such that $K_n \subseteq K_m$ whenever $n \leq m$, and $K = \operatorname{cl} \bigcup_{n\in\mathbb{N}} K_n$. Then $\varphi^n := \delta_{K_n}$ converges to $\varphi := \delta_K$ in the sense of Mosco. We recall that, quite generally, the Mosco convergence of $\delta_{K_n}$ to $\delta_K$ is synonymous with the Mosco convergence of $K_n$ to $K$; see [2, Propositions 3.21 and 3.22].

THEOREM 2. *Let the assumption of Theorem* 1 *hold. Assume, for simplicity, that* $\sigma = \tau = 1$. *Let* $\overline{u}$ *denote the unique solution of*

$$\overline{u} = \overline{u} - g(\overline{u}) + J^{\varphi}(g(\overline{u}) + h(\overline{u})) \tag{4.2}$$

*and for all* $n \in \mathbb{N}$ *let*

$$u^{n+1} := u^n - g(u^n) + J^{\varphi^n}(g(u^n) + h(u^n)), \quad u^1 \in H \text{ arbitrary}. \tag{4.3}$$

*Then the sequence* $\{u^n\}$ *converges to* $\overline{u}$.

PROOF. Let $\Phi^n(u) := u - g(u) + J^{\varphi^n}(g(u) + h(u))$ and let $\Phi(u)$ be given by (3.6). Then from (4.2) and (4.3) follows (*cf.* the proof of Theorem 1):

$$\begin{aligned}\left\|u^{n+1} - \overline{u}\right\| &= \|\Phi^n(u^n) - \Phi(\overline{u})\| \le \|\Phi^n(u^n) - \Phi^n(\overline{u})\| + \|\Phi^n(\overline{u}) - \Phi(\overline{u})\| \\ &\le \|\Phi^n(u^n) - \Phi^n(\overline{u})\| + \left\|(J^{\varphi^n} - J^{\varphi})(g(\overline{u}) + h(\overline{u}))\right\| \\ &\le (k_1 + k_2)\,\|u^n - \overline{u}\| + \varepsilon_n,\end{aligned}$$

where $k_1 + k_2 < 1$ and $\varepsilon_n := \left\|(J^{\varphi^n} - J^{\varphi})(g(\overline{u}) + h(\overline{u}))\right\| \to 0$ from (4.1). The result follows then from [14, p. 394].

REMARK 4. No algorithm is known for computing the proximity mapping $J^{\varphi}$ for arbitrary convex functions $\varphi$. However, if $\varphi$ is a convex function from $\mathbb{R}^N$ to $\mathbb{R}$ finite everywhere, we can use Auslender's algorithm [3] to compute $J^{\varphi}$. If we take a problem of the form (1.3), then the proximity mapping coincides with the orthogonal projection onto the convex subset $K$. In many cases explicit expressions for such projections can be given, for instance if $K$ is a polyhedral set.

## 5. Extension

Let $M$ be a maximal monotone subset of $H \times H$. This means that for every $(y, \eta) \in H \times H$

$$(y,\eta) \in M \iff \big(\langle \xi - \eta, x - y\rangle \ge 0 \;\forall\, (x,\xi) \in M\big).$$

Then we can generalize our approach by replacing the conditions $\xi \in \partial\varphi(x)$ and/or $x \in \partial\varphi^*(\xi)$ by $(x, \xi) \in M$. So we consider the problem of finding $\overline{u} \in H$ such that

$$(g(\overline{u}), h(\overline{u})) \in M. \tag{5.1}$$

This extends problem (1.1) – or rather the equivalent (1.1a) – since the graph of $\partial\varphi$ is maximal monotone in $H \times H$.

$M$ defines two maximal monotone multi-valued mappings $\delta, \delta^*\colon H \rightrightarrows H$ such that

$$\xi \in \delta(x) \iff (x,\xi) \in M \iff x \in \delta^*(\xi).$$

If we define $J^{\delta} := (\mathbf{I} + \delta)^{-1}$ and $J^{\delta^*} := (\mathbf{I} + \delta^*)^{-1}$, then $J^{\delta}$ and $J^{\delta^*}$ are single-valued on all of $H$ and non-expansive [4]. The mappings $J^{\delta}$ and $J^{\delta^*}$ replace $J^{\varphi}$ and $J^{\varphi^*}$. By analogy with (3.6) we now define $\Phi\colon H \to H$ by

$$\Phi(u) := u - g(u) + J^{\delta}(g(u) + h(u)).$$

Then it is obvious that

$$v = \Phi(u) \iff (g(u) - u + v, h(u) + u - v) \in M$$

and in particular

$$\overline{u} = \Phi(\overline{u}) \iff \overline{u} \text{ solves } (5.1).$$

The solution set of (5.1) remains unchanged if we replace $h(\cdot)$ by $\tau h(\cdot)$, $g(\cdot)$ by $\sigma g(\cdot)$, $\delta(\cdot)$ by $\tau\delta(\frac{\cdot}{\sigma})$, $\delta^*(\cdot)$ by $\sigma\delta^*(\frac{\cdot}{\tau})$, for $\sigma > 0$, $\tau > 0$. This is the same scaling procedure as described before in Remark 1. Furthermore Lemma 1 carries over to $J^\delta$. Therefore Theorem 1 remains valid for problem (5.1).

With $M$ and $\delta$ as before, let $\{M^n\}_{n\in\mathbf{N}}$ be a sequence of maximal monotone subsets of $H \times H$. For each $n$ let $\delta^n$ be the maximal monotone mapping associated with $M^n$. We assume that the sequence $\{\delta^n\}$ graph-converges to $\delta$. This means that for every $(x, \xi) \in M$ there exists a sequence $\{(x_n, \xi_n)\}_{n\in\mathbf{N}}$ converging to $(x, \xi)$ such that $(x_n, \xi_n) \in M^n$ for all $n$. Then we know from [2, Proposition 3.60], that

$$J^{\delta^n}(u) \to J^\delta(u) \quad \forall u \in H.$$

With this the proof of Theorem 2 carries over to the present setting and we obtain the following result.

THEOREM 3. *Let $g, h$ satisfy the hypothesis of Theorem 1 with $\sigma = \tau = 1$. Then the iterates*

$$u^{n+1} := u^n - g(u^n) + J^{\delta^n}(g(u^n) + h(u^n))$$

*converge to $\overline{u}$, the unique solution of* (5.1).

Just as an example we mention that the Yosida approximants

$$\delta^n := \tfrac{1}{\lambda_n}(\mathbf{I} - (\mathbf{I} + \lambda_n\delta)^{-1})$$

graph-converge to $\delta$ for $\lambda_n \downarrow 0$ ([2, Proposition 3.56]).

We did not consider here the possibility that $-h$ is also maximal monotone. This case, with $g = \mathbf{I}$, has been studied in [10] or [17].

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