Combinatorics, Probability and Computing (2023), **32**, pp. 122–133
doi:10.1017/S0963548322000165

ARTICLE

Clustered colouring of graph classes with bounded treedepth or pathwidth

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(Received 14 February 2021; revised 20 January 2022; accepted 8 June 2022; first published online 5 July 2022)

Abstract
The *clustered chromatic number* of a class of graphs is the minimum integer k such that for some integer c every graph in the class is k-colourable with monochromatic components of size at most c. We determine the clustered chromatic number of any minor-closed class with bounded treedepth, and prove a best possible upper bound on the clustered chromatic number of any minor-closed class with bounded pathwidth. As a consequence, we determine the fractional clustered chromatic number of every minor-closed class.

Keywords: graph; graph colouring; clustered colouring; minor; treedepth; pathwidth; fractional colouring

2020 MSC Codes: Primary: 05C83, Secondary: 05C15

1. Introduction

This paper studies improper vertex colourings of graphs with bounded monochromatic degree or bounded monochromatic component size. This topic has been extensively studied recently [1–6, 7, 8, 9–18, 19–21]; see [22] for a survey.

A *k-colouring* of a graph G is a function that assigns one of k colours to each vertex of G. In a coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour.

A colouring has *defect d* if each monochromatic component has maximum degree at most d. The *defective chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\Delta(\mathcal{G})$, is the minimum integer k such that, for some integer d, every graph in $\mathcal{G}$ is k-colourable with defect d.

A colouring has *clustering c* if each monochromatic component has at most c vertices. The *clustered chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\star(\mathcal{G})$, is the minimum integer k such that, for some integer c, every graph in $\mathcal{G}$ has a k-colouring with clustering c. We shall consider such colourings, where the goal is to minimise the number of colours, without optimising the clustering value.

Every colouring of a graph with clustering c has defect $c-1$. Thus, $\chi_\Delta(\mathcal{G}) \leqslant \chi_\star(\mathcal{G})$ for every class $\mathcal{G}$.

The following is a well-known and important example in defective and clustered graph colouring. Let T be a rooted tree. The *depth* of T is the maximum number of vertices on a root–to–leaf path in T. The *closure* of T is obtained from T by adding an edge between every ancestor and descendant in T. For $h, k \geqslant 1$, let $C\langle h, k\rangle$ be the closure of the complete k-ary tree of depth h, as illustrated in Figure 1.

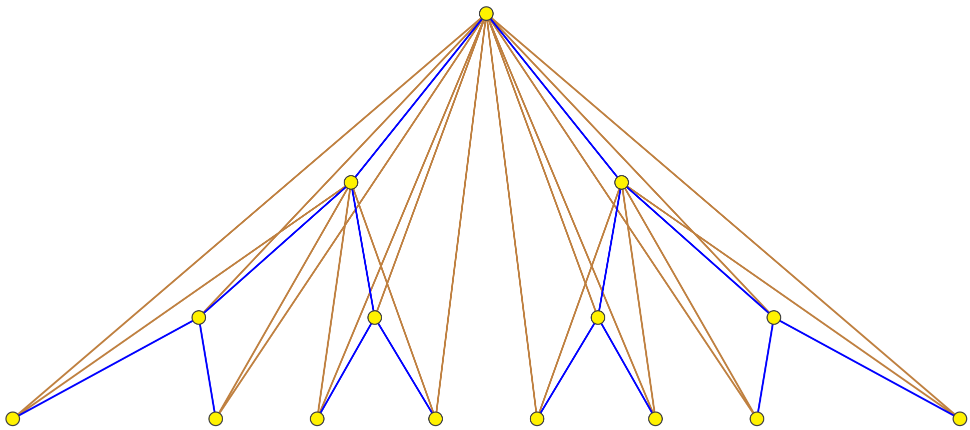

Figure 1. The standard example $C\langle 4, 2\rangle$.

It is well known and easily proved (see [22]) that there is no $(h-1)$-colouring of $C\langle h, k\rangle$ with defect $k-1$, which implies there is no $(h-1)$-colouring of $C\langle h, k\rangle$ with clustering k. This says that if a graph class $\mathcal{G}$ includes $C\langle h, k\rangle$ for all k, then the defective chromatic number and the clustered chromatic number are at least h. Put another way, define the *tree-closure-number* of a graph class $\mathcal{G}$ to be

$$\text{tcn}(\mathcal{G}) := \min\{h : \exists k\, C\langle h, k\rangle \notin \mathcal{G}\} = \max\{h : \forall k\, C\langle h, k\rangle \in \mathcal{G}\} + 1;$$

then

$$\chi_\star(\mathcal{G}) \geqslant \chi_\Delta(\mathcal{G}) \geqslant \text{tcn}(\mathcal{G}) - 1.$$

Our main result, Theorem 1 below, establishes a converse result for minor-closed classes with bounded treedepth. First we explain these terms. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from some subgraph of G by contracting edges. A class of graphs $\mathcal{M}$ is *minor-closed* if for every graph $G \in \mathcal{M}$ every minor of G is in $\mathcal{M}$, and $\mathcal{M}$ is *proper* minor-closed if, in addition, some graph is not in $\mathcal{M}$. The *connected treedepth* of a graph H, denoted by $\overline{\text{td}}(H)$, is the minimum depth of a rooted tree T such that H is a subgraph of the closure of T. This definition is a variant of the more commonly used definition of the *treedepth* of H, denoted by $\text{td}(H)$, which equals the maximum connected treedepth of the connected components of H. (See [23] for background on treedepth.) If H is connected, then $\text{td}(H) = \overline{\text{td}}(H)$. In fact, $\text{td}(H) = \overline{\text{td}}(H)$ unless H has two connected components H_1 and H_2 with $\text{td}(H_1) = \text{td}(H_2) = \text{td}(H)$, in which case $\overline{\text{td}}(H) = \text{td}(H) + 1$. It is convenient to work with connected treedepth to avoid this distinction. A class of graphs has *bounded treedepth* if there exists a constant c such that every graph in the class has treedepth at most c.

Theorem 1. *For every minor-closed class $\mathcal{G}$ with bounded treedepth,*

$$\chi_\Delta(\mathcal{G}) = \chi_\star(\mathcal{G}) = \text{tcn}(\mathcal{G}) - 1.$$

Our second result concerns pathwidth. A *path-decomposition* of a graph G consists of a sequence $(B_1, \ldots, B_n)$, where each B_i is a subset of $V(G)$ called a *bag*, such that for every vertex $v \in V(G)$, the set $\{i \in [1, n] : v \in B_i\}$ is an interval, and for every edge $vw \in E(G)$ there is a bag B_i containing both v and w. Here $[a, b] := \{a, a+1, \ldots, b\}$. The *width* of a path decomposition $(B_1, \ldots, B_n)$ is $\max\{|B_i| : i \in [1, n]\} - 1$. The *pathwidth* of a graph G is the minimum width of a path-decomposition of G. Note that paths (and more generally caterpillars) have pathwidth 1.

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A class of graphs has *bounded pathwidth* if there exists a constant c such that every graph in the class has pathwidth at most c.

Theorem 2. *For every minor-closed class $\mathcal{G}$ with bounded pathwidth,*

$$\chi_\Delta(\mathcal{G}) \leqslant \chi_\star(\mathcal{G}) \leqslant 2\mathrm{tcn}(\mathcal{G}) - 2.$$

Theorems 1 and 2 are, respectively, proved in Sections 2 and 3. These results are best possible and partially resolve a number of conjectures from the literature, as we now explain.

Ossona de Mendez et al. [20] studied the defective chromatic number of minor-closed classes. For a graph H, let $\mathcal{M}_H$ be the class of H-minor-free graphs (that is, not containing H as a minor). Ossona de Mendez et al. [20] proved the lower bound, $\chi_\Delta(\mathcal{M}_H) \geqslant \overline{\mathrm{td}}(H) - 1$ and conjectured that equality holds.

Conjecture 3 ([20]). *For every graph H,*

$$\chi_\Delta(\mathcal{M}_H) = \overline{\mathrm{td}}(H) - 1.$$

Conjecture 3 is known to hold in some special cases. Edwards et al. [8] proved it if $H = K_t$; that is, $\chi_\Delta(\mathcal{M}_{K_t}) = t - 1$, which can be thought of as a defective version of Hadwiger's Conjecture; see [21] for an improved bound on the defect in this case. Ossona de Mendez et al. [20] proved Conjecture 3 if $\overline{\mathrm{td}}(H) \leqslant 3$ or if H is a complete bipartite graph. In particular, $\chi_\Delta(\mathcal{M}_{K_{s,t}}) = \min\{s, t\}$.

Norin et al. [19] studied the clustered chromatic number of minor-closed classes. They showed that for each $k \geqslant 2$, there is a graph H with treedepth k and connected treedepth k such that $\chi_\star(\mathcal{M}_H) \geqslant 2k - 2$. Their proof in fact constructs a set $\mathcal{X}$ of graphs in $\mathcal{M}_H$ with bounded pathwidth (at most $2k - 3$ to be precise) such that $\chi_\star(\mathcal{X}) \geqslant 2k - 2$. Thus, the upper bound on $\chi_\star(\mathcal{G})$ in Theorem 2 is best possible.

Norin et al. [19] conjectured the following converse upper bound (analogous to Conjecture 3):

Conjecture 4 ([19]). *For every graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant 2\overline{\mathrm{td}}(H) - 2.$$

While Conjectures 3 and 4 remain open, Norin et al. [19] showed in the following theorem that $\chi_\Delta(\mathcal{M}_H)$ and $\chi_\star(\mathcal{M}_H)$ are controlled by the treedepth of H:

Theorem 5 ([19]). *For every graph H, $\chi_\star(\mathcal{M}_H)$ is tied to the (connected) treedepth of H. In particular,*

$$\overline{\mathrm{td}}(H) - 1 \leqslant \chi_\star(\mathcal{M}_H) \leqslant 2^{\overline{\mathrm{td}}(H)+1} - 4.$$

Theorem 1 gives a much more precise bound than Theorem 5 under the extra assumption of bounded treedepth.

Our third main result concerns fractional colourings. For real $t \geqslant 1$, a graph G is *fractionally t-colourable with clustering c* if there exist $Y_1, Y_2, \ldots, Y_s \subseteq V(G)$ and $\alpha_1, \ldots, \alpha_s \in [0, 1]$ such that[1]:

- Every component of $G[Y_i]$ has at most c vertices,
- $\sum_{i=1}^{s} \alpha_i \leqslant t$,
- $\sum_{i:v\in Y_i} \alpha_i \geqslant 1$ for every $v \in V(G)$.

The *fractional clustered chromatic number* $\chi^f_\star(\mathcal{G})$ of a graph class $\mathcal{G}$ is the infimum of $t > 0$ such that there exists $c = c(t, \mathcal{G})$ such that every $G \in \mathcal{G}$ is fractionally t-colourable with clustering c.

[1] If $c = 1$, then this corresponds to a (proper) fractional t-colouring, and if the α_i are integral, then this yields a t-colouring with clustering c.

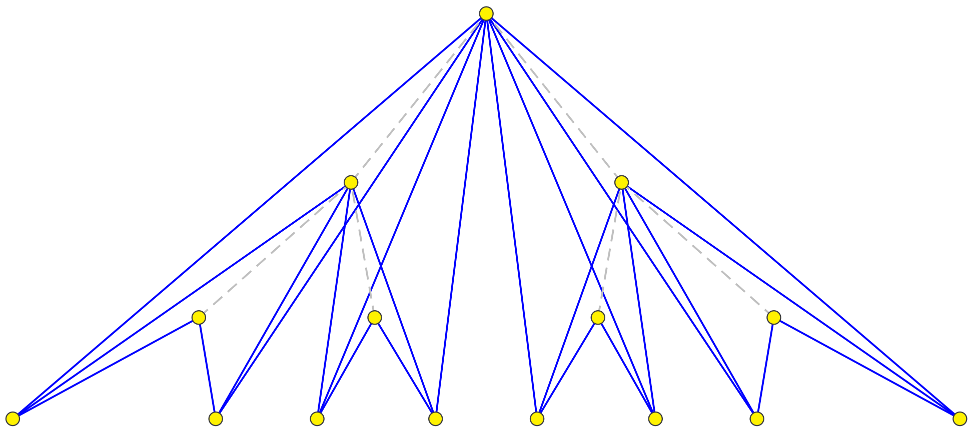

Figure 2. The weak closure $W\langle 4, 2\rangle$.

Fractionally t-colourable with defect d and *fractional defective chromatic number* $\chi^f_\Delta(\mathcal{G})$ are defined in exactly the same way, except the condition on the component size of $G[Y_i]$ is replaced by "the maximum degree of $G[Y_i]$ is at most d".

The following theorem determines the fractional clustered chromatic number and fractional defective chromatic number of any proper minor-closed class.

Theorem 6. *For every proper minor-closed class* $\mathcal{G}$*,*

$$\chi^f_\Delta(\mathcal{G}) = \chi^f_\star(\mathcal{G}) = \text{tcn}(\mathcal{G}) - 1.$$

This result is proved in Section 4.

We now give an interesting example of Theorem 6.

Corollary 7. *For every surface* Σ*, if* $\mathcal{G}_\Sigma$ *is the class of graphs embeddable in* Σ*, then*

$$\chi^f_\Delta(\mathcal{G}_\Sigma) = \chi^f_\star(\mathcal{G}_\Sigma) = 3.$$

Proof. Note that $C\langle 3, k\rangle$ is planar for all k. Thus, $\text{tcn}(\mathcal{G}_\Sigma) \geqslant 4$. Say Σ has Euler genus g. It follows from Euler's formula that $K_{3,2g+3} \notin \mathcal{G}_\Sigma$. Since $K_{3,2g+3} \subseteq C\langle 4, 2g+3\rangle$, we have $C\langle 4, 2g+3\rangle \notin \mathcal{G}_\Sigma$. Thus, $\text{tcn}(\mathcal{G}_\Sigma) = 4$. The result follows from Theorem 6. □

In contrast to Corollary 7, Dvořák and Norin [7] proved that $\chi_\star(\mathcal{G}_\Sigma) = 4$. Note that Archdeacon [2] proved that $\chi_\Delta(\mathcal{G}_\Sigma) = 3$; see [5] for an improved bound on the defect.

2. Treedepth

Say G is a subgraph of the closure of some rooted tree T. For each vertex $v \in V(T)$, let T_v be the maximal subtree of T rooted at v (consisting of v and all its descendants), and let $G[T_v]$ be the subgraph of G induced by $V(T_v)$.

The *weak closure* of a rooted tree T is the graph G with vertex set $V(T)$, where two vertices $v, w \in V(T)$ are adjacent in G whenever v is a leaf of T and w is an ancestor of v in T. As illustrated in Figure 2, let $W\langle h, k\rangle$ be the weak closure of the complete k-ary tree of height h.

Note that $W\langle h, k\rangle$ is a proper subgraph of $C\langle h, k\rangle$ for $h \geqslant 3$. On the other hand, Norin et al. [19] showed that $W\langle h, k\rangle$ contains $C\langle h, k-1\rangle$ as a minor for all $h, k \geqslant 2$. Therefore, Theorem 1 is an immediate consequence of the following lemma.

Lemma 8. *For all $d, k, h \in \mathbb{N}$ there exists $c = c(d, k, h) \in \mathbb{N}$ such that for every graph G with treedepth at most d, either G contains a $W\langle h, k\rangle$-minor or G is $(h-1)$-colourable with clustering c.*

Proof. Throughout this proof, d, k and h are fixed, and we make no attempt to optimise c.

We may assume that G is connected. So G is a subgraph of the closure of some rooted tree of depth at most d. Choose a tree T of depth at most d rooted at some vertex r, such that G is a subgraph of the closure of T, and subject to this, $\sum_{v\in V(T)} \text{dist}_T(v, r)$ is minimal. Suppose that $G[T_v]$ is disconnected for some vertex v in T. Choose such a vertex v at maximum distance from r. Since G is connected, $v \neq r$. By the choice of v, for each child w of v, the subgraph $G[T_w]$ is connected. Thus, for some child w of v, there is no edge in G joining v and $G[T_w]$. Let u be the parent of v. Let T' be obtained from T by deleting the edge vw and adding the edge uw, so that w is a child of u in T'. Note that G is a subgraph of the closure of T' (since v has no neighbour in $G[T_w]$). Moreover, $\text{dist}_{T'}(x, r) = \text{dist}_T(x, r) - 1$ for every vertex $x \in V(T_w)$, and $\text{dist}_{T'}(y, r) = \text{dist}_T(y, r)$ for every vertex $y \in V(T) \setminus V(T_w)$. Hence, $\sum_{v\in V(T')} \text{dist}_{T'}(v, r) < \sum_{v\in V(T)} \text{dist}_T(v, r)$, which contradicts our choice of T. Therefore, $G[T_v]$ is connected for every vertex v of T.

Consider each vertex $v \in V(T)$. Define the *level* $\ell(v) := \text{dist}_T(r, v) \in [0, d-1]$. Let T_v^+ be the subtree of T consisting of T_v plus the vr-path in T, and let $G[T_v^+]$ be the subgraph of G induced by $V(T_v^+)$. For a subtree X of T rooted at vertex v, define the *level* $\ell(X) := \ell(v)$.

A *ranked graph* (for fixed d) is a triple $(H, L, \preceq)$ where:

- H is a graph,
- $L : V(H) \to [0, d-1]$ is a function,
- $\preceq$ is a partial order on $V(H)$ such that $L(v) < L(w)$ whenever $v \prec w$.

Here and throughout this proof, $v \prec w$ means that $v \preceq w$ and $v \neq w$. Up to isomorphism, the number of ranked graphs on n vertices is at most $2^{\binom{n}{2}}\, d^n\, 3^{\binom{n}{2}}$. For a vertex v of T, a ranked graph $(H, L, \preceq)$ is said to be *contained in* $G[T_v^+]$ if there is an isomorphism ϕ from H to some subgraph of $G[T_v^+]$ such that:

(A) for each vertex $v \in V(H)$ we have $L(v) = \ell(\phi(v))$, and

(B) for all distinct vertices $v, w \in V(H)$ we have that $v \prec w$ if and only if $\phi(v)$ is an ancestor of $\phi(w)$ in T.

Say $(H, L, \preceq)$ is a ranked graph and $i \in [0, d-1]$. Below we define the *i-splice* of $(H, L, \preceq)$ to be a particular ranked graph $(H', L', \preceq')$, which (intuitively speaking) is obtained from $(H, L, \preceq)$ by copying k times the subgraph of H induced by the vertices v with $L(v) > i$. Formally, let

$$
\begin{aligned}
V(H') &:= \{(v, 0) : v \in V(H), L(v) \in [0, i]\} \cup \\
&\qquad \{(v, j) : v \in V(H), L(v) \in [i+1, d], j \in [1, k]\}. \\
E(H') &:= \{(v, 0)(w, 0) : vw \in E(H), L(v) \in [0, i], L(w) \in [0, i]\} \cup \\
&\qquad \{(v, 0)(w, j) : vw \in E(H), L(v) \in [0, i], L(w) \in [i+1, d], j \in [1, k]\} \cup \\
&\qquad \{(v, j)(w, j) : vw \in E(H), L(v) \in [i+1, d], L(w) \in [i+1, d], j \in [1, k]\}.
\end{aligned}
$$

Define $L'((v, j)) := L(v)$ for every vertex $(v, j) \in V(H')$. Now define the following partial order $\preceq'$ on $V(H')$:

- $(v, j) \preceq' (v, j)$ for all $(v, j) \in V(H')$;
- if $v \prec w$ and $L(v), L(w) \in [0, i]$, then $(v, 0) \prec' (w, 0)$;
- if $v \prec w$ and $L(v) \in [0, i]$ and $L(w) \in [i+1, d]$, then $(v, 0) \prec' (w, j)$ for all $j \in [1, k]$; and
- if $v \prec w$ and $L(v), L(w) \in [i+1, d]$, then $(v, j) \prec' (w, j)$ for all $j \in [1, k]$.

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Note that if $(v, a) \prec'(w, b)$, then $a \leqslant b$ and $v \prec w$ (implying $(L(v) < L(w))$. It follows that $\prec'$ is a partial order on $V(H')$ such that $L'((v, a)) < L'((w, b))$ whenever $(v, a) \prec'(w, b)$. Thus, $(H', L', \preceq')$ is a ranked graph.

For $\ell \in [0, d-1]$, let

$$N_\ell := (d+1)(h-1)(k+1)^{d-1-\ell}.$$

For each vertex v of T, define the *profile* of v to be the set of all ranked graphs $(H, L, \preceq)$ contained in $G[T_v^+]$ such that $|V(H)| \leqslant N_{\ell(v)}$. Note that if v is a descendant of u, then the profile of v is a subset of the profile of u. For $\ell \in [0, d-1]$, if $N = N_\ell$ then let

$$M_\ell := 2^{2^{\binom{N}{2}} d^N 3^{\binom{N}{2}}}.$$

Then there are at most M_ℓ possible profiles of a vertex at level ℓ.

We now partition $V(T)$ into subtrees. Each subtree is called a *group*. (At the end of the proof, vertices in a single group will be assigned the same colour.) We assign vertices to groups in non-increasing order of their distance from the root. Initialise this process by placing each leaf v of T into a singleton group. We now show how to determine the group of a non-leaf vertex. Let v be a vertex not assigned to a group at maximum distance from r. So each child of v is assigned to a group. Let Y_v be the set of children y of v, such that the number of children of v that have the same profile as y is in the range $[1, k-1]$. If $Y_v = \emptyset$ start a new singleton group $\{v\}$. If $Y_v \neq \emptyset$ then merge all the groups rooted at vertices in Y_v into one group including v. This defines our partition of $V(T)$ into groups. Each group X is *rooted* at the vertex in X closest to r in T. A group Y is *above* a distinct group X if the root of Y is on the path in T from the root of X to r.

The next claim is the key to the remainder of the proof.

Claim 1. *Let $uv \in E(T)$ where u is the parent of v, and u is in a different group to v. Then for every ranked graph $(H, L, \preceq)$ in the profile of v, the $\ell(u)$-splice of $(H, L, \preceq)$ is in the profile of u.*

Proof. Since $(H, L, \preceq)$ is in the profile of v, there is an isomorphism ϕ from H to some subgraph of $G[T_v^+]$ such that for each vertex $x \in V(H)$ we have $L(x) = \ell(\phi(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in T.

Since u and v are in different groups, there are k children $y_1, \ldots, y_k$ of u (one of which is v) such that the profiles of $y_1, \ldots, y_k$ are equal. Thus, $(H, L, \preceq)$ is in the profile of each of $y_1, \ldots, y_k$. That is, for each $j \in [1, k]$, there is an isomorphism ϕ_j from H to some subgraph of $G[T_{y_j}^+]$ such that for each vertex $x \in V(H)$ we have $L(x) = \ell(\phi_j(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi_j(x)$ is an ancestor of $\phi_j(y)$ in T.

Let $(H', L', \preceq')$ be the $\ell(u)$-splice of $(H, L, \preceq)$. We now define a function ϕ' from $V(H')$ to $V(G[T_u^+])$. For each vertex $(x, 0)$ of H' (thus with $x \in V(H)$ and $L(x) \in [0, \ell(u)]$), define $\phi'((x, 0)) := \phi(x)$. For every other vertex (x, j) of H' (thus with $x \in V(H)$ and $L(x) \in [\ell(u)+1, d-1]$ and $j \in [1, k]$), define $\phi'((x, j)) := \phi_j(x)$.

We now show that ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$. Consider an edge $(x, a)(y, b)$ of H'. Thus, $xy \in E(H)$. It suffices to show that $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$. First suppose that $a = b = 0$. So $L(x) \in [0, \ell(u)]$ and $L(y) \in [0, \ell(u)]$. Thus $\phi'((x, a)) = \phi(x)$ and $\phi'((y, b)) = \phi(y)$. Since ϕ is an isomorphism to a subgraph of $G[T_v^+]$, we have $\phi(x)\phi(y) \in E(G[T_v^+])$, which is a subgraph of $G[T_u^+]$. Hence, $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$, as desired. Now suppose that $a = 0$ and $b \in [1, k]$. Thus, $\phi'((x, a)) = \phi(x)$ and $\phi'((y, b)) = \phi_b(y)$. Moreover, both $\ell(\phi(x))$ and $\ell(\phi_b(x))$ equal $L(x) \in [0, \ell(u)]$. There is only vertex z in T_v^+ with $\ell(z)$ equal to a specific number in $[0, \ell(u)]$. Thus, $\phi'((x, a)) = \phi(x) = \phi_b(x)$ $(= z)$. Since ϕ_b is an isomorphism to a subgraph of $G[T_{y_b}^+]$, we have $\phi_b(x)\phi_b(y) \in E(G[T_{y_b}^+])$, which is a subgraph of $G[T_u^+]$. Hence, $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$, as desired. Finally, suppose that $a = b \in [1, k]$. Thus, $\phi'((x, a)) = \phi_a(x)$ and $\phi'((y, b)) = \phi_b(y) = \phi_a(y)$. Since ϕ_a is an isomorphism to a subgraph of $G[T_{y_a}^+]$, we have $\phi_a(x)\phi_a(y) \in E(G[T_{y_a}^+])$, which is a subgraph of $G[T_u^+]$. Hence, $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$, as desired. This shows that ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$.

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We now verify property (A) for $(H', L', \preceq')$. For each vertex $(x, 0)$ of H' (thus with $x \in V(H)$ and $L(x) \in [0, \ell(u)]$) we have $L'((x, 0)) = L(x) = \ell(\phi(x)) = \ell(\phi'((x, 0)))$, as desired. For every other vertex (x, j) of H' (thus with $x \in V(H)$ and $L(x) \in [\ell(u)+1, d-1]$ and $j \in [1, k]$) we have $L'((x, j)) = L(x) = \ell(\phi_j(x)) = \ell(\phi'((x, j)))$, as desired. Hence, property (A) is satisfied for $(H', L', \preceq')$.

We now verify property (B) for $(H', L', \preceq')$. Consider distinct vertices $(x, a), (y, b) \in V(H')$. First suppose that $a = 0$ and $b = 0$. Then $(x, a) \prec'(y, b)$ if and only if $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in T if and only if $\phi'((x, a))$ is an ancestor of $\phi'((y, b))$ in T, as desired. Now suppose that $a = 0$ and $b \in [1, k]$. Then $(x, a) \prec'(y, b)$ if and only if $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi_b(y)$ in T if and only if $\phi'((x, a))$ is an ancestor of $\phi'((y, b))$ in T, as desired. Now suppose that $a = b \in [1, k]$. Then $(x, a) \prec'(y, b)$ if and only if $x \prec y$ if and only if $\phi_a(x)$ is an ancestor of $\phi_b(y)$ in T if and only if $\phi'((x, a))$ is an ancestor of $\phi'((y, b))$ in T, as desired. Finally, suppose that $a, b \in [1, k]$ and $a \neq b$. Then (x, a) and (y, b) are incomparable under $\prec'$, and $\phi'((x, a))$ and $\phi'((y, b))$ in T are unrelated in T, as desired. Hence, property (B) is satisfied for $(H', L', \preceq')$.

So ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$ satisfying properties (A) and (B). Thus $(H', L', \preceq')$ is contained in $G[T_u^+]$, as desired. Since $(H, L, \preceq)$ is in the profile of v, we have $|V(H)| \leqslant (d+1)(h-1)(k+1)^{h-\ell(v)}$. Since $|V(H')| \leqslant (k+1)|V(H)|$ and $\ell(u) = \ell(v) - 1$, we have $|V(H')| \leqslant (d+1)(h-1)(k+1)^{h+1-\ell(v)} = (d+1)(h-1)(k+1)^{h-\ell(u)}$. Thus, $(H', L', \preceq')$ is in the profile of u. □

The proof now divides into two cases. If some group X_0 is adjacent in G to at least $h-1$ other groups above X_0, then we show that G contains $W\langle h, k\rangle$ as a minor. Otherwise, every group X is adjacent in G to at most $h-2$ other groups above X, in which case we show that G is $(h-1)$-colourable with bounded clustering.

Finding the minor

Suppose that some group X_0 is adjacent in G to at least $h-1$ other groups $X_1, \ldots, X_{h-1}$ above X_0. We now show that G contains $W\langle h, k\rangle$ as a minor; refer to Figure 3. For $i \in [1, h-1]$, since X_i is above X_0, the root v_i of X_i is on the v_0r-path in T. Without loss of generality, $v_0, v_1, \ldots, v_{h-1}$ appear in this order on the v_0r-path in T. For $i \in [1, h-1]$, let w_i be a vertex in X_i adjacent to some vertex z_i in X_0; since G is a subgraph of the closure of T, w_i is on the v_0r-path in T. For $i \in [0, h-2]$, let u_i be the parent of v_i in T (which exists since $v_{h-2} \neq r$). So u_i is not in X_i (but may be in X_{i+1}). Note that $v_0, u_0, w_1, v_1, u_1, \ldots, w_{h-2}, v_{h-2}, u_{h-2}, w_{h-1}, v_{h-1}$ appear in this order on the v_0r-path in T, where $v_0, v_1, \ldots, v_{h-1}$ are distinct (since they are in distinct groups).

Let P_j be the z_jr-path in T for $j \in [1, h-1]$. Let H_0 be the graph with $V(H_0) := V(P_1 \cup \ldots \cup P_{h-1})$ and $E(H_0) := \{z_jw_j : j \in [1, h-1]\}$. Define the function $L_0 : V(H_0) \to [0, d-1]$ by $L_0(x) := \ell(x)$ for each $x \in V(H_0)$. Define the partial order $\preceq_0$ on $V(H_0)$, where $x \prec_0 y$ if and only if x is ancestor of y in T. Thus, $(H_0, L_0, \preceq_0)$ is a ranked graph. By construction, $(H_0, L_0, \preceq_0)$ is contained in $G[T_{v_0}^+]$. Since H_0 has less than $(d+1)(h-1)$ vertices, H_0 is in the profile of v_0. For $i = 0, 1, \ldots, h-2$, let $(H_{i+1}, L_{i+1}, \prec_{i+1})$ be the $\ell(u_i)$-splice of $(H_i, L_i, \prec_i)$.

By induction on i, using Claim 1 at each step and since $G[T_{u_i}^+] \subseteq G[T_{v_{i+1}}^+]$, we conclude that for each $i \in [0, h-1]$, the ranked graph $(H_i, L_i, \preceq_i)$ is in the profile of v_i. In particular, $(H_{h-1}, L_{h-1}, \prec_{h-1})$ is in the profile of v_{h-1}, and H_{h-1} is isomorphic to a subgraph of G. Note that each vertex of H_{h-1} is of the form $(((\ldots(x, d_1), d_2), \ldots), d_{h-1})$ for some $x \in V(H_0)$ and $d_1, \ldots, d_{h-1} \in [0, k]$. For brevity, call such a vertex $x\langle d_1, \ldots, d_{h-1}\rangle$. Note that if $x = w_j$ for some $j \in [1, h-1]$, then $d_1 = \ldots = d_j = 0$ (since w_j is above u_i whenever $i < j$, and $(H_{i+1}, L_{i+1}, \prec_{i+1})$ is the $\ell(u_i)$-splice of $(H_i, L_i, \preceq_i)$).

For $x \in V(H_0)$, let Λ_x be the set of vertices $x\langle d_1, \ldots, d_{h-1}\rangle$ in H_{h-1}. By construction, no two vertices in Λ_x are comparable under $\preceq_{h-1}$. Therefore, by property (B), $V(T_a) \cap V(T_b) = \emptyset$ for all distinct $a, b \in \Lambda_x$. In particular, $V(T_a) \cap V(T_b) = \emptyset$ for all distinct $a, b \in \Lambda_{v_0}$. As proved above,

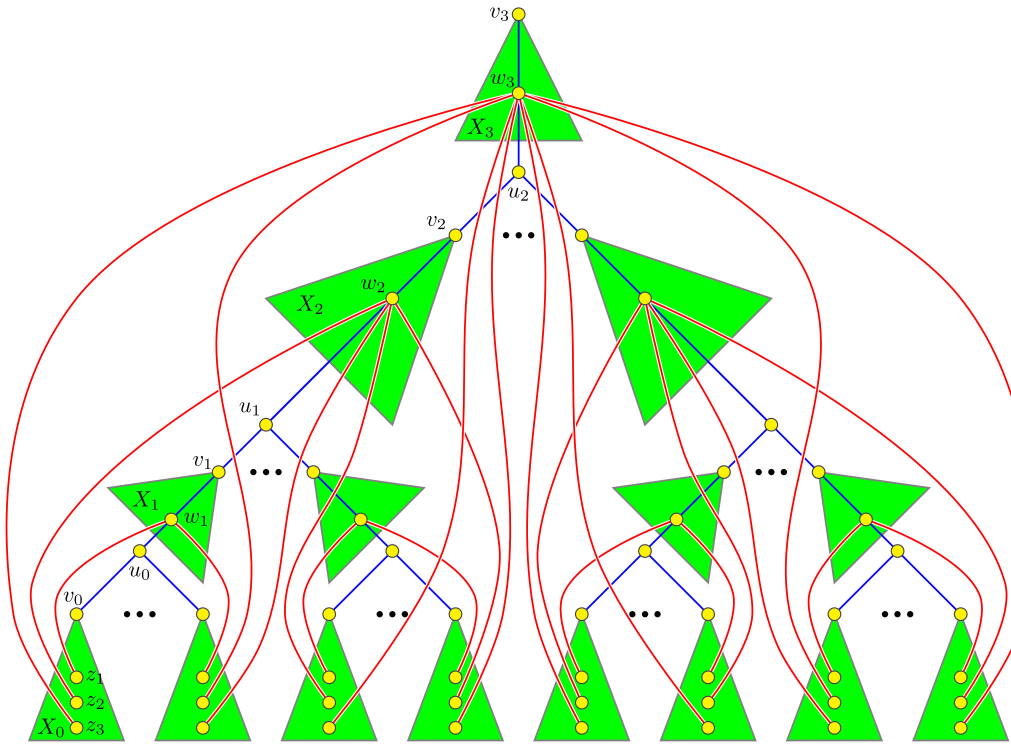

Figure 3. Construction of a $W\langle 4, k\rangle$ minor (where u_i might be in X_{i+1}).

$G[T_a]$ is connected for each $a \in V(T)$. Let G' be the graph obtained from G by contracting $G[T_a]$ into a single vertex $\alpha\langle d_1, \ldots, d_{h-1}\rangle$, for each $a = v_0\langle d_1, \ldots, d_{h-1}\rangle \in \Lambda_{v_0}$. So G' is a minor of G.

Let U be the tree with vertex set

$$\{\langle d_1, \ldots, d_{h-1}\rangle : \exists j \in [0, h-1]\ d_1 = \ldots = d_j = 0 \text{ and } d_{j+1}, \ldots, d_{h-1} \in [1, k]\},$$

where the parent of $(0, \ldots, 0, d_{j+1}, d_{j+2}, \ldots, d_{h-1})$ is $(0, \ldots, 0, d_{j+2}, \ldots, d_{h-1})$. Then U is isomorphic to the complete k-tree of height h rooted at $\langle 0, \ldots, 0\rangle$. We now show that the weak closure of U is a subgraph of G', where each vertex $\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ of U with $j \in [1, h-1]$ is mapped to vertex $w_j\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ of G', and each other vertex $\langle d_1, \ldots, d_{h-1}\rangle$ of U is mapped to $\alpha\langle d_1, \ldots, d_{h-1}\rangle$ of G'. For all $d_1, \ldots, d_{h-1} \in [1, k]$ and $j \in [1, h-1]$ the vertex $z_j\langle d_1, \ldots, d_{h-1}\rangle$ of G is contracted into the vertex $\alpha\langle d_1, \ldots, d_{h-1}\rangle$ of G'. By construction, $z_j\langle d_1, \ldots, d_{h-1}\rangle$ is adjacent to $w_j\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ in G. So $\alpha\langle d_1, \ldots, d_{h-1}\rangle$ is adjacent to $w_j\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\rangle$ in G'. This implies that the weak closure of U (that is, $W\langle h, k\rangle$) is isomorphic to a subgraph of G', and is therefore a minor of G.

Finding the colouring

Now assume that every group X is adjacent in G to at most $h-2$ other groups above X. Then $(h-1)$-colour the groups in order of distance from the root, such that every group X is assigned a colour different from the colours assigned to the neighbouring groups above X. Assign each vertex within a group the same colour as that assigned to the whole group. This defines an $(h-1)$-colouring of G.

https://doi.org/10.1017/S0963548322000165 Published online by Cambridge University Press

Consider the function $s : [0, d-1] \to \mathbb{N}$ recursively defined by

$$s(\ell) := \begin{cases} 1 & \text{if } \ell = d-1 \\ (k-1) \cdot M_{\ell+1} \cdot s(\ell+1) & \text{if } \ell \in [0, d-2]. \end{cases}$$

Then every group at level ℓ has at most $s(\ell)$ vertices. By construction, our $(h-1)$-colouring of G has clustering $s(0)$, which is bounded by a function of d, k and h, as desired. □

3. Pathwidth

The following lemma of independent interest is the key to proving Theorem 2. Note that Eppstein [24] independently discovered the same result (with a slightly weaker bound on the path length). The decomposition method in the proof has been previously used, for example, by Dujmović, Joret, Kozik, and Wood [25, Lemma 17].

Lemma 9. *Every graph with pathwidth at most w has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^w$ vertices.*

Proof. We proceed by induction on $w \geqslant 1$. Every graph with pathwidth 1 is a caterpillar, and is thus properly 2-colourable. Now assume $w \geqslant 2$ and the result holds for graphs with pathwidth at most $w-1$. Let G be a graph with pathwidth at most w. Let $(B_1, \ldots, B_n)$ be a path-decomposition of G with width at most w. Let $t_1, t_2, \ldots, t_m$ be a maximal sequence such that $t_1 = 1$ and for each $i \geqslant 2$, t_i is the minimum integer such that $B_{t_i} \cap B_{t_{i-1}} = \emptyset$. For odd i, colour every vertex in B_{t_i} 'red'. For even i, colour every vertex in B_{t_i} 'blue'. Since $B_{t_i} \cap B_{t_{i-1}} = \emptyset$ for $i \geqslant 2$, no vertex is coloured twice. Let G' be the subgraph of G induced by the uncoloured vertices. By the choice of B_{t_i}, for $i \geqslant 2$ each bag B_j with $j \in [t_{i-1}+1, t_i-1]$ intersects $B_{t_{i-1}}$. Thus, $(B_1 \cap V(G'), \ldots, B_n \cap V(G'))$ is a path-decomposition of G' of width at most $w-1$. By induction, G' has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^{w-1}$ vertices. Since $B_{t_i} \cup B_{t_{i+2}}$ separates $B_{t_i+1} \cup \ldots \cup B_{t_{i+2}-1}$ from the rest of G, each monochromatic component of G is contained in $B_{t_i+1} \cup \ldots \cup B_{t_{i+2}-1}$ for some $i \in [0, n-2]$. Consider a monochromatic path P in $G[B_{t_i+1} \cup \ldots \cup B_{t_{i+2}-1}]$. Then P has at most $w+1$ vertices in $B_{t_{i+1}}$. Note that $P - B_{t_{i+1}}$ is contained in G'. Thus, P consists of up to $w+2$ monochromatic subpaths in G' plus $w+1$ vertices in $B_{t_{i+1}}$. Hence, P has at most $(w+2)(w+3)^{w-1} + (w+1) < (w+3)^w$ vertices. □

Nešetřil and Ossona de Mendez [23] showed that if a graph G contains no path on k vertices, then $\mathrm{td}(G) < k$ (since G is a subgraph of the closure of a DFS spanning tree with height at most k). Thus Lemma 9 implies:

Corollary 10. *Every graph with pathwidth at most w has a vertex 2-colouring such that each monochromatic component has treedepth at most $(w+3)^w$.*

Proof of Theorem 2. Let $\mathcal{G}$ be a minor-closed class of graphs, each with pathwidth at most w. Let h be the minimum integer such that $C\langle h, k\rangle \notin \mathcal{G}$ for some $k \in \mathbb{N}$. Consider $G \in \mathcal{G}$. Thus, $W\langle h, k+1\rangle$ is not a minor of G (since $C\langle h, k\rangle$ is a minor of $W\langle h, k+1\rangle$, as noted above). By Corollary 10, G has a vertex 2-colouring such that each monochromatic component H of G has treedepth at most $(w+3)^w$. Thus, $W\langle h, k+1\rangle$ is not a minor of H. By Lemma 8, H is $(h-1)$-colourable with clustering $c((w+3)^w, k+1, h)$. Taking a product colouring, G is $(2h-2)$-colourable with clustering $c((w+3)^w, k+1, h)$. Hence, $\chi_\Delta(\mathcal{G}) \leqslant \chi_\star(\mathcal{G}) \leqslant 2h-2$. □

Note that Lemma 9 cannot be extended to the setting of bounded tree-width graphs: Esperet and Joret (see [[14], Theorem 4.1]) proved that for all positive integers w and d there exists a graph G with tree-width at most w such that for every w-colouring of G there exists a monochromatic component of G with diameter greater than d (and thus with a monochromatic path on more than d vertices, and thus with treedepth at least $\log_2 d$).

4. Fractional colouring

This section proves Theorem 6. The starting point is the following key result of Dvořák and Sereni [26].[2]

Theorem 11 ([26]). *For every proper minor-closed class $\mathcal{G}$ and every $\delta > 0$ there exists $d \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $s \in \mathbb{N}$ and $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that:*

- $\operatorname{td}(G[X_i]) \leqslant d$, *and*
- *every $v \in V(G)$ belongs to at least $(1 - \delta)s$ of these sets.*

We now prove a lower bound on the fractional defective chromatic number of the closure of complete trees of given height.

Lemma 12. *Let $\mathcal{C}_h := \{C\langle h, k\rangle\}_{k\in\mathbb{N}}$. Then $\chi^f_\Delta(\mathcal{C}_h) \geqslant h$.*

Proof. We show by induction on h that if $C\langle h, k\rangle$ is fractionally t-colourable with defect d, then $t \geqslant h - (h - 1)d/k$. This clearly implies the lemma. The base case $h = 1$ is trivial.

For the induction step, suppose that $G := C\langle h, k\rangle$ is fractionally t-colourable with defect d. Thus, there exist $Y_1, Y_2, \ldots, Y_s \subseteq V(G)$ and $\alpha_1, \ldots, \alpha_s \in [0, 1]$ such that:

- every component of $G[Y_i]$ has maximum degree at most d,
- $\sum_{i=1}^s \alpha_i \leqslant t$, and
- $\sum_{i:v\in Y_i} \alpha_i \geqslant 1$ for every $v \in V(G)$.

Let r be the vertex of G corresponding to the root of the complete k-ary tree and let $H_1, \ldots, H_k$ be the components of $G - r$. Then each H_i is isomorphic to $C\langle h - 1, k\rangle$. Let $J_0 := \{j : r \in Y_j\}$, and let $J_i := \{j : Y_j \cap V(H_i) \neq \emptyset\}$ for $i \in [1, k]$. Denote $\sum_{j\in J_i} \alpha_j$ by $\alpha(J_i)$ for brevity. Thus, $\alpha(J_0) \geqslant 1$. For $i \in [1, k]$, the subgraph H_i is $\alpha(J_i)$-colourable with defect d, and thus $\alpha(J_i) \geqslant h - 1 - (h - 2)d/k$ by the induction hypothesis. Thus,

$$(k - d)\alpha(J_0) + \sum_{i=1}^{k} \alpha(J_i) \geqslant (k - d) + k(h - 1) - (h - 2)d = kh - (h - 1)d.$$

If $j \in J_0$ then Y_j intersects at most d of $H_1, \ldots, H_k$ (since $G[Y_j]$ has maximum degree at most d). Thus, every α_j appears with coefficient at most k in the left side of the above inequality, implying

$$(k - d)\alpha(J_0) + \sum_{i=1}^{k} \alpha(J_i) \leqslant k \sum_{i=1}^{s} \alpha_i \leqslant kt.$$

Combining the above inequalities yields the claimed bound on t. □

Proof of Theorem 6. By Lemma 12,

$$\chi^f_\star(\mathcal{G}) \geqslant \chi^f_\Delta(\mathcal{G}) \geqslant \operatorname{tcn}(\mathcal{G}) - 1.$$

It remains to show that $\chi^f_\star(\mathcal{G}) \leqslant \operatorname{tcn}(\mathcal{G}) - 1$. Equivalently, we need to show that for all $h, k \in \mathbb{N}$ and $\varepsilon > 0$, if $C\langle h, k\rangle \notin \mathcal{G}$ then there exists c such that every graph in $\mathcal{G}$ is fractionally $(h - 1 + \varepsilon)$-colourable with clustering c. This is trivial for $h = 1$, and so we assume $h \geqslant 2$.

[2] Dvořák and Sereni [26] expressed their result in the terms of "treedepth fragility". The sentence "proper minor-closed classes are fractionally treedepth-fragile" after Theorem 31 in [26] is equivalent to Theorem 11. Informally speaking, Theorem 11 shows that the fractional "treedepth" chromatic number of every minor-closed class equals 1.

https://doi.org/10.1017/S0963548322000165 Published online by Cambridge University Press

Let $d \in \mathbb{N}$ satisfy the conclusion of Theorem 11 for the class $\mathcal{G}$ and $\delta = 1 - \frac{1}{1+\varepsilon/(h-1)}$. Choose $c = c(d, k+1, h)$ to satisfy the conclusion of Lemma 8. We show that c is as desired.

Consider $G \in \mathcal{G}$. By the choice of d there exists $s \in \mathbb{N}$ and $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that:

- $\mathrm{td}(G[X_i]) \leqslant d$, and
- every $v \in V(G)$ belongs to at least $(1-\delta)s$ of these sets.

Since $C\langle h, k\rangle \notin \mathcal{G}$, we have $W\langle h, k+1\rangle \notin \mathcal{G}$, and by the choice of c, for each $i \in [1, s]$ there exists a partition $(Y_i^1, Y_i^2, \ldots, Y_i^{h-1})$ of X_i such that every component of $G[Y_i^j]$ has at most c vertices. Every vertex of G belongs to at least $(1-\delta)s$ sets Y_i^j where $i \in [1, s]$ and $j \in [1, h-1]$. Considering these sets with equal coefficients $\alpha_i^j := \frac{1}{(1-\delta)s}$, we conclude that G is fractionally $\frac{h-1}{1-\delta}$-colourable with clustering c, as desired (since $\frac{h-1}{1-\delta} = h - 1 + \varepsilon$). □

Acknowledgement

This work was partially completed while SN was visiting Monash University supported by a Robert Bartnik Visiting Fellowship. SN thanks the School of Mathematics at Monash University for its hospitality. Thanks to the referee for several helpful comments.

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Cite this article: Norin S, Scott A, and Wood DR (2023). Clustered colouring of graph classes with bounded treedepth or pathwidth. *Combinatorics, Probability and Computing* **32**, 122–133. https://doi.org/10.1017/S0963548322000165