WEIGHTED FUNCTION ALGEBRAS ON GROUPS
AND SEMIGROUPS

Heneri A.M. Dzinotyiweyi

For a locally compact topological group admitting a weight function,
we establish necessary and sufficient criteria for all the weighted
continuous functions to be weakly almost periodic. Among other
results, we show that weak almost periodicity of all $\dot\omega$-weighted
continuous functions on a discrete semigroup S , can be very
different drom the phenomenon of regularity of multiplication in
the weighted algebra $\ell^1(S,\omega)$.

1. Introduction

Let S be a locally compact topological semigroup, ω a continuous
weight function on S and $C(S,\omega)$ the space of all continuous ω-weighted
functions on S . When S is a group (and $\omega = 1$), it is well known
that all functions in $C(S,1)$ are weakly almost periodic if and only if
S is compact - see for example [1] or [3]. In this paper we establish
necessary and sufficient criteria for all functions in $C(S,\omega)$ to be
weakly almost periodic for the case where S is a group or a cancellative
discrete semigroup.

When S is discrete, it is known that all functions in $C(S,1)$ are
weakly almost periodic if and only if the algebra $\ell^1(S)$ has regular
multiplication - see for example [8] or [3, Theorem 4.5.5(ii)]. One may
ask whether a similar result holds for the corresponding weighted algebras.
We show that the situation for weighted algebras can be very different

Among other results, we also prove that, if S is a group, the

weighted group algebra $L^1(G,\omega)$ contains a weakly compact element if and only if G is compact. This improves earlier results of Ghahramani ([5] and [6]).

In section 2 we introduce our notation and prove some elementary properties of a weight function. Our main results are given in section 3.

2. Notation and some elementary properties of weight functions.

Throughout this paper, let S denote a *topological semigroup*. A positive function ω defined on S is called a *weight function* if

$$\omega(xy) \leq \omega(x)\,\omega(y) \quad \text{for all} \quad x,y \in S.$$

It is interesting to note that measurable weight functions on locally compact groups are bounded away from both zero and infinity on compact sets. This is what our next result says.

PROPOSITION 2.1. *Let K be a compact subset of a locally compact group G and ω a measurable weight function on G. Then there exists $a,b \in \mathbb{R}$ such that*

$$0 < a \leq \omega(x) \leq b \quad \text{for all} \quad x \in K.$$

Proof. First we establish the existence of b. To this end, let λ be a Haar measure on G and set $U_n := \{x \in G : \omega(x) < n\}$. Clearly $\bigcup_{n=1}^{\infty} U_n = G$ and each U_n is measurable. Hence $\lambda(U_m) > 0$ for some $n = m$. The interior of U_m^2 is non-empty - see for example ([3], pp. 17 and 18). Let $z \in \text{int}(U_m^2)$ and set $V := \text{int}(z^{-1}U_m^2)$, and note that V is an open neighbourhood of the identity. By the compactness of K we can find a finite number of points $y_1,\dots,y_t$ in $K \cup K^{-1}$ such that

$$K \cup K^{-1} \subset Vy_1 \cup \dots \cup Vy_t.$$

Let $b := m^2\omega(z^{-1}).\max\{\omega(y_i) : 1 \leq i \leq t\}$. Then if $x \in K \cup K^{-1}$ we have $x = vy_i$ for some $v \in V$ and $i \in \{1,\dots,t\}$, and so

$$\omega(x) = \omega(vy_i) \leq \omega(v)\,\omega(y_i) \leq m^2\omega(z^{-1})\omega(y_i) \leq b.$$

Thus $\omega(x) \leq b$, for all $x \in K \cup K^{-1}$.   (*)

Next let $a := \inf\{\omega(x) : x \in K\}$ and suppose $a = 0$. Then there exists a sequence $\{x_n\}$ in K such that $\omega(x_n) \to 0$. Since $1 \leqslant \omega(x_n) \omega(x_n^{-1})$, we must have $\omega(x_n^{-1}) \to \infty$, which contradicts (*). Hence we must have $a > 0$.

REMARK. Of course there are many measurable weight functions that are not continuous. For example on the additive group $\mathbb{R}$ with the usual topology one may consider the weight function ω given by

$$\omega(x) := \begin{cases} 1 & \text{if } x \in [0,1] \\ 2 & \text{elsewhere .} \end{cases}$$

PROPOSITION 2.2. *Let* ω *be a weight function on* S *such that* $\{x \in S : \omega(x) < \epsilon\}$ *is finite for some* $\epsilon > 0$. *Then* $\omega(x) \geqslant 1$, *for all* $x \in S$.

Proof. Suppose $\omega(x) < 1$ for some $x \in S$. Choose $p \in \mathbb{N}$ such that $\omega(x^p) < \epsilon$. Then

$$\omega(x^{p+n+1}) \leqslant \omega(x^{p+n})\omega(x) < \omega(x^{p+n}) \leqslant \omega(x^p) \omega(x)^n < \epsilon .$$

Thus $\{x^{p+n} : n \in \mathbb{N}\}$ is an infinite subset of $\{y \in S : \omega(y) < \epsilon\}$ — contrary to the hypothesis. Thus the result follows.

COROLLARY 2.3. *If* ω *is a measurable weight function on a compact topological group* G , *then* $\omega(x) \geqslant 1$ *for all* $x \in G$.

Proof. This follows trivially from Propositions 2.1 and 2.2.

2.4. Let $\mathbb{C}(S)$ be the set of all complex-valued continuous functions on S , $C(S)$ the space of all bounded functions in $\mathbb{C}(S)$ under the supremum norm $\| \|_S$ and ω a continuous weight function on S . We define the space of weighted continuous functions $C(S,\omega)$ by

$$C(S,\omega) := \{f \in \mathbb{C}(S) : f\omega \in C(S)\}$$

with the norm given by $\|f\|_\omega := \|f\omega\|_S$.

Let

$$WAP(S,\omega) := \{f \in C(S,\omega) : \{_x f : x \in S\} \text{ is a relatively weakly compact subset of } C(S,\omega)\}$$

and

$AP(S,\omega) := \{f \in C(S,\omega) : \{_xf : x \in S\}$ is a relatively norm
compact subset of $C(S,\omega)\}$,

where

$$_xf(y) := f(xy) \quad \text{for all} \quad x,y \in S .$$

When $\omega = 1$ we get the familiar spaces $WAP(S) = WAP(S,1)$,
$AP(S) = AP(S,1)$ and $C(S) = C(S,1)$.

PROPOSITION 2.5. *A function* f *belongs to* $WAP(S,\omega)$ *if and only if*
$\{_xf.\omega : x \in S\}$ *is a relatively weakly compact subset of* $C(S)$.

Proof. A standard argument (which we omit) shows that the map
$f \to f.\omega$ of $C(S,\omega)$ into $C(S)$ is weakly a topological homomorphism and
hence the result follows.

2.6. For a discrete semigroup S with weight function ω , the set
$\ell^1(S,\omega)$ of all complex-valued functions f on S such that $f.\omega \in \ell^1(S)$,
is a Banach algebra with convolution as multiplication and norm given by

$$\|f\| := \|f\omega\|_{\ell^1(S)} .$$

If ω is a measurable weight function on a locally compact topological
group G , we have the weighted convolution algebra $L^1(G,\omega)$ of
(equivalence classes of) all measurable complex-valued functions f on G
such that $f.\omega \in L^1(G)$ under the norm $\|f\| := \|f.\omega\|_{L^1(G)} .$

2.7. The reader can easily note that our definitions for the spaces
$WAP(S,\omega)$ and $AP(S,\omega)$ contain the requirement that the functions
$f \in C(S,\omega)$ involved be such that $_xf \in C(S,\omega)$ for all $x \in S$. This is
related to the finiteness of $\Omega(x)$ where

$$\Omega(x) := \sup \left\{ \frac{\omega(y)}{\omega(xy)} : y \in S \right\}$$

for all $x \in S$ - see for example Lemma 3.1.

3. The main results

LEMMA 3.1. *Let* ω *be a continuous weight function on* S . *Then*
$_xf \in C(S,\omega)$, *for all* $f \in C(S,\omega)$ *and* $x \in S$, *if and only if* $\Omega(x)$

is finite.

Proof. To establish the necessary condition, we note that $\omega^{-1} \in C(S,\omega)$ and so if $x \in S$ we have $_x\omega^{-1} \in C(S,\omega)$. Hence

$$\infty > \| _x\omega^{-1} \|_\omega = \sup \left\{ \frac{\omega(y)}{\omega(xy)} : y \in S \right\} = \Omega(x) .$$

Conversely, suppose $\Omega(x)$ is finite, for all $x \in S$, and let $f \in C(S,\omega)$. Thus, for all $x,y \in S$, we have that

$$\left| _xf(y) \ \omega(y) \right| = \left| f(xy) \ \omega(y) \right| = \left| f(xy) \ \omega(xy) \right| \frac{\omega(y)}{\omega(xy)} \leqslant \| f\|_\omega \ \Omega(x)$$

and so

$$\| _xf \|_\omega < \infty \quad \text{or} \quad _xf \in C(S,\omega) .$$

COROLLARY 3.2. *If* ω *is a continuous weight function on a locally compact topological group* G *, then* $_xf \in C(G,\omega)$ *for all* $f \in C(G,\omega)$ *and* $x \in G$ *.*

Proof. For all x and y in G, we have

$$\frac{\omega(y)}{\omega(xy)} = \frac{\omega(x^{-1}xy)}{\omega(xy)} \leqslant \frac{\omega(x^{-1}) \ \omega(xy)}{\omega(xy)} = \omega(x^{-1})$$

and so $\Omega(x) \leqslant \omega(x^{-1})$. By 3.1 our result follows.

The reader is asked to bear the preceding result in mind throughout the proof of our next theorem.

THEOREM 3.3. *Let* ω *be a continuous weight function on a locally compact topological group* G. *Then, the following are equivalent*

(i) $WAP(G,\omega) = C(G,\omega)$ *;*

(ii) *either* *(a)* G *is compact*

  or   *(b)* G *is a countable discrete group such that*

$$\lim_n \lim_m \frac{\omega(y_m)}{\omega(x_ny_m)} = 0 \ , \ \text{whenever} \ \{x_n\} \ \text{and} \ \{y_m\}$$

  are sequences of distinct points of G *.*

Proof. Suppose (i) holds and set $V_k := \{x \in G : \omega(x) \leqslant k\}$. We claim that either V_k is finite $(k \in I\!N)$ or G is compact. To establish our claim, suppose for some k (now fixed) V_k is infinite and

G is not compact. Then, by an argument similar but simpler to that we employed in ([4], Theorem 7.3), there exist sequences $\{x_n\} \subset V_k$ and $\{y_m\} \subset G$, and a positive function h in $C(G)$ such that

$$h(x_n y_m) = \begin{cases} 0 & \text{if} \quad n < m \\ 1 & \text{if} \quad n \geqslant m \ . \end{cases}$$

Then $f := h\omega^{-1}$ is a function in $C(G,\omega)$ and

$$(*) \quad x_n f(y_m) \ \omega(y_m) = f(x_n y_m) \ \omega(y_m) = \frac{h(x_n y_m) \ \omega(y_m)}{\omega(x_n y_m)} = \begin{cases} 0 & \text{if} \quad n < m \\ C(n,m) & \text{if} \quad n \geqslant m \ ; \end{cases}$$

where

$$C(n,m) := \frac{\omega(y_m)}{\omega(x_n y_m)} \geqslant \frac{\omega(y_m)}{\omega(x_n) \ \omega(y_m)} = \frac{1}{\omega(x_n)} \geqslant \frac{1}{k} \ .$$

So, by Grothendieck ([7], Theorem 6), equation (*) implies that $\{_x f . \omega : x \in G\}$ is not relatively weakly compact. Recalling Proposition 2.5 we have $f \in C(G,\omega) \setminus WAP(G,\omega)$, which contradicts (i). Hence each V_k is finite or G is compact.

Next suppose G is not compact and choose $p \in \mathbb{N}$ such that $\omega(1) < p$. Then V_p is a finite neighbourhood of the identity, so G is discrete. As $G = \bigcup_{k=1}^{\infty} V_k$ we also have G countable. Since each V_k is finite, if $\{z_n\}$ a sequence of distinct points of G , then

$$(1) \qquad\qquad\qquad \omega(z_n) \to \infty \quad \text{as} \quad n \to \infty$$

Now let $\{x_n\}$ and $\{y_n\}$ be any sequences of distinct points in G . The function $f := \omega^{-1}$ is in $C(G,\omega)$ and hence in $WAP(G,\omega)$. Let

$$d := \overline{\lim_n} \ \overline{\lim_m} \ f(x_n y_m) \ \omega(y_m) = \overline{\lim_n} \ \overline{\lim_m} \ \frac{\omega(y_m)}{\omega(x_n y_m)}$$

(where d may be infinite). From (1), we get

$$(2) \qquad \lim_m \lim_n {}_{x_n} f(y_m) \ \omega(y_m) = \lim_m \lim_n \frac{\omega(y_m)}{\omega(x_n y_m)} = \lim_m 0 = 0$$

Since $\{_x f . \omega : x \in G\}$ is relatively weakly compact (by (i) and

Proposition 2.5), Grothendieck ([7], Theorem 6) and (2) imply that $d = 0$

Consequently

$$\lim_n \lim_m \frac{\omega(y_m)}{\omega(x_n y_m)} = 0 \ .$$

Thus (i) implies (ii).

Now suppose (ii) (a) holds. Then

$$WAP(G) = C(G) = C(G,\omega)$$

and the map $f \to f . \omega$ of $C(G)$ into $C(G,\omega)$ is weakly continuous. Hence
$\{_x f . \omega : x \in G\}$ is a relatively weakly compact subset of $C(G,\omega)$, for all
$f \in C(G,\omega)$. Thus $WAP(G,\omega) = C(G,\omega)$, by Proposition 2.5.

Next suppose (ii) (b) holds. Let f be any non-negative function
in $C(G,\omega)$ and set $M := \| f \|_\omega$. Then if $\{x_n\}$ and $\{y_m\}$ are any
sequences, as specified in (ii) (b), we have

$$(3) \qquad 0 \leqslant {}_{x_n} f(y_m) \ \omega(y_m) = f(x_n y_m) \ \omega(x_n y_m) \ \frac{\omega(y_m)}{\omega(x_n y_m)} \leqslant M \ \frac{\omega(y_m)}{\omega(x_n y_m)}$$

and

$$(4) \qquad 0 < \frac{1}{\omega(x_n)} = \frac{\omega(x_n y_m)}{\omega(x_n) \omega(x_n y_m)} \leqslant \frac{\omega(x_n) \omega(y_m)}{\omega(x_n) \omega(x_n y_m)} = \frac{\omega(y_m)}{\omega(x_n y_m)}$$

Hence

$$\lim_n \lim_m {}_{x_n} f(y_m) \ \omega(y_m) = 0 \quad \text{by condition (ii) (b) and (3)}.$$

(4) and condition (ii) (b) imply that $\omega(z_n) \to 0$ as $n \to \infty$ for any

sequence of distinct points $\{z_n\}$. Consequently, this and (3) give

$$\lim_m \lim_n {}_{x_n} f(y_m) \ \omega(y_m) = 0 \ .$$

Hence $f \in WAP(G,\omega)$ by ([7], Theorem 6) and Proposition 2.5.

Thus (ii) implies (i), and our proof is complete.

A mild adjustment of our preceding proof yields the following result,

on noting that $\Omega(x)$ is finite when either condition holds.

PROPOSITION 3.4. *Let* S *be infinite, discrete and such that* $x^{-1}\{y\}$ *and* $\{y\}x^{-1}$ *are finite, for all* $x,y \in S$. *If* ω *is a weight function on* S , *then the following are equivalent :*

(i) $WAP(S,\omega) = C(S,\omega)$

(ii) S *is countable and*

$$\lim_n \lim_m \frac{\omega(y_m)}{\omega(x_n y_m)} = 0$$

for any sequences $\{x_n\}$ *and* $\{y_m\}$ *of distinct points of* S .

Proof. First we note that (i) implies that $_x f \in C(S,\omega)$ whenever $f \in C(S,\omega)$ and $x \in S$, by our definition of $WAP(S,\omega)$. Now suppose (ii) holds and, on the contrary, $\Omega(x) = \infty$, for some $x \in S$. Then by definition of $\Omega(x)$ (see 2.7), there exists a sequence $\{y_m\}$ of distinct points of S such that

$$\frac{\omega(y_m)}{\omega(xy_m)} \to \infty \quad \text{as} \quad m \to \infty .$$

Since $a^{-1}\{b\}$, $\{b\}a^{-1}$ are finite for all $a,b \in S$, we can find a sequence $\{a_n\} \subset S$ such that $a_n x \neq a_k x$ whenever $n \neq k$. Let $z_n := a_n x$ and note that $\{z_n\}$ is a sequence of distinct points and so

$$0 = \lim_n \lim_m \frac{\omega(y_m)}{\omega(z_n y_m)} \geq \lim_n \lim_m \frac{\omega(y_m)}{\omega(a_n)\,\omega(xy_m)} = \infty$$

- which is impossible! Hence $\Omega(x) < \infty$ and so $_x f \in C(S,\omega)$, for all $f \in C(S,\omega)$ and $x \in S$. The rest of the proof can now be modelled on that of Theorem 3.3 .

The analogue of Theorem 3.3 for weighted almost periodic functions yields the following stronger result.

THEOREM 3.5. *Let* ω *be a continuous weight function on a locally compact topological group* G . *Then the following are equivalent:*

(i) $AP(G,\omega) = C(G,\omega)$

(ii) G *is compact.*

Proof. Suppose (i) holds. By Theorem 3.3 G is compact or discrete. If G is discrete then $V_k := \{x \in G : \omega(x) < k\}$ is finite, for all $k \in \mathbb{N}$. Since $\|f\|_\omega := \|f.\omega\|_G < \infty$, we must have $f \in C_o(G)$ for all $f \in C(G,\omega)$

By Proposition 2.2 we have $\omega \geqslant 1$ and so

$$\|f\|_\omega \geqslant \|f\|_G .$$

The map $f \to f$ of $C(G,\omega)$ into $C_o(G)$ is thus (norm) continuous. Hence $AP(G,\omega) \subseteq AP(G)$ and $AP(G)$ contains a non-zero element of $C_o(G)$. The latter implies that G is compact. Thus (i) implies (ii).

By a standard argument the reader can easily verify that (ii) implies (i).

We recall that multiplication in a Banach algebra is said to be *regular* if the two Arens products on its bidual coincide. If they do not coincide, we have *irregular multiplication*. For a weighted semigroup algebra $\ell^1(S,\omega)$, Craw and Young (see for example [2] or Theorem 1.2 in Chapter 6 of [3]) obtained the following partial characterization of regular multiplication.

PROPOSITION 3.6. *Let ω be a weight function on a cancellative semigroup S . Then the following are equivalent:*

(i) $\ell^1(S,\omega)$ has irregular multiplication

(ii) there exist sequences $\{x_n\}$ and $\{y_m\}$ of distinct points of S such that the double sequence $\left\{\dfrac{\omega(x_n y_m)}{\omega(x_n)\ \omega(y_m)}\right\}_{n,m}$ has a non-zero repeated limit.

The phenomena of weak almost periodicity of all functions in $C(S,\omega)$ and regularity of multiplication in $\ell^1(S,\omega)$ are independent, even for discrete cancellative S . This is the message of our next theorem.

THEOREM 3.7. *Let S be a free semigroup generated by an infinite countable set, endowed with the discrete topology. Then*

(i) S admits a weight function ω such that $WAP(S,\omega) = C(S,\omega)$

but $\ell^1(S,\omega)$ *has irregular multiplication*

(ii) S admits a weight function ρ *such that* $WAP(S,\rho) \neq C(S,\rho)$
but $\ell^1(S,\rho)$ *has regular multiplication.*

Proof. (i) Let $D := \{d_k : k \in I\!\!N\}$ be an enumeration of a free generating set for S . Every point x in S can be uniquely expressed in the form $x = d_{k_1} \ldots d_{k_p}$ and we define

$$\omega(x) := \exp(k_1 + \ldots + k_p) \ .$$

Clearly $\omega(xy) = \omega(x) \omega(y)$ $(x,y \in S)$ and $\omega(x_n) \to \infty$ for any sequence $\{x_n\}$ of distinct points of S . In particular, if $\{y_m\}$ is another sequence of distinct points of S , then

$$(5) \qquad \lim_n \lim_m \frac{\omega(y_m)}{\omega(x_n y_m)} = \lim_{n \to \infty} \frac{1}{\omega(x_n)} = 0$$

and

$$(6) \qquad \frac{\omega(x_n y_m)}{\omega(x_n)\ \omega(y_m)} = 1 \ .$$

From (5) and Proposition 3.4, we have that $WAP(S,\omega) = C(S,\omega)$; while from (6) and Proposition 3.6, we have that multiplication in $\ell^1(S,\omega)$ is irregular.

(ii) From ([2], page 353), S admits a weight function ρ such that $\ell^1(S,\rho)$ has regular multiplication, $\rho(x_n) \to \infty$ as $n \to \infty$ for any sequence of distinct points of S , and $\rho(xy) = \rho(x) + \rho(y) - 1$ for for all $x,y \in S$. So if $\{y_m\}$ is another sequence of distinct points of S , we get

$$(7) \qquad \lim_n \lim_m \frac{\rho(y_m)}{\rho(x_n y_m)} = \lim_n \lim_m \frac{\rho(y_m)}{\rho(x_n) + \rho(y_m) - 1} = 1$$

From (7) and Proposition 3.4, it follows that $WAP(S,\rho) \neq C(S,\rho)$.

Let ω be a continuous weight function on a locally compact topological group G . In [6], the author raised the following two questions: Is every right compact element of $L^1(G,\omega)$ a left compact

element? Does the existence of a non-xero right compact element imply

that all elements of $L^1(G,\omega)$ are right compact? Affirmative answers to
both questions follow trivially (or from standard results) as a
consequence of the following extension of the main result of [5] and [6].

THEOREM 3.8. *Let ω be a continuous weight function on a locally*
compact topological group G. Then the following are equivalent:

(i) *there exists a (non-zero) weakly compact right multiplier of*
$L^1(G,\omega)$;

(ii) G *is compact* .

Proof. Suppose (i) holds. Then ([6], Theorem 3.1) says that the
function

$$x \to f(x) := \int \frac{\omega(xy)}{\omega(x)} |g(y)| \ d\lambda(y)$$

defined on G , for some non-zero $g \in L^1(G,\omega)$, vanishes at infinity;
where λ denotes Haar measure. But

$$\frac{\omega(xy)}{\omega(x)} = \frac{\omega(xy)}{\omega(xy\ y^{-1})} \geqslant \frac{\omega(xy)}{\omega(xy)\ \omega(y^{-1})} = \omega(y^{-1})^{-1} > 0 \quad (y \in G)$$

and so

$$f(x) \geqslant \int \omega(y^{-1})^{-1} |g(y)| \ d\lambda(y) > 0 \quad (x \in G) \ .$$

Hence $f(x)$ vanishing at infinity can only happen when G is compact.
That (ii) implies (i) is evident (see for example [6]).

REMARKS. In fact the conclusion of Theorem 3.8 holds more generally
for the case where ω is any measurable weight function, on noting that
with the assistance of Proposition 2.1 one can similarly obtain the
result ([6], Theorem 3.1) for such ω .

For a comprehensive study of convolution measure algebras on
topological semigroups, we refer the interested reader to our book [3].

References

[1] R.B. Burckel, *Weakly almost periodic functions on semigroups.*
(Gordon and Breach, New York, 1970).

[2]   I.G. Craw and N.J. Young, Regularity of multiplication in weighted
          group and semigroup algebras.   *Quart. J. Math. Oxford Ser (2)*
          (1974), 351-358.

[3]   H.A.M. Dzinotyiweyi, *The analogue of the group algebra for topological
          semigroups*. (Pitman Advanced Publishing Program, Boston, London,
          Melbourne, 1984).

[4]   H.A.M. Dzinotyiweyi, Nonseparability of quotient spaces of function
          algebras on topological semigroups. *Trans. Amer. Math. Soc.*
          272 (1982), 223-235.

[5]   F. Ghahramani, Compact elements of weighted group algebras. *Pac. J.
          Math.* (to appear).

[6]   F. Ghahramani, Weighted group algebra as an ideal in its second dual
          space. *Proc. Amer. Math. Soc.* 90 (1984), 71-76.

[7]   A. Grothendieck, Critères de compacité dans les espaces fonctionnels
          généraux. *Amer. J. Math.* 75 (1952), 168-186.

[8]   N.J. Young, Semigroup algebras having regular multiplication. *Studia
          Math.* 47 (1973), 191-196.

Department of Mathematics
University of Zimbabwe
P.O. Box MP 167, Mount Pleasant
Harare, Zimbabwe.