SUMS OF RECIPROCAL POWERS OF TERMS IN ARITHMETIC SEQUENCE

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A note* by N. Kimura gives the sums

$$\sum_{\mu=0}^{\infty} \frac{1}{(a+\mu)^p (a+\mu+1)^p} = S_p, \quad \text{defined for } -a \neq 0, 1, 2, \ldots,$$

explicitly linearly in terms of the sums $\sum_{\mu=0}^{\infty} \frac{1}{(\mu+a)^{2p}}$ for positive integers p.

We note here that by a similar simple method, the Bernoulli numbers and Euler numbers may be related similarly to these sums.

First, from the expansions

$$\sum_{\mu=0}^{\infty} t^{\mu} \{x^{\mu} + (1-x)^{\mu}\} = \frac{1}{1-tx} + \frac{1}{1-t(1-x)}$$

$$= (2-t) \sum_{\mu=0}^{\infty} t^{\mu} \{1 - tx(1-x)\}^{\mu}$$

$$= (2-t) \sum_{\nu=0}^{\infty} t^{\nu} \sum_{\mu \geqq \nu/2}^{\nu} (-1)^{\nu-\mu} \binom{\mu}{\nu-\mu} x^{\nu-\mu} (1-x)^{\nu-\mu}$$

we have, for $x = -a$,

$$\frac{1}{a^p} + \frac{(-1)^p}{(a+1)^p} = \sum_{\nu \geqq p/2}^{p} \frac{p}{\nu} \binom{\nu}{p-\nu} \frac{1}{a^{\nu}(a+1)^{\nu}} .$$

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If $S_p = \sum_{\mu=0}^{\infty} \frac{1}{(a+\mu)^p (a+\mu+1)^p}$, then

$$\frac{1}{a^{2p+1}} = \sum_{\nu=p+1}^{2p+1} \frac{2p+1}{\nu} \binom{\nu}{2p+1-\nu} S_\nu$$

and

$$\frac{1}{a^{2p}} + 2 \sum_{\mu=1}^{\infty} \frac{1}{(a+\mu)^{2p}} = \sum_{\nu=p}^{2p} \frac{2p}{\nu} \binom{\nu}{2p-\nu} S_\nu .$$

In the linear sense, these invert Kimura's relations.

If we write

$$\sum_{\mu=0}^{\infty} \frac{1}{(a+\mu)^2} = \frac{1}{a} + \frac{1}{2a^2} + \sum_{p=1}^{k-1} \frac{b_p}{a^{2p+1}} + \frac{C_k(a)}{a^{2k+1}} ,$$

where $C_k(a) \to b_k$ as $a \to \infty$,

$$\text{then} \quad \frac{S_2}{2} = \sum_{p=1}^{k-1} b_p \sum_{\nu=p+1}^{2p+1} \frac{2p+1}{\nu} \binom{\nu\ \nu}{2p+1-\nu} S_\nu + \frac{C_k(a)}{a^{2k+1}} .$$

Then

$$\sum_{\nu \geq (p/2)}^{p} \frac{2\nu+1}{p} \binom{p}{2\nu+1-p} b_\nu = 0 ,$$

with $b_1 = \frac{1}{6}$, defines uniquely the Bernoulli numbers $\{b_\nu\}$.

We also have

$$\sum_{\nu=0}^{\infty} t^\nu \{x^{\nu+1} - (1-x)^{\nu+1}\} = \frac{x}{1-tx} - \frac{1-x}{1-t(1-x)}$$

$$= (2x-1) \sum_{\nu=0}^{\infty} t \sum_{\mu \geq (\nu/2)} (-1)^{\nu-\mu} \binom{\mu}{\nu-\mu} x^{\nu-\mu} (1-x)^{\nu-\mu}$$

whence

$$\frac{1}{(k-1)^{p+1}} + \frac{(-1)^p}{(k+1)^{p+1}} = \sum_{\nu \geq p/2}^{p} \binom{\nu}{p-\nu} \frac{2^{2\nu-p+1} k}{(k^2-1)^{\nu+1}} .$$

Here, $\frac{1}{k} = - \sum_{\nu=1}^{n} \frac{(-1)^{\nu} k}{(k^2-1)^{\nu}} + (-1)^n \frac{1}{k(k^2-1)^n}$.

If $k_{\mu} = a + 2\mu + 1$, and $\sigma_p = \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu} k_{\mu}}{(k_{\mu}^2 - 1)^p}$

we may set $\sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{a+2\mu+1} = \sum_{p=0}^{k-1} \frac{e_p}{2a^{2p+1}} + \frac{C_k^*(a)}{2a^{2k+1}}$

where $C_k^*(a) \to e_k$ if $a \to \infty$. Simplifying

$$\sum_{p=0}^{k} \frac{e_p}{2} \sum_{\nu=p}^{2p} \binom{\nu}{2p-\nu} 2^{2\nu-2p+1} \sigma_{\nu+1} + 0\left(\frac{1}{a^{2k+3}}\right)$$

$$= \sum_{p=0}^{k} \frac{e_p}{2a^{2p+1}} , \text{ we have } \sum_{\nu \geq p/2}^{p} \binom{p}{2\nu-p} 2^{2p-2\nu} e_{\nu} = (-1)^p$$

These linear relations, with $e_1 = -1$, define the Euler numbers uniquely.

Using the definitions $\log\left(\frac{k+1}{k}\right) = \int_0^x \frac{2dx}{1-x^2} = 2x + \int_0^x \frac{2x^2 dx}{1-x^2}$

for $x = \frac{1}{2k+1}$, and integrating by parts, we extend these results easily to Stirling's expansion of the logarithm of the factorial function, and its derivative

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$$- C + \sum_{\mu=1}^{\infty} \left\{ \frac{1}{\mu} - \frac{1}{a+\mu} \right\} = \log a + \frac{1}{2a} - \frac{1}{12a^2} - \frac{b_2}{4a^4} - \dots ,$$

treated as an asymptotic series in the above sense.

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*N. Kimura. This Bulletin (1962) 5 3, pp. 305-309.

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