SQUARING THE SQUARE

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1. Introduction. It is the object of this paper to describe in more detail than has hitherto been done the general methods by which a square may be dissected into smaller unequal non-overlapping squares. Examples of such dissections are given.

The problem of finding a method for constructing a "simple" perfect rectangle whose sides are in any given rational ratio remains unsolved. It is found however that the theory developed to deal with the case in which the ratio is 1 can also be applied to construct a family of simple perfect dissections for the particular case in which the ratio is 15/17. (A squared rectangle is "simple" if it contains no smaller squared rectangle of order >1).

2. Self-dual maps. It is remarked in the companion paper, [1], that most of the general methods for the construction of the perfect squares depend on the properties of self-dual maps. (For an exception, see [2], [3] and [4].) In the notation of that paper, if the edge W_j of M is equivalent under the symmetry of $Z(M)$ to W^*_j, then in the full flow with polar edge W_j we have

$$(1) \qquad \begin{aligned}(W_j.W_j) &= C - (W^*_j.W^*_j) \\ &= C - (W_j.W_j)\end{aligned}$$

by [1], equation (3).

Hence the squared rectangle whose c-net is the 1-section of M and the poles of whose p-net are the ends of W_j has its horizontal side equal to its vertical side; it is a squared square ([1], Sec. (5)).

It is not of course perfect, because of the symmetry of $Z(M)$. We shall see however that in some cases it can be modified so as to give a perfect square.

It was shown in [1] that such "self-dual edges" occur in all planar reflexes but not (in general) in central reflexes. Besides the reflexes there is one other kind of self-dual map in which a self-dual edge can occur. This is the case in which the map M is transformed into its dual map by a rotation through an angle $\pi/2$ about an axis X through the centre of the sphere. Evidently the points at which X cuts the sphere cannot be vertices of M or M^*. Neither can they be interior points of edges of $Z(M)$. Each must therefore be one of the points w_j. If they are taken to be w_1 and w_2, then W_1 and W_2 are evidently self-dual edges of M. We shall call the axis X a *dualizing tetrad axis*.

It was shown in [1] that by taking an edge of the girdle of a planar reflex as polar edge we obtain a flow representing a diagonally symmetric squared square. In a similar way it can be shown that if the polar edge of a flow is

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transformed into its dual edge by a rotation through $\pi/2$ about a dualizing tetrad axis then the corresponding squared square has the symmetry of the swastika. It is thus far from perfect.

3. Externally equivalent networks. We are left with the problem of turning a symmetric squared square into an unsymmetric one. We shall discuss one solution of the problem in the present section.

In this paper, if P_i and P_j are the two vertices of an electrical network, we shall denote the transpedance $(P_iP_j.P_iP_j)$, or $(ij.ij)$, by $V(P_iP_j)$ or V_{ij}. We write $V_{rr} = 0$.

We shall prove two general propositions about transpedances. They are as follows.

$$(2)\qquad \begin{aligned} &\text{(i)} \quad 2(rs.rt) = V_{rs} + V_{rt} - V_{st};\\ &\text{(ii)} \quad 2(rs.tu) = V_{ru} + V_{st} - V_{rt} - V_{su}. \end{aligned}$$

We prove (i) as follows.

$$(rs.rt) = (rs.rs) + (rs.rt) = (rt.rt) + (ts.rt),$$

by formulae (C) and (E) of the companion paper. Hence, by these same formulae,

$$\begin{aligned} 2(rs.rt) &= (rs.rs) + (rt.rt) - ((sr.st) + (rt.st))\\ &= V_{rs} + V_{rt} - V_{st}. \end{aligned}$$

We may now prove (ii) as follows.

$$\begin{aligned} 2(rs.tu) &= 2(rs.ru) - 2(rs.rt)\\ &= (V_{rs} + V_{ru} - V_{su}) - (V_{rs} + V_{rt} - V_{st})\\ &= V_{ru} + V_{st} - V_{rt} - V_{su}. \end{aligned}$$

Suppose that N and N' are two connected electrical networks such that $C(N) = C(N')$. Suppose further that there is a set $A = \{A_1, A_2, \ldots, A_n\}$ of vertices of N and a set $\{A'_1, A'_2, \ldots, A'_n\}$ of vertices of N' such that

$$(3)\qquad V(A_iA_j) = V(A'_iA'_j)$$

for any members A_i and A_j of A.

Then by (2) we have

$$(4)\qquad (A_iA_j.A_kA_l) = (A'_iA'_j.A'_kA'_l),$$

where A_i, A_j, A_k and A_l are arbitrary members of A.

By a flow F in N with *polar set* A we mean a flow of current in N which satisfies Kirchhoff's Laws everywhere except at some members of A. In such a flow F let the sum of the currents flowing from A_j in the edges of N incident with A_j be denoted by I_j. The full flow in N with poles A_j and A_n will be denoted by F_j $(j = 1, 2, \ldots, n-1)$. Because of the linearity of the Kirchhoff equations we may write

$$F = \sum_{j=1}^{n-1}\left(\frac{I_j}{C(N)}.F_j\right).$$

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We mean by this that any current or potential difference x in F is equal to the sum

$$\sum_{j=1}^{n-1}\left(\frac{I_j}{C(N)}\cdot x_j\right),$$

where x_j is the corresponding current or potential difference in F_j.

We denote by F'_j the full flow in N' with poles A'_j and A'_n ($j = 1, 2, \ldots, n-1$). We write

$$F' = \sum_{j=1}^{n-1}\left(\frac{I_j}{C(N)}\cdot F'_j\right)$$

Then, since $C(N') = C(N)$ the sum of the currents of F' flowing from A'_j in the edges of N incident with A'_j is I_j. Also by (4) the potential drop from A'_i to A'_j in F' is equal to the potential drop from A_i to A_j in F.

We call F' the flow in N' *corresponding* to F.

Now let us suppose that N is part of a connected network L and that each vertex of N which is incident with an edge of L not belonging to N belongs to A. We define an *external transpedance* of L as a transpedance $(rs.tu)$ with the four corresponding vertices P_r, P_s, P_t, P_u so chosen that those in N belong to A.

We consider the effect of replacing N by N' in L. More precisely we suppose that N' has no point in common with L, we identify A_i with A'_i for each i, and then we suppress all the edges and vertices of N not belonging to A. We denote the resulting graph by L'. Transpedances of L' will be distinguished by primes.

THEOREM. *The complexity and external transpedances of L are invariant under the operation of replacing N by N'.*

Consider a full flow Φ in L with positive pole P_r and negative pole P_s, such that if either of these vertices is in N it belongs to A. The currents of Φ in N define a flow F in N with polar set A. Let F' be the corresponding flow in N'. Let us replace N by N' and F by F'. Then by the definition and properties of F' there results a flow Φ' in L' which satisfies Kirchhoff's Laws everywhere except at P_r and P_s. The potential difference between two vertices P_i and P_j common to L and L' is unaltered by this process.[1] Also the total current flowing from P_r is the same in L' as in L. We conclude that Φ' is the full flow in L', with positive pole P_r and negative pole P_s, multiplied by $C(L)/C(L')$. Hence

$$\frac{(rs.tu)}{C(L)} = \frac{(rs.tu)'}{C(L')} \tag{5}$$

for each external transpedance $(rs.tu)$ of L.

[1]If P_j denotes A_i in L it is taken to denote A'_i in L'. In this case we still say that P_j is common to L and L'.

Let q denote the number of edges of L not in N. If $q = 0$, then $C(L) = C(L')$ since $L = N$ and $L' = N'$. We assume as an inductive hypothesis that $C(L) = C(L')$ whenever q is less than some positive integer m. We go on to consider the case $q = m$.

In this case let E be an edge of L not in N. Let its ends be P_x and P_y. Let L_1 and L'_1 be the networks obtained from L and L' respectively by omitting the edge E. Let L_2 and L'_2 be the networks obtained from L_1 and L'_1 respectively by identifying P_x and P_y.

Suppose P_x and P_y are not both in N. Clearly L_2 and L'_2 are connected. Also L'_2 is obtained from L_2 by the operation of replacing N by N'. Hence $C(L'_2) = C(L_2)$ by the inductive hypothesis. Thus $(xy.xy)' = C(L'_2) = C(L_2) = (xy.xy)$ by the definition of complexity and transpedances in terms of determinants given in the companion paper. Hence $C(L') = C(L)$, by (5).

Next suppose that P_x and P_y are both in N. Then L_1 and L'_1 are connected and L'_1 is obtained from L_1 by replacing N by N'. Hence, by the inductive hypothesis, the transpedance $(P_xP_y.P_xP_y)$ has the same value in L'_1 as in L_1. But by the definitions this transpedance has the same value in L_1 as in L, and the same value in L'_1 as in L'. Hence $(xy.xy)' = (xy.xy)$ and $C(L') = C(L)$ as before.

It follows by induction that $C(L') = C(L)$ for all values of q. The theorem now follows from (5).

As examples of externally equivalent networks we have various pairs of equivalent p-nets ([5], Sec. 7). We return to these later. These are cases in which $n = 2$, n being the number of members of A.

Another case is obtained by taking N and N' to be a rotor and its mirror image. By saying that N is a rotor we mean that it has n-fold rotational symmetry in which A is a set of corresponding points. In this case the external equivalence follows immediately from the symmetry. Rotors are discussed in Sec. (7) of [5].

Before explaining a third case we note another general property of transpedances. Let N be an electrical network. Let E be an edge of N, of conductance 1, with ends P_x and P_y. Let N' be the network derived from N by suppressing E. We suppose that N' is connected, so that $C(N') > 0$. We distinguish transpedances of N' by primes. Then for any transpedance $(rs.tu)'$ of N' we have

(6) $$(rs.tu)' = (rs.tu) - ((rs.tu)(xy.xy) - (rs.xy)(tu.xy))/C,$$

where C is the complexity of N.

To prove this, let F_{xy} be the full flow in N with positive pole P_x and negative pole P_y. We define F_{rs} analogously. We consider the flow

$$F = \left(1 - \frac{(xy.xy)}{C}\right) F_{rs} + \frac{(rs.xy)}{C} F_{xy}.$$

It is easily verified that by suppressing E we obtain from F a flow F' in N' which satisfies Kirchhoff's Laws everywhere except at P_r and P_s. Hence

$F = \lambda F'_{rs}$, where F'_{rs} is the full flow in N' with positive pole P_r and negative pole P_s. By considering the total current flowing from P_r in F we find that $\lambda C(N') = C - (xy.xy)$. But it follows from the definition of complexity and transpedances as determinants that $C - (xy.xy) = C(N')$. Hence $\lambda = 1$ and therefore $F = F'_{rs}$. Formula (6) follows.

The third case of externally equivalent networks, in which $n = 3$, arises as follows. Consider two consecutive edges of the girdle of a planar reflex M. Let their common end be A, and the other two ends B and C. Then by the results of [1],

(7) $$V(AB) = V(AC) = \tfrac{1}{2}C(M).$$

Let us now suppress AB and AC, and distinguish transpedances referring to the new network, M_1 say, by primes. We have

$$V'(AB) = V(AB) - (V(AB).V(AC) - (AB.AC)^2)/C(M),$$

by (6). We have used the fact, evident from the definitions, that a transpedance $(xy.rs)$ is independent of c_{xy} and c_{rs}. Hence by (7) and (6),

$$V'(AB) = V(AC) - (V(AC).V(AB) - (AC.AB)^2)/C(M) = V'(AC).$$

We take N and N' to have each the same structure as M_1. But whereas in N, A_1, A_2 and A_3 are taken to be A, B, and C respectively, in N', A'_1, A'_2 and A'_3 are taken to be A, C, and B respectively.

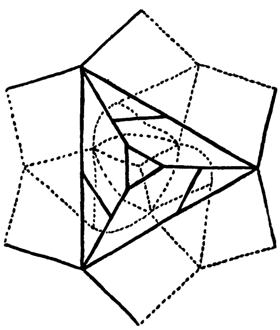

FIGURE 1

Our first method for the construction of perfect squares is that briefly described in the companion paper. We construct a planar reflex in which the part of the network on one side of the girdle is one of two externally equivalent networks, the vertices of the polar set but no others being on the girdle. If we replace this part by the other network of the pair, the squared rectangles obtained by taking edges of the girdle as polar edges will still be squared squares, by the Theorem, but in general there will be no evident reason why these squared squares should not be perfect. Figure 1 shows a planar reflex in which the part of the network on one side of the girdle is a rotor. The reflex is represented as projected in the equatorial plane. The part of the network in one hemisphere is represented by full lines, that in the other by broken lines.

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The particular case in which the girdle has four edges only is easily seen to be the case of Figure V of [5]. Any squared square whose polar edge belongs to the girdle has two elements which are bisected by one of the diagonals of the whole square, and the remainder of the squared square consists of two congruent rectangles dissected into squares. The externally equivalent networks are here equivalent p-nets of squared rectangles. In order that the squared square may be perfect it is necessary that the two p-nets shall be perfect. Further, no element of one may be equal to any element of the other: i.e., in the terminology of [5], the p-nets must be totally different. Unfortunately only very clumsy methods of constructing totally different p-nets are known, though many simple cases have been discovered empirically. Most of the perfect squares of this type can be reduced in order by making a corner of one rectangle overlap an element in the other. One that cannot be so reduced is described in [6]. Its reduced side is 1015 and its full side is 1015^2. Another perfect square of this class is described by R. Sprague in [2]. Its order can be reduced by overlapping.

4. Use of planar reflexes. A general method for constructing a pair of totally different perfect rectangles is to take a rotor of four-fold symmetry and polar set $\{A_1, A_2, A_3, A_4\}$ and another of three-fold symmetry and polar set $\{A_1, A_2, A_3\}$, the two having no other vertices in common. We take two of these four vertices as poles for one p-net. For the other we take the same poles and replace each rotor by its mirror image.

In general there is no evident reason why the two equivalent squared rectangles thus obtained should not be perfect and totally different. I know of only one case in which the necessary calculations have been performed. In this case the rectangles are totally different and have reduced sides 115407650 and 160618071. In the notation of C. J. Bouwkamp[2] they are:
(48217845, 55448257, 56951969), (30183899, 18033946), (10803534, 15373190, 27767821, 1503712), (58455681), (12149953, 12117871, 4569656), (7548215, 12394631), (3712938, 11106732, 4846416), (16831932, 13489890, 8331174, 3680856), (7393794), (12817296, 32191572), (5587698, 2743476), (2844222, 11842800, 6556980), (3342042, 7181904, 2536962, 428982), (2107980, 6752922), (4644942), (20173974), (19374276), (16334112, 2245656), (14088456) and:
(64077519, 52804978, 43735574), (9069404, 34666170), (24019929, 18016469, 19837984), (51330131, 12747388), (6003460, 10191494, 1821515), (8369979, 7530738, 5758782), (38582743, 4188034), (1771956, 7679724, 11964522, 19008750), (9302694), (14286054, 8463453), (3394926, 4284798), (5822601, 10317432, 5021040), (4131168, 4062330, 8055822), (68838, 3993492), (5296392, 3924654), (20108655), (17997156, 1011594), (16985562), (15613824).

We can construct a perfect square from these two rectangles as explained above. Or we may modify the method by making the rectangles overlap in

[2]See [7]. The notation is also explained in [1], Sec. (5).

a suitable corner element. Thus, taking the corner element to be that of side 48217845, we obtain a perfect square of the 85th order and of reduced side 227807876.

Another perfect square of this "overlapping" type is of the 29th order and has reduced side 1424. It was discovered empirically. It is:
(193, 285, 186, 273, 462), (99, 87), (101, 92,) (360), (9, 119, 348), (110), (229), (133, 329), (51, 82), (791, 158, 13), (64), (33,49), (81, 16), (65), (633).

An extreme case of this overlapping method arises when the corner element to be overlapped has a side in common with the rectangle to which it belongs. Then when we overlap the two rectangles and suppress the overlapped element we get a square of the type shown in [5], Fig. 9. Perfect squares of this type seem fairly common. There is the one of [5], Fig. 9, which appears to be the perfect square of smallest known order (26); there is another of reduced side 1015, of full side $(1015)^2$, and of the 28th order, given by:
(280, 372, 363), (188, 92), (93, 270), (119, 261, 84), (177), (165, 23), (142), (163, 120, 167, 183, 382), (43, 30, 47), (13, 17), (219), (215), (199),
and a third, also of the 28th order, with reduced side 1073 and full side $(1073)^2$ given by:
(244, 153, 248, 169, 259), (91, 62), (79, 90), (29, 33), (364), (360), (349), (465, 252, 156, 89, 111), (67, 22), (133), (135, 88), (221), (213, 39), (174).

The second of these squares is given in [6].

It is interesting to note that there are two quite different perfect squares of the 28th order having reduced side 1015 and full side $(1015)^2$. Their first publication seems to have been in a note by A. H. Stone, ([6]). One is completely described above; the other has been completely described by C. J. Bouwkamp ([7], p. 75).

We go on to consider the case in which the planar reflex has just 6 edges in the girdle. In general the corresponding symmetrical squared square will have just three elements which are bisected by the symmetry diagonal. (See Fig. 6 of [5].) Evidently perfect squares derived from such a planar reflex by replacing the part of the network on one side of the girdle by an externally equivalent network contain smaller squared rectangles. That is they are not simple. Such a smaller squared rectangle is formed by the middle diagonal element and the elements above the bisecting diagonal.

The structure, but not the elements, of one such perfect square is given in [5] ((8.2), second paragraph). The elements have been published since by C. J. Bouwkamp.[3] Its reduced side is 1813.

Another example, due to R. L. Brooks, is as follows:
(2378, 1163, 1098), (65, 1033), (737, 491), (249, 242), (7, 235), (478, 259), (256), (324, 944), (219, 296), (1030, 829, 519, 697), (620), (341, 178), (163, 712, 1564), (201, 440, 157, 31), (126, 409), (283), (1231), (992, 140), (852).
Its reduced side is 4639. Both these squares are of the 39th order.

[3][7], p. 75.

In [5], Sec. 9, a certain infinite sequence of perfect squares is discussed. Its members are all of the above type.

Consider the case where the reflex has eight edges in its girdle. In general the symmetry diagonal of a corresponding squared square bisects four of the elements (Fig. 2). In this case we may hope to derive simple perfect squares.

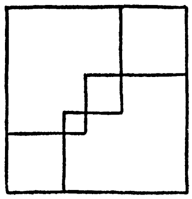

FIGURE 2

One simple perfect square of this type (due to C. A. B. Smith and W. T. Tutte) is known. It is of the 52nd order. In Bouwkamp's notation it is as follows: (51573, 41645, 88851), (9928, 31717), (41320, 20181), (14795, 5386), (3778, 12450, 15489), (9164), (9411, 3039), (245, 8919), (15040), (18528, 14970, 26382, 47499), (32624, 8696), (6174, 12156), (192, 5982), (23928), (3558, 11412), (18138, 10056, 12030), (4176, 10721, 22897), (8082, 1974), (11635, 6545) (56552, 20693, 5527), (5090, 12176), (15166, 7086), (1780, 45719), (43939), (35859).

The reduced side of this perfect square is 182069.

The externally equivalent networks used for the perfect squares of reduced sides 1813, 4639 and 182069 are rotors.

5. Overlaps. In our discussion above of the case of a girdle with four edges we used the device of making two equivalent squared rectangles overlap in a corner element. A similar but less trivial process is applicable when the number of edges in the girdle exceeds four.

Suppose we have a c-net N, and suppose P_i, P_j, P_k, P_l are four vertices of N, the pairs (P_i,P_j) and (P_k,P_l) being joined by edges. Suppose further that

$$\text{(i)} \quad (ij.kl) = 0$$

and

$$\text{(ii)} \quad V_{kl} = \tfrac{1}{2}C(N).$$

Equation (ii) states that the edge P_kP_l, taken as polar edge corresponds to a squared square.

We obtain a new "electrical" network N_0 from N by changing the conductance of the edge joining P_k and P_l from 1 to -1. Then $C(N_0) = C(N) - 2V_{kl} = 0$ by the definitions, and the properties of determinants. We may therefore no longer assert that the Kirchhoff equations for a flow in N_0 have a unique solution.

Consider the full flow in N with positive pole P_i and negative pole P_j. In virtue of (i) this remains a flow, satisfying Kirchhoff's Laws except at P_i and P_j, in N_0. We denote it by F. Next consider the full flow in N with positive

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pole P_k and negative pole P_l. This will give rise to a flow F_0 in N_0 when we reverse the current in the edge whose conductance is changed to -1. We note that by (ii), F_0 must satisfy Kirchhoff's Laws *everywhere*. It has in fact no poles.

It follows that for any λ, $F + \lambda F_0$ is a flow in N_0 with positive pole P_i and negative pole P_j in which the current entering at P_i and the potential difference between P_i and P_j are independent of λ, being $C(N)$ and V_{ij} (referring to N) respectively (by (i)). An example of such an N_0 is given in Fig. 3. It corresponds to the generalized squared rectangle of Fig. 4. It will be noticed that the edge of N_0 of conductance -1 corresponds to a square region Y in which two elements corresponding to edges of conductance 1 overlap. By dissecting perfectly the square X in different ways and arranging in each case that the region of overlap coincides with a corner element of this dissection (which, together with the "element" corresponding to the conductance -1, is then suppressed) it is possible to obtain a number of perfect dissections of the rectangle whose sides are in the ratio 15:17. By suitable choices of the perfect square involved we can even obtain simple perfect dissections of this rectangle.

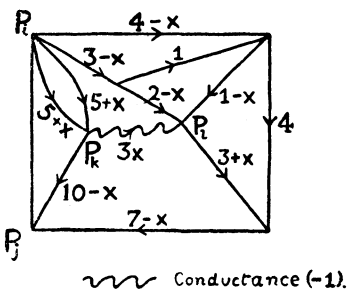

FIGURE 3

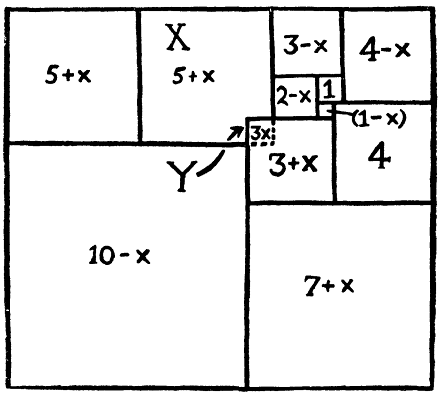

FIGURE 4

Now suppose that in N_0, P_k is joined to a vertex P_m other than P_l by an edge of conductance 1. Then in the flow $F + \lambda F_0$ the potential difference between P_l and P_m is given (in terms of N) by

$$(8) \qquad (ij.lm) + \lambda(kl.lm).$$

Now, provided that P_k can be joined to P_l by a simple arc in N passing through P_m, we have $(kl.lm) \neq 0$ (by the results of [1] Sec.(5)). Hence we can choose λ so that the potentials of P_l and P_m become equal. The resulting flow continues to obey Kirchhoff's Laws (for poles P_i, P_j) when P_l and P_m are identified. Further, after this identification is made we can suppress two edges joining P_k and $P_l = P_m$, one of conductance 1 and the other of conductance -1 without affecting the matrix $\{c_{rs}\}$. If the resulting electrical network

N_1, in which each conductance is now $+ 1$, is the p-net (with poles P_i,P_j) of a squared rectangle, the sides of that rectangle must be in the same ratio as those of the rectangle corresponding to the poles P_i,P_j in the p-net N. We describe this operation as *overlapping* the edges P_kP_l and P_lP_m of N. In order that the rectangle derived from N_1 shall not be "trivially imperfect"[4] it is necessary (in general) that P_k shall be incident with at least five edges in N, and that N shall contain no quadrilateral constituted by P_kP_l, P_lP_m and two other edges.[5] Usually we are given that N is planar; we can then arrange that N_1 is planar by taking P_lP_k and P_lP_m to be consecutive in the cyclic sequence at P_l of the edges incident with P_l.

The dissection of the rectangle with sides in the ratio 15:17 by the use of the network of Fig. 3 can be discussed in terms of this overlapping operation. We first replace one of the edges P_iP_k by a p-net of a perfect square and then we overlap the edge P_kP_l with one of the edges of this p-net.

Let us now return to the equations (i) and (ii). If in addition the network N satisfies (iii), $V_{ij} = \frac{1}{2}C(N)$ we say that the edges P_iP_j and P_kP_l are *squarely conjugate.* Then P_iP_j, as well as P_kP_l corresponds to a squared square. Hence the edge P_iP_j in any planar network N_1 formed from N by overlapping P_kP_l also corresponds to a squared square.

For an example we refer to Theorem VIII of the companion paper. We see that in the girdle of any planar reflex any two members of the same class S_1 or S_2 are squarely conjugate. By the Theorem of the present paper this property is unaffected if the reflex is modified by replacing the part of the network on one side of the girdle by an externally equivalent network.

It is easily seen, by considering the flows F and F_0 that any two members of a set of mutually squarely conjugate edges remain squarely conjugate when any other member of the set is overlapped. Also as a general result any zero current in the p-net of a squared square which corresponds to an edge squarely conjugate to the polar edge in the corresponding c-net can be eliminated by overlapping the corresponding edge.

By applying this method to the members of S_1 in the girdle of a suitable modified planar reflex (with $n > 2$) it is possible to obtain a simple perfect square having no crosses.[6]

This method is essentially the same as that given in Sec. (8.4) of [5], but the present account is more general. A recent criticism by C. J. Bouwkamp[7] of the method as described in [5] has now been withdrawn.[8]

[4]If a p-net has a part, not containing a pole, joined to the rest by only two edges, or if it has a pair of vertices joined by two (or more) edges, these two edges will clearly have numerically equal currents. If these two currents are non-zero we say that the p-net and its corresponding squared rectangle are "trivially imperfect".

[5]We ignore the exceptional case in which P_l and the fourth member P_q of a quadrilateral $P_kP_lP_mP_q$ have equal potentials in $F + \lambda F_0$.

[6]See [1], Sec. (5). It is easily seen that a zero current in a p-net corresponds to a cross in the corresponding squared rectangle, unless this zero current belongs to a part of the p-net joined to the rest at only one vertex P and containing no pole which is not P.

[7][7], p. 75. [8][8] and [9].

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Two examples of simple crossless perfect squares obtained by this sort of overlapping have been published in [8] and [9]. We give below a perfect square containing one cross, obtained by eliminating one of the zero currents in a modified planar reflex. This is of particular interest in that the externally equivalent networks involved are not rotors. They are derived from the planar reflex of Fig. 2 of [1] according to the method described in Sec. (2). Fig. 2 of [1] is lettered in accordance with that description. The perfect square, due to R. L. Brooks, is of the 38th order and has reduced side 4920. It is:

(1348, 1092, 893, 1587), (199, 694), (256, 420, 615), (1440, 164), (584), (120, 984, 1177), (281, 454), (108, 173), (692), (627), (217, 527, 240), (47, 1130), (2132, 534, 310), (287), (224, 900), (758), (104, 1026), (82, 922), (840).

6. Use of a dualizing tetrad axis. The next method to be described utilizes a planar reflex which has a dualizing tetrad axis perpendicular to the dualizing plane. (It can be shown that such a map is also a central reflex.) A reflex of this type, from which perfect squares have been derived, is shown in Fig. 5. It will suffice to describe the application of the method to this reflex.

If X and Y are any two vertices of the reflex, we denote by $F(XY)$ the full flow with positive pole X and negative pole Y.

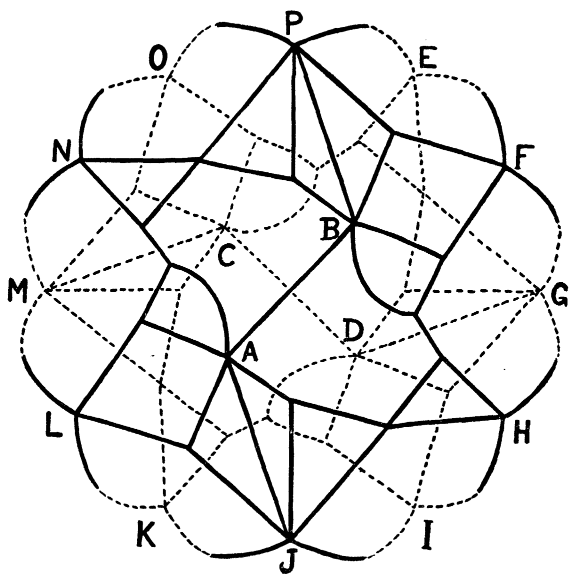

FIGURE 5

As we saw in Sec. (1) the flows $F(AB)$ and $F(CD)$ correspond to squared squares (with the symmetry of the swastika). Also $(AB.CD) = 0$ by Theorem IV of the companion paper. Thus the edges AB and CD are squarely conjugate.

Choose two diametrically opposite edges such as GH and MN in the girdle.

We proceed to show that AB and CD remain squarely conjugate when the edges GH and MN are suppressed and the two vertices G and H identified. To do this we consider the linear combination of flows $F(AB) + bF(GH) + cF(MN)$. We try to arrange that

$$(9) \qquad \begin{aligned} (AB.GH) + b(GH.GH) + c(MN.GH) &= 0, \\ (AB.MN) + b(GH.MN) + c(MN.MN) &= cC \end{aligned}$$

where C is the complexity of the reflex. Since $(MN.GH) = 0$ (Theorem VIII of the companion paper) and since $(GH.GH)$ and $C - (MN.MN)$ are non-zero, ([1], Sec. (5)), we can do this by setting $b = -(AB.GH)/(GH.GH)$ and $c = (AB.MN)/(C - (MN.MN))$, that is $b = c = -(AB.GH)/\frac{1}{2}C$ since the map is a planar reflex and $(AB.GH) = -(AB.MN)$ by symmetry.

It is easily seen that when (9) holds, and we perform the operation of suppressing the edges GH and MN and identifying the vertices G,H we obtain from $F(AB) + bF(GH) + cF(MN)$ a flow in the new network which satisfies Kirchhoff's Laws everywhere except at the vertices A and B. The total current flowing from A in this flow is C. The currents in AB and CD in this flow are the same as in $F(AB)$ since $b = c$ and (by symmetry) $(AB.GH) = -(AB.MN)$ and $(CD.GH) = -(CD.MN)$. Hence AB and CD remain squarely conjugate. (To prove that CD still corresponds to a squared square we use a similar argument with $F(CD)$ replacing $F(AB)$.)

We can apply a similar operation to any other pair of diametrically opposite edges of the girdle. Indeed we can operate simultaneously on several such pairs provided that all the edges of the girdle concerned belong to the same class S_1 or S_2. Then the operations do not interfere with one another (a consequence of Theorem VIII of the companion paper). In an actual computation the edges EF, GH, IJ, KL, MN, and OP were suppressed, and the pairs (E,F), (G,H) and (I,J) were each identified. The flow AB was then found to represent a simple perfect square of the 70th order and of reduced side 384948. This square is:

(74378, 83540, 71817, 71781, 83432), (60130, 11651), (11723, 60094), (65216, 9162), (95083), (56054, 48371), (41113, 48772, 78710), (72887, 48383), (33454, 7659), (32106, 62977), (26493, 29938), (24504, 23879), (9708, 23746), (23048, 3445), (19603, 28332, 65393, 30871), (19549, 4330), (97391), (14038), (1181, 21911, 14692, 13994, 19928, 8729), (20730), (37061), (34522, 59326), (8060, 5934), (7219, 7473), (25862), (49606, 254), (111, 7949), (7838), (15787), (44887, 108934, 24804), (84130), (30446, 19160), (75076, 22315), (64047), (52761).

As AB and CD are squarely conjugate, this square contains a cross corresponding to the edge CD. This cross can be eliminated by overlapping CD as explained in the preceding section. One method of doing this converts the above perfect square into one of the 69th order having reduced side 7919535.

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It is:
(1543151, 1726140, 1477594, 1469823, 1702827), (1236819, 233004), (248546, 1229048), (1360162, 182989), (1935831), (1177173, 980502), (887428, 985625, 1573316), (1507596, 1029739), (789231, 98197), (614300, 1294531), (496131, 587691), (477857, 551882), (404571, 91560), (313011, 560159, 1367466, 653231), (222716, 566515), (1985453), (430799, 121083), (470434, 247148), (343799), (807307), (396716, 34083), (714235, 1233527), (362633, 224917, 243260, 113587), (584021), (137716, 87201), (68858, 174402), (50515, 105544), (947580), (279946), (961572, 2272111, 519292), (1752819), (598613, 348967), (1523173, 462280), (1310539), (1060893).

This square is simple and has no cross.

Postscript

Mr. T. H. Willcocks, of 24 Pembroke Rd., Clifton, Bristol, England, has written to say that he published in the *Fairy Chess Review* (August, 1948), a dissection of a square into 24 unequal squares, the reduced side being 175. It is hoped that a note by him on this and other dissections will appear in this Journal shortly.

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