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NOTES ON LOCAL INTEGRAL EXTENSION DOMAINS

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1. Introduction. All rings in this paper are assumed to be commutative with identity, and the undefined terminology is the same as that in [**3**].

In 1956, in an important paper [**2**], M. Nagata constructed an example which showed (among other things): (i) a maximal chain of prime ideals in an integral extension domain R' of a local domain (R, M) need not contract in R to a maximal chain of prime ideals; and, (ii) a prime ideal P in R' may be such that height $P <$ height $P \cap R$. In his example, R' was the integral closure of R and had two maximal ideals. In this paper, by using Nagata's example, we show that there exists a finite *local* integral extension domain of $D = R[X]_{(M,X)}$ for which (i) and (ii) hold (see (2.8.1) and (2.10)).

The fact that these two properties can be transferred from a semi-local integral extension domain of R to a local integral extension domain of D is because of a construction in the proof of a more general result, (2.4). In (2.4), it is shown that given an arbitrary maximal chain of prime ideals of length n in an arbitrary integral extension domain of an arbitrary local domain (R, M), there corresponds a maximal chain of prime ideals of length $n + 1$ in a quasi-local integral extension domain of D, and this new chain contracts in D to a maximal chain of prime ideals if and only if the original chain contracts in R to a maximal chain of prime ideals. From this, the above mentioned (2.8.1) and (2.10) readily follows.

Finally, in (2.13), it is shown that the construction can also be used to go in the opposite direction (from D to finite integral extension domains of R). Specifically, (2.13) shows that if D has any one of a number of properties concerning chains of prime ideals, then each local domain B_P also has this property, where B is a finite integral extension domain of R and P is a prime ideal in B.

2. Local integral extension domains. We begin this section with a definition due to I. Kaplansky.

(2.1) *Definition* [**1**, p. 16]. An integral domain B is an *S-domain* in case height $QB[X] = 1$, for all height one prime ideals Q in B. B is a *strong S-domain* in case B/P is an S-domain, for all prime ideals P in B.

For example, all Noetherian domains are strong S-domains [**1**, Theorem 148].

The following proposition, which is of some importance, is probably known,

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but the author knows of no reference for it. Therefore, since it is needed in the proof of (2.4), a proof of the proposition will be given.

(2.2) PROPOSITION. *Let Q be a prime ideal in an integral extension domain B of a Noetherian domain A. Then* height $QB[X]$ = height Q, depth $QB[X]$ = depth $Q + 1$, $QB[X] \cap A[X] = (Q \cap A)A[X]$, *and B and B_Q are strong S-domains.*

Proof. It will first be shown that B is a strong S-domain. For this, if P is a prime ideal in B, then, with $B_1 = B/P$ and $A_1 = A/(P \cap A)$, A_1 and B_1 satisfy the same conditions as A and B. Therefore it suffices to prove that B is an S-domain—that is, if p is a height one prime ideal in B, then height $pB[X] = 1$. Now height $pB[X] = 1$, if, for each prime ideal m in the integral closure B' of B such that $m \cap B = p$, B'_m is a valuation ring [**10**, Theorem 8]. To see that this holds, note that height $m = 1$ (since height $p = 1$), so height $m \cap A' = 1$ [**3**, (10.14)], where A' is the integral closure of A. Let $C = B'_{(A'-(m \cap A'))}$, so C is the integral closure of the valuation ring $A'_{(m \cap A')}$ in the quotient field of B. Also, mC is a maximal ideal, so C_{mC} is a valuation ring [**11**, Corollary 2, p. 27]. Finally, $B'_m = C_{mC}$, so height $pB[X] = 1$, hence B is a strong S-domain.

It is clear that if C is a strong S-domain, then so is C_S, for all m.c. sets S in C, so B_Q is a strong S-domain. Also, by [**1**, Theorem 39], height $QB[X]$ = height Q and height Q^* = height $Q + 1$, for all prime ideals Q^* in B such that $QB[X] \subset Q^*$ and $Q^* \cap B = Q$. Therefore altitude $B[X]$ = altitude $B + 1$, so depth $QB[X]$ = altitude $B[X]/QB[X]$ = altitude $(B/Q)[X]$ = depth $Q + 1$ (since B/Q is a strong S-domain).

Finally, $(B/Q)[X] \cong B[X]/QB[X]$ is integral over

$$D_1 = A[X]/(QB[X] \cap A[X])$$

and $(B/Q)[X]$ is integral over $(A/(Q \cap A))[X] \cong A[X]/(Q \cap A)A[X] = D_2$ (say). Therefore, since D_1 is a homomorphic image of D_2, $QB[X] \cap A[X] = (Q \cap A)A[X]$, completing the proof.

The following definition is needed in order to avoid continual repetitions in the remainder of this paper.

(2.3) *Definition.* It will be said that an integral domain A has a *mcpil n* in case there exists a *m*aximal *c*hain of *p*rime *i*deals of *l*ength n in A (that is, a chain of prime ideals $(0) = p_0 \subset p_1 \subset \ldots \subset p_n$ such that p_n is maximal and height $p_i/p_{i-1} = 1$, for $i = 1, \ldots, n$).

The next result is the main theorem in this paper. Before stating it, it should be mentioned that the same conclusions hold for any of the rings $R[X]_N$, where N is a maximal ideal in $R[X]$ such that $N \cap R = M$, as is readily seen by the proof of (2.4).

(2.4) THEOREM. *Let B be an integral extension domain of a local domain (R, M), and let $(0) \subset Q_1 \subset \ldots \subset Q_n$ be a mcpil n in B. Then there exists a* *quasi-local integral extension domain* L *of* $D = R[X]_{(M,X)}$ *which has a mcpil* $n + 1$, *say* $(0) \subset P_1 \subset \ldots \subset P_{n+1}$, *such that* height P_i = height Q_i, depth P_i = depth $Q_i + 1$, *and* $P_i \cap D = (Q_i \cap R)D$, *for* $i = 1, \ldots, n$. *Therefore* $(0) \subset P_1 \cap D \subset \ldots \subset P_{n+1} \cap D$ *is a mcpil* $n + 1$ *if and only if* $(0) \subset Q_1 \cap R \subset \ldots \subset Q_n \cap R$ *is a mcpil* n.

Proof. Let $S = B[X]_{(B[X]-(M,X))}$, and let $L = D + J$, where J is the Jacobson radical of S. Let $P_i = Q_iS \cap L$ $(i = 1, \ldots, n)$, and let $P_{n+1} = (Q_n, X)S \cap L$ $(= J)$. Then S is integral over D, so L is a quasi-local integral extension domain of D. Moreover, $(0) \subset Q_1B[X] \subset \ldots \subset Q_nB[X] \subset (Q_n, X)B[X]$ is a mcpil $n + 1$ in $B[X]$ (since B is a strong S-domain (2.2)), so $(0) \subset Q_1S \subset \ldots \subset Q_nS \subset (Q_n, X)S$ is a mcpil $n + 1$ in S. Therefore, since depth P_n = depth $Q_nS = 1$ (since S is integral over L and depth $Q_nB[X] = 1$ (2.2)) and since $L_{P_n} = S_{Q_nS}$ (since J is the conductor of L in S), $(0) \subset P_1 \subset \ldots \subset P_{n+1}$ is a mcpil $n + 1$ in L and height P_i = height Q_iS = height Q_i (2.2), for $i = 1, \ldots, n$. Also, by integral dependence and (2.2), for $i = 1, \ldots, n$, depth P_i = depth $Q_iS \leqq$ depth $Q_iB[X]$ = depth $Q_i + 1$, and if depth $Q_i = d$ and $Q_i \subset Q_{i+1}' \subset \ldots \subset Q_{i+d}'$ is a saturated chain of prime ideals in B of length d, then $Q_iS \subset Q_{i+1}'S \subset \ldots \subset Q_{i+d}'S \subset (Q_{i+d}', X)S$, so depth $Q_iS \geqq$ depth $Q_i + 1$, hence depth P_i = depth $Q_i + 1$. Moreover, since $Q_iB[X] \cap R[X] = (Q_i \cap R)R[X]$ (2.2), it readily follows that $P_i \cap D = (Q_i \cap R)D$, for $i = 1, \ldots, n$. The last statement now follows, since $(0) \subset Q_1 \cap R \subset \ldots \subset Q_n \cap R$ is a mcpil n if and only if $(0) \subset (Q_1 \cap R)D \subset \ldots \subset (Q_n \cap R)D = MD \subset (M, X)D$ is a mcpil $n + 1$.

A few comments on (2.4) will now be made.

(2.5) *Remark.* With the notation of (2.4), the following statements hold:

(2.5.1) If B is quasi-local, then S (in the proof of (2.4)) is quasi-local, so S and $(0) \subset Q_1S \subset \ldots \subset Q_nS \subset (Q_n, X)S$ could be used in place of L and $(0) \subset P_1 \subset \ldots \subset P_{n+1}$.

(2.5.2) If R, instead of being local, is semi-local with exactly h maximal ideals, then the same conclusions in (2.4) hold, except that L is quasi-semi-local with exactly h maximal ideals.

(2.5.3) If B is a finite integral extension domain of R, then L is a finite local (Noetherian) integral extension domain of D.

(2.5.4) It is an open problem if there exists a finite integral extension domain $A \subseteq B$ of R such that $(0) \subset Q_1 \cap A \subset \ldots \subset Q_n \cap A$ is a mcpil n and height $Q_i \cap A$ = height Q_i (and depth $Q_i \cap A$ = depth Q_i), for $i = 1, \ldots, n$. However, this is easily shown to hold if, for each i, $Q_i \not\subseteq \cup \{Q_i' \in \text{Spec } B;\ Q_i' \cap R = Q_i \cap R \text{ and } Q_i' \neq Q_i\}$. (This holds, for example, if B is contained in the integral closure of a finite integral extension domain of R.) If such A exists, then it may be assumed that L is a finite local integral extension domain of D in (2.4).

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(2.5.5) Let $P_1' = XS \cap L$ and $P_{i+1}' = (Q_i, X)S \cap L$, for $i = 1, \ldots, n$. Then $(0) \subset P_1' \subset \ldots \subset P_n'$ is a saturated chain of prime ideals of length n in L such that, for $i = 0, 1, \ldots, n - 1$ and $Q_0 = P_0 = (0)$, height $P_{i+1}' =$ height $Q_i + 1$, depth $P_{i+1}' =$ depth Q_i, and $P_{i+1}' \cap D = (Q_i \cap R, X)D = (P_i \cap D, X)D$. Moreover, $(0) \subset P_1' \subset \ldots \subset P_n' \subset P_{n+1}'$ is a mcpil $n + 1$ in L if and only if depth $Q_{n-1} = 1$, and in this case, this chain contracts in D to a mcpil $n + 1$ if and only if the $P_i \cap D$ form a mcpil $n + 1$ in D.

Proof of (2.5.5). For $i = 0, 1, \ldots, n$ and $Q_0 = (0)$, height $(Q_i, X)S =$ height $Q_i + 1$, by [**1**, Theorem 39]. Also, $L_{P_n'} = S_{(Q_{n-1}, X)S}$ (since J is the conductor of L in S), so height $P_{i+1}' =$ height $(Q_i, X)S =$ height $Q_i + 1$, for $i = 0, 1, \ldots, n - 1$, and $(0) \subset P_1' \subset \ldots \subset P_n'$ is a saturated chain of prime ideals (since clearly $(0) \subset XS \subset (Q_1, X)S \subset \ldots \subset (Q_{n-1}, X)S$ is a saturated chain of prime ideals). Further, it is clear that depth $P_{i+1}' =$ depth $(Q_i, X)S =$ depth Q_i, for $i = 0, 1, \ldots, n$, so depth $P_n' =$ depth Q_{n-1}, hence $(0) \subset P_1' \subset \ldots \subset P_n' \subset P_{n+1}'$ is a mcpil $n + 1$ if and only if depth $Q_{n-1} = 1$ (since L is quasi-local). Moreover, since $Q_iS \cap D = (Q_i \cap R)D$, it readily follows that $P_{i+1}' \cap D = (Q_i \cap R, X)D = (P_i \cap D, X)D$, for $i = 0, 1, \ldots, n$ (and with $Q_0 = P_0 = (0)$), so the last statement is clear from this and the fact that $(0) \subset Q_1 \cap R \subset \ldots \subset Q_n \cap R$ is a mcpil n if and only if $(0) \subset XD \subset (Q_1 \cap R, X)D \subset \ldots \subset (Q_n \cap R, X)D$ is a mcpil $n + 1$.

(2.6) *Remark.* In (2.5.5), use was made of the fact that if q is a prime ideal in a strong S-domain A, then $P = (q, X)A[X]$ is a prime ideal such that height $P =$ height $q + 1$ and depth $P =$ depth q. Note, on the other hand, if $X \notin Q$ is a given prime ideal in $A[X]$ and $Q' = (Q, X)A[X]$ is a prime ideal, then height $Q' >$ height $Q + 1$ is possible, even if A is Noetherian. For example, let (R, M) be a local domain and let p be a prime ideal in R such that height $M/p = 1$, height $M >$ height $p + 1$, and $M = (p, y)$R, for some $y \in M$. (Such R, M, and p exist, as can be seen using [**3**, Example 2, pp. 203-205] in the case $m > 0$.) Let $Q = (p, X - y)R[X]$, so Q is a prime ideal and $Q \cap R = p$. Also, $Q' = (Q, X)R[X]$ $(= (M, X)R[X])$ is prime and $Q' \cap R = M$. Therefore height $Q =$ height $p + 1 <$ height $M <$ height $M + 1 =$ height Q'.

As mentioned in (2.5.4), I do not know if, in general, L can be chosen to be a finite local integral extension domain of D in (2.4). However, if just one prime ideal in B is given (instead of a chain of prime ideals), then L can be so chosen, as will now be shown.

(2.7) Corollary. *With R, B, and D as in* (2.4), *let Q be a prime ideal in B, let* height $Q = h$, *and let* depth $Q = d$. *Then there exists a finite local integral extension domain L of D which has a prime ideal P such that* height $P = h$, depth $P = d + 1$, *and* $P \cap D = (Q \cap R)D$, *so* height $P \cap D =$ height $Q \cap R$.

Proof. By [**5**, (2.9)], there exists a simple integral extension domain $R[c]$ of R which has a prime ideal p such that $p \cap R = Q \cap R$, height $p = h$, and depth

$p = d$. Therefore, by (2.5.3), there exists a local integral extension domain L of D which has a mcpil $h + d + 1$, say $(0) \subset P_1 \subset \ldots \subset P_{h+d+1}$, such that height $P_h = h$, depth $P_h = d + 1$, and $P_h \cap D = (p \cap R)D = (Q \cap R)D$, so height $P \cap D =$ height $Q \cap R$.

(2.8) *Remark*. With the notation of (2.7), the following statements hold:

(2.8.1) In [**3**, Example 2, pp. 203-205], it is shown that, for all $m \geqq 0$ and $r \geqq 1$, there exists a local domain R whose integral closure has a maximal ideal N such that height $N = m + 1$ and height $N \cap R = r + m + 1$. Therefore, by (2.7), there exists a finite local integral extension domain L of D such that L has a prime ideal P such that height $P = m + 1$ and height $P \cap D = m + r + 1$.

(2.8.2) It is readily seen, much as in (2.5.5), that there exists a minimal prime divisor P' of $(P, X)L$ such that height $P' = h + 1$, depth $P' = d$, and $P' \cap D = (Q \cap R, X)D$, so height $P' \cap D =$ height $(Q \cap R) + 1$.

For the next corollary, the following definition is needed.

(2.9) *Definition* [**7**, Section 4]. Let $\mathscr{C}$ be the class of local domains R such that there exists a mcpil n in some integral extension domain of R if and only if there exists a mcpil n in R.

A number of facts concerning $\mathscr{C}$ and the rings which are in $\mathscr{C}$ are given in [**7**, Section 4]. In particular, it is known that for all local domains (R, M), $R[X]_{(M,X)} \in \mathscr{C}$ [**7**, (4.1.2)].

The following corollary shows that [**6**, Question 3.15)] has a negative answer.

(2.10) Corollary. *There exist local domains $R \subset S$ such that S is a finite integral extension of R, $R \in \mathscr{C}$, and there exists a mcpil n in S which doesn't contract in R to a maximal chain of prime ideals.*

Proof. By [**3**, Example 2, pp. 203-205], there exists a local domain (R_0, M_0) whose integral closure is a finite R_0-algebra and which has a mcpil m which doesn't contract in R_0 to a maximal chain of prime ideals. Therefore, by (2.5.3), there exists a finite local integral extension domain S of $R = R_0[X]_{(M_0,X)}$ which has a mcpil $m + 1$ which doesn't contract in R to a maximal chain of prime ideals.

It is still an open problem if R in (2.10) can be Henselian. As pointed out in [**6**, p. 87], if such Henselian R exist, then the Chain Conjecture (the integral closure of a local domain is catenary) fails to hold.

Since $D \in \mathscr{C}$ [**7**, (4.1.2]), given the mcpil $n + 1$ in L in (2.4), there exists a mcpil $n + 1$ in D. Is it true that there exists a mcpil n + 1 in D, say $(0) \subset p_1 \subset \ldots \subset p_{n+1} = (M, X)D$, such that height $p_i =$ height P_i and depth $p_i =$ depth P_i $(i = 1, \ldots, n + 1)$, where the P_i are as in (2.4)?

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Another application of the construction in the proof of (2.4) will be given in (2.13). In order to keep from overburdening the statement of (2.13), the following definition is needed.

(2.11) *Definition.* Let (*) denote a property of a local domain (R, M) such that: (i) if S is a finite local integral extension domain of R and R has (*), then S has (*); (ii) if $R(X) = R[X]_{MR[X]}$ has (*), then R has (*); and, (iii) if R has (*), then R_P has (*), for all prime ideals P in R.

(2.12) *Remark.* (2.12.1) Examples of conditions which satisfy (*) are f.c.c (first chain condition for prime ideals); s.c.c. (second chain condition for prime ideals); pseudo-geometric; and, GB (that is, adjacent prime ideals in arbitrary integral extension domains of R contract to adjacent prime ideals in R—(see [**8**])).

(2.12.2) Examples of conditions which satisfy (i) and (ii) in (2.11), but for which it is unknown if (iii) holds, are: H_i (prime ideals of height $= i$ have depth $=$ altitude $R - i$ (see [**9**])); C_i (R is H_i, H_{i+1}, and every maximal ideal in the integral closure of R/p has height $=$ altitude R/p, for each prime ideal p in R such that height $p = i$ (see [**9**])); and, D_i (prime ideals of depth $= i$ have height $=$ altitude $R - i$ (see [**5**])).

(2.13) ties the construction in the proof of (2.4) to the conditions (*).

(2.13) PROPOSITION. *Let (R, M) be a local domain and let $D = R[X]_{(M,X)}$. If D has a* (*) *property* (2.11), *then, for all finite integral extension domains B of R and for all prime ideals P in B, B_P has the same* (*) *property.*

Proof. Assume that D has a property (*) and let B be a finite integral extension domain of R. If B is local, then B has property (*). (For, $R(X)$ does, by (iii), so R does, by (ii), hence B does, by (i).) Therefore, by (iii), B_P has property (*), for all prime ideals P in B.

Therefore, assume that B has more than one maximal ideal. Let P be a prime ideal in B and let N be a maximal ideal in B such that $P \subseteq N$. Then, by (iii), it suffices to show that B_N has property (*). For this, let $S = B[X]_{(R[X]-(M,X))}$, and let $L = D + J$, where J is the Jacobson radical of S. Then L is a finite local integral extension domain of D, so L has property (*), by (i). Also, since J is the conductor of L in S, $L_{NS \cap L} = S_{NS}$, so S_{NS} has property (*), by (iii). Finally, $S_{NS} = B_N(X)$, so B_N has property (*), by (ii), completing the proof.

Concerning the properties (*), if the defining conditions in (2.11) are assumed to be properties of a quasi-local domain R, and if (i) is changed to: if S is a quasi-local integral extension domain of R and R has (*), then S has (*); then, whenever D has a property (*), then so does B_P, for *all* integral extension domains B of R and prime ideals P in B. The proof is the same as the proof of (2.13).

The following important known result follows quite easily from (2.13).

(2.14) COROLLARY [**4**, Theorem 2.21]. *If* (R, M) *is a local domain such that* $D = R[X]_{(M,X)}$ *satisfies the f.c.c., then* R *satisfies the s.c.c.*

Proof. If B is a finite integral extension domain of R and Q is a maximal ideal in B, then, since D satisfies the f.c.c., B_Q satisfies the f.c.c., by (2.12.1) and (2.13). Also, a mcpil n up to Q in B gives rise to a mcpil $n + 1$ in a finite local integral extension domain L of D (2.5). Now D satisfies the f.c.c., so L does (by (2.12.1) and (2.11) (i)), hence $n + 1 =$ altitude L, so $n =$ altitude R. Therefore B satisfies the f.c.c., so R satisfies the s.c.c. [**3**, (34.3)].

As a final comment on (2.13), we note that if it can be shown that (2.11) (iii) holds for H_1, then, using (2.13), it can be shown that the Catenary Chain Conjecture (the integral closure of a catenary local domain is catenary) holds.

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