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OSSERMAN PSEUDO-RIEMANNIAN MANIFOLDS OF SIGNATURE (2, 2)

NOVICA BLAŽIĆ, NEDA BOKAN and ZORAN RAKIĆ

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Abstract

A pseudo-Riemannian manifold is said to be timelike (spacelike) Osserman if the Jordan form of the Jacobi operator $\mathscr{K}_X$ is independent of the particular unit timelike (spacelike) tangent vector X. The first main result is that timelike (spacelike) Osserman manifold (M, g) of signature $(2, 2)$ with the diagonalizable Jacobi operator is either locally rank-one symmetric or flat. In the nondiagonalizable case the characteristic polynomial of $\mathscr{K}_X$ has to have a triple zero, which is the other main result. An important step in the proof is based on Walker's study of pseudo-Riemannian manifolds admitting parallel totally isotropic distributions. Also some interesting additional geometric properties of Osserman type manifolds are established. For the nondiagonalizable Jacobi operators some of the examples show a nature of the Osserman condition for Riemannian manifolds different from that of pseudo-Riemannian manifolds.

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1. Introduction

Let M be an n-dimensional pseudo-Riemannian manifold with the metric tensor $\langle\cdot,\cdot\rangle$ of signature (p, q). Let S_p^+ (S_p^-) be the set of all unit spacelike (timelike) tangent vectors $X \in T_p$ at $p \in M$. Let S_p^ϵ $(\epsilon = \pm)$ be unit timelike (spacelike) sphere and for $X \in S_p^\epsilon$ denote by $T_X(S_p^\epsilon)$ the orthogonal complement of $X \in S_p^\epsilon$. The Jacobi operator, $R_X : Y \mapsto R(Y, X)X$, is a symmetric endomorphism of T_pM which restricts to the endomorphism $\mathscr{K}_X$ of $T_X S_p^\epsilon$ for $X \in S_p^\epsilon$. Note $\mathscr{K}_X$ is not necessarily diagonalizable unless $T_X S_p^\epsilon$ has definite induced metric.

For Riemannian manifolds Osserman made the following conjecture in [22].

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CONJECTURE (Osserman). If the eigenvalues of the Jacobi operator $\mathcal{K}_X$ are independent of the choice of unit vectors $X \in T_pM$ and of the choice $p \in M$, then either M is locally a rank-one symmetric space or M is flat.

The Jacobi operator of pseudo-Riemannian manifolds may not be diagonalizable. If the Jacobi operator is not diagonalizable, then M is not necessarily a locally rank-one symmetric space (it may be a rank-two symmetric space—see Example 1) or even a locally symmetric space (but may be simple harmonic—Example 3). These examples motivated the authors to introduce the notion of timelike and spacelike Osserman manifolds and to study their characterizations and possible complete classification. Also the examples show that Osserman conjecture in this form does not hold. In some of the examples the characteristic polynomial is constant, but the corresponding minimal polynomial changes from point to point, because of the nature of pseudo-Riemannian manifolds.

REMARK. Note that if $T_X S_p^\epsilon$ has definite metric then since $\mathcal{K}_X$ is diagonalizable, the Osserman condition at p is then equivalent to that the Jordan form of $\mathcal{K}_X$ is independent of X. The following definition is a natural generalization of the Osserman condition in the pseudo-Riemannian case.

DEFINITION. (1) M is *timelike (spacelike) Osserman* at p if the Jordan form of $\mathcal{K}_X$ is independent of $X \in S_p^+$ ($X \in S_p^-$).
(2) M is *pointwise timelike (spacelike) Osserman* if M is timelike (spacelike) Osserman at each $p \in M$.
(3) M is *timelike (spacelike) Osserman (globally Osserman)* if the Jordan form of $\mathcal{K}_X$ is independent of $p \in M$.

REMARK. If M is a 4-dimensional manifold of signature $(2, 2)$, the Osserman condition is equivalent to the constancy of the minimal polynomial of the Jacobi operator $\mathcal{K}_X$.

Chi [10] has proved the Osserman conjecture for $n \neq 4k$, $k > 1$. He has also obtained some related results [11, 12]. The pointwise Osserman conjecture was studied by Gilkey [17] and its relations with global Osserman condition by Gilkey, Swann and Vanhecke [18]. Osserman Lorentzian manifolds were studied by García-Rio, Kupeli and Vásquez-Abal [15] and the first two authors and Gilkey in [5]. Timelike Osserman Lorentzian manifolds of dimension $n \geq 2$ and spacelike Osserman of dimension $m \leq 4$ are of constant sectional curvature, which is shown in [15]; by using a different method it was proved true for arbitrary $m, n \geq 2$ in [5].

In this paper we study spacelike (timelike) Osserman manifolds and characterize them. More precisely, we prove the following theorem.

THEOREM 1.1 (Main Theorem). *Let M be a 4-dimensional pseudo-Riemannian manifold of signature* $(2, 2)$. *Then the following conditions are equivalent*:

(a) *M is timelike Osserman.*

(b) *M is spacelike Osserman.*

(c) *The universal covering space $\tilde{M}$ of M is one of the following manifolds*:

  (1) *$\tilde{M}$ is a manifold of constant sectional curvature.*

  (2) *$\tilde{M}$ is a Kähler manifold of constant holomorphic sectional curvature.*

  (3) *$\tilde{M}$ is a para-Kähler manifold of constant para-holomorphic sectional curvature.*

  (4) *The Jacobi operator of $\tilde{M}$ is nondiagonalizable, and its characteristic polynomial has a triple zero α and its curvature is given by* (8.1) *and* (8.2).

We remark that pseudo-Riemannian manifolds (M, g) satisfying the Osserman condition are not completely classified. More details of existing classifications of Kähler and para-Kähler pseudo-Riemannian space forms, based on the established classification of the universal covering space $\tilde{M}$, are given in Section 2.

Geometry of timelike and spacelike Osserman manifolds with the nondiagonalizable Jacobi operator has been studied in this paper as well as in [7, 6]. The manifolds have to have the characteristic polynomial with a triple zero α. So far the existence of the manifolds when $\alpha \neq 0$ has been an open problem.

For Riemannian manifolds a nice open problem whether Osserman manifolds are necessarily locally homogeneous was stated by Vanhecke (for more details see [27]). In the pseudo Riemannian setting this is not true for timelike (spacelike) Osserman manifolds (see [7]).

Let us mention that a timelike (spacelike) Osserman manifold (M, g) is an Einstein self-dual or anti-self-dual manifold (see [2]). Moreover, it admits a foliation by two-dimensional totally geodesic isotropic submanifolds. Related problems were studied from another point of view by Akivis and Konnov in [1].

Our paper is organized as follows. In Section 2 we give some basic notions and notations that we use throughout the paper. We also recall some basic facts related to certain known classification results for Kähler and para-Kähler pseudo-Riemannian space forms. In Section 3 we study the traces of $\mathscr{K}_X$ and $\mathscr{K}_X^2$ of the Jacobi operators to establish the relations between the components of the curvature tensor in the case of 4-dimensional manifolds of signature $(2, 2)$. We prove that timelike (spacelike) Osserman manifolds are Einsteinian too. Section 4 is devoted to the linear algebra of symmetric operators in pseudounitary spaces, and especially to symmetric operators in dimension three. In Section 5 we determine the components of the curvature tensor, assuming timelike Osserman condition is fulfilled. It is shown that all timelike (spacelike) Osserman manifolds are curvature homogeneous. The proof of the Main Theorem is a consequence of the results in Section 6–Section 8. In

Section 6 we investigate the case when the Jacobi operator is diagonalizable. We show that these manifolds are locally rank-one symmetric or flat spaces and we may say that the Osserman conjecture holds. We also prove the existence and the integrability of an almost complex and a para-complex structure in this case. In Section 7 we show that there does not exist a manifold whose characteristic polynomial of the Jacobi operator has a complex root. In Section 8 we investigate the case when the characteristic polynomial of the nondiagonalizable Jacobi operator has a multiple root. It is shown that the characteristic polynomial is necessarily with a triple root α. The most interesting part of the proof is the nonexistence of timelike Osserman manifolds in that class whose characteristic polynomials have roots $\alpha, \alpha, 4\alpha \neq 0$. It is based on Walker's local classification of the manifolds which admit a parallel null distribution and some unexpected cancellations which took place along the necessary computations. We also describe the curvature tensors of the manifold and we find that they are, in both cases, very similar. We also consider the examples of timelike Osserman manifolds for $\alpha = 0$ to show that these manifolds have very interesting and rich geometry. In Section 9, we prove, using some results of Wu [30], the existence of a timelike Osserman locally rank-two symmetric space endowed with an integrable para-quaternionic structure with holonomy algebra given by (9.3) and (9.4). Nonsymmetric Ricci flat Osserman manifolds that are not even locally homogeneous are also discussed.

2. Preliminaries

Let M be a pseudo-Riemannian manifold of dimension n, $p + q = n$ with the metric $\langle\cdot,\cdot\rangle$ of signature (p, q). For convenience we also use the notation g for the metric. Let $\epsilon_i = -1$ for $i = 1, \dots, p$ and $\epsilon_i = +1$ for $i = p+1, \dots, n$. We denote by $e_1, \dots, e_n$ an orthonormal basis of M,

$$\langle e_i, e_j\rangle = \epsilon_i \delta_{ij}. \tag{2.1}$$

Let TM be the tangent bundle of M and X, Y, Z arbitrary vector fields. If ∇ is the Levi-Civita connection, then $R(X, Y) : T_pM \to T_pM$ is the pseudo-Riemannian curvature operator given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.2}$$

Let ω, Ω be the connection 1-forms and the curvature 2-forms respectively. Then we have

$$\nabla e_i = \sum \omega_i^s e_s, \quad i, s = 1, \dots, n, \tag{2.3}$$

$$\Omega_j^i = \frac{1}{2}\sum R_{klj}^i \theta^k \wedge \theta^l. \tag{2.4}$$

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To explain the geometry of spacelike and timelike Osserman manifolds of signature (2, 2) we recall here some basic facts related to certain known classification results for the universal coverings of Kähler and para-Kähler pseudo-Riemannian space forms. The curvature tensor of pseudo-Riemannian manifold (M^n, g) of signature $(s, n-s)$ of constant sectional curvature c is

$$R(u, v)w = c\{g(v, w)u - g(u, w)v\}, \tag{2.5}$$

for $u, v, w \in T_pM$. Two such complete, connected, simply connected, manifolds of the same sectional curvature c are isometric. Particularly, it is interesting for us that there exists a classification of space forms M_2^4 for $c \neq 0$ (see [29]). Their universal pseudo-Riemannian coverings S_2^4 and H_2^4 are spheres in the pseudo-Euclidean spaces $\mathbb{R}_2^5$ and $\mathbb{R}_3^5$ respectively [29].

Analogous to the projective space $\mathbb{CP}^n$, the indefinite projective space of signature $(2s, 2n-2s)$ can be constructed (for details see for example [4]). It is a Kähler space form of the constant sectional curvature c, $c \neq 0$, and its curvature tensor is

$$\begin{aligned} R(u, v)w = (c/4)\{&g(v, w)u - g(u, w)v + g(Jv, w)Ju - g(Ju, w)Jv \\ &- 2g(Ju, v)Jw\}, \end{aligned} \tag{2.6}$$

for all $u, v, w \in T_pM$. We also have that every connected, simply connected, complete indefinite Kähler manifold of complex dimension n, of signature $(2s, 2n-2s)$ and of constant holomorphic sectional curvature c, $c \neq 0$, is holomorphically isometric to $\mathbb{CP}_s^n(c)$ (see [4, Theorem 3.4]). Therefore only $\mathbb{CP}_1^2(c)$ is interesting for us.

The tangent bundle TS^n of the n-sphere can be equipped with a pseudo-Riemannian metric g of signature (n, n) and a para-complex structure such that $P^n(B) = (TS^n, g, F)$ is of constant para-holomorphic sectional curvature c, $c \neq 0$. For $n > 1$, $P^n(B)$ is complete, connected and simply connected (see [14]). Two such complete, connected, simply connected manifolds of the same para-holomorphic sectional curvature c are F-isometric. Particularly, for $n = 2$, a complete, connected, para-Kähler manifold of constant para-holomorphic sectional curvature c, $c \neq 0$, is F-holomorphically isometric to the space $P^2(B) \approx TS^2$ or to the space $P^2(B)/\mathbb{Z}_2 \approx T\mathbb{RP}^2$ (see [14, Corollary 2]). Their curvature tensor is

$$\begin{aligned} R(x, y)z = (c/4)\{&g(v, w)u - g(u, w)v - g(Jv, w)Ju \\ &+ g(Ju, w)Jv + 2g(Ju, v)Jw\}. \end{aligned} \tag{2.7}$$

It is interesting to notice that indefinite Kähler manifolds with vanishing holomorphic sectional curvature and para-Kähler manifolds with vanishing para-holomorphic sectional curvature are also flat. But the complete classification of flat pseudo-Riemannian manifolds is not known (see [29, Section 37, page 334]).

3. Characteristic polynomial and its coefficients

In this section we study the traces of $\mathcal{K}_X$ and $\mathcal{K}_X^2$ for an arbitrary pseudo-Riemannian manifold with a metric of signature (p, q). We prove that a timelike Osserman manifold with a metric of arbitrary signature (p, q) is an Einstein space. Then we use it to establish the relations between the components of the curvature tensor in the case of 4-dimensional manifold of signature $(2, 2)$.

Let $i < j$ be two fixed indices and $\delta \in \{1, -1\}$. We now choose orthogonal vectors X and Y such that

$$X = \alpha e_i + \beta e_j, \qquad |X|^2 = \alpha^2\epsilon_i + \beta^2\epsilon_j = \delta, \tag{3.1}$$

$$Y = \beta\epsilon_j e_i - \alpha\epsilon_i e_j, \qquad |Y|^2 = \epsilon_i\epsilon_j\delta. \tag{3.2}$$

In our study of timelike Osserman spaces we assume $\delta = -1$. For $\delta = +1$, we can obtain similar results for spacelike Osserman spaces. Let us now make a hyperbolic rotation in the i-th and j-th coordinate to create a new orthonormal frame $\{E_i\}$, or more precisely,

$$E_1 = e_1, \dots, E_i = \alpha e_i + \beta e_j, \dots, E_j = \beta\epsilon_j e_i - \alpha\epsilon_i e_j, \dots, E_n = e_n. \tag{3.3}$$

Because of (3.1) we have

$$\operatorname{tr}\mathcal{K}_X = \beta^2(\rho_{jj} - \epsilon_i\epsilon_j\rho_{ii}) + 2\alpha\beta\rho_{ji} - \epsilon_i\rho_{ii}, \tag{3.4}$$

where, $\rho_{kl} = \rho(e_k, e_l) = \sum_m \epsilon_m\langle R(e_m, e_k)e_l, e_m\rangle$ is the corresponding Ricci curvature.

PROPOSITION 3.1. *If M is a timelike Osserman manifold with a metric of arbitrary signature (p, q), then M is an Einstein space. Specially, if M is a 4-dimensional timelike Osserman manifold with the metric of signature $(2, 2)$, then the orthogonal sectional curvatures are the same.*

PROOF. Since $\operatorname{tr}\mathcal{K}_X$ should be independent on β the relation (3.4) yields that M satisfies the Einsteinian condition $\rho(X, Y) = c\langle X, Y\rangle$ when X or Y is a timelike vector. Since M satisfies the Einsteinian condition for all timelike vectors, then from [13, Theorem 3.1] it follows that M is Einsteinian. The second statement is direct consequence of the Einsteinian condition for timelike vectors. □

We now recall the well-known formula

$$\operatorname{tr}\mathcal{K}_X^2 = -2\sigma_2 + (\operatorname{tr}\mathcal{K}_X)^2, \tag{3.5}$$

where σ_2 is the coefficient of λ^{n-2} of the characteristic polynomial. Using the relation (3.5) and the Einsteinian condition one can find $\operatorname{tr}\mathcal{K}_X^2$. One can see that $\operatorname{tr}\mathcal{K}_X^2$ is a polynomial in $\beta^4, \beta^2, \alpha\beta^3$ and $\alpha\beta$. Since $\operatorname{tr}\mathcal{K}_X^2$ has to be independent of the unit vector X, it yields that the coefficients with $\beta^4, \beta^2, \alpha\beta^3$ are vanishing.

4. Symmetric operators in 3-dimensional pseudo-Riemannian spaces

Let Ω be a 3-dimensional pseudo-Riemannian space, that is, $\mathbb{R}^3$ endowed with a bilinear, symmetric, nondegenerate form of signature $(1, 2)$. There exists in Ω a main (pseudo-orthogonal) basis e_1, e_2, e_3 such that we have $\langle x, x\rangle = -\xi_1^2 + \xi_2^2 + \xi_3^2$ for $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$.

The matrix of an arbitrary symmetric operator $\mathscr{A}$ with respect to a main basis is the following one

$$\begin{bmatrix} a_{11} & -a_{12} & -a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}. \tag{4.1}$$

We use the theory of the reduction of symmetric operators in the pseudo-unitary space (see [20]), to see that the following theorem holds.

THEOREM 4.1. *Let $\mathscr{K}$ be a symmetric operator of Ω. Then there exists a main basis in Ω such that the matrix of $\mathscr{K}$ is consequently one of the following*

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \tag{4.2}$$

$$\begin{bmatrix} \epsilon(\alpha - 1/2) & \epsilon(1/2) & 0 \\ -\epsilon(1/2) & \epsilon(\alpha + 1/2) & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \epsilon = \pm \tag{4.3}$$

$$\begin{bmatrix} \alpha & 0 & \sqrt{2}/2 \\ 0 & \alpha & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & \alpha \end{bmatrix}, \tag{4.4}$$

$$\begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \beta \neq 0 \tag{4.5}$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$, depending on the minimal polynomial $\mu_k(\lambda)$.

PROOF. The theorem follows from the analysis of all the possibilities for the minimal polynomial $\mu_k(\lambda)$ and from [20]. The following four cases are possible.

(i1) All ireducible factors of $\mu_k(\lambda)$ are linear in $\mathbb{R}[X]$ and the matrix of $\mathscr{K}$ in some main basis is given by (4.2).

(i2) If there exists an ireducible factor of $\mu_k(\lambda)$ of degree 2, then there exists a main basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$ such that the matrix of $\mathscr{K}$ is of the form (4.3).

(i3) If $\mu_k(\lambda) = (\lambda - \alpha)^3$, then there exists a main basis $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ such that the matrix of the operator $\mathcal{K}$ is given by (4.4).

(i4) If $\mu_k(\lambda)$ has a complex root $z = \alpha + i\beta$, $\beta \neq 0$, then there exists a basis $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ such that the matrix of $\mathcal{K}$ is of the form (4.5). □

Theorem 4.1 enables us to introduce the type of a 4-dimensional timelike Osserman manifold with signature (2, 2) as follows.

DEFINITION 4.2. Let M be a 4-dimensional timelike (spacelike) Osserman manifold of signature (2, 2). We say that M is of type

(Ia) if $\mathcal{K}_X$ is diagonalizable;
(Ib) if the characteristic polynomial of the Jacobi operator $\mathcal{K}_X$ has a complex root;
(II) if the minimal polynomial of the Jacobi operator $\mathcal{K}_X$ has a double root α;
(III) if the minimal polynomial of the Jacobi operator $\mathcal{K}_X$ has a triple root α.

5. The components of the curvature tensor

In this section, as the first step, we find the components of the curvature tensor in a main basis at a fixed point, studying all the possibilities for the matrix of the Jacobi operator that appear in Theorem 4.1. Then we can extend it to a smooth local moving frame in a neighbourhood U of an arbitrary point. The Jacobi operator is of the same type along U and the moving frame forms a main basis at every point of U. To establish the existence of such a moving frame we will choose a neighbourhood U to be contractable. Then the existence follows from the fact that a vector bundle over a contractable base has to be trivial.

Of course, we assume from now on that M is a 4-dimensional timelike Osserman manifold with the metric $\langle\cdot,\cdot\rangle$ of signature (2, 2). All the components of curvature tensor in a main basis for a timelike Osserman manifold are constant, see Theorem 5.1.

In this section we find the full curvature tensor of the endomorphism $\mathcal{K}_X$ for all types of manifolds from Definition 4.2. More precisely, we have

THEOREM 5.1. *Let M be a 4-dimensional timelike Osserman manifold with the metric of signature* (2, 2).

(a) *If M is of type* (Ia), *then the only non-vanishing components of the curvature tensor* (*with respect to main basis from Theorem* 4.1) *are*:

$$R_{3443} = R_{2112} = a, \quad R_{2332} = R_{4114} = -c, \quad R_{4224} = R_{3113} = -b,$$
$$R_{1234} = x = \varepsilon(b + c - 2a)/3, \quad R_{1423} = y = \varepsilon(b + a - 2c)/3, \quad \textit{where } \varepsilon = \pm 1.$$

(b) *If M is of type* (Ib) *or* (II), *then there exists a main basis such that the only non-vanishing components of the curvature tensor are*:

$$\begin{aligned} &R_{1221} = R_{4334} = a, && R_{2113} = R_{2443} = -\gamma, && R_{1234} = x = (-2a + b + c)/3, \\ &R_{1331} = R_{4224} = -b, && R_{1224} = R_{1334} = \bar{\gamma}, && R_{1423} = y = (a + b - 2c)/3, \\ &R_{1441} = R_{3223} = -c, && R_{1342} = -x - y, && \bar{\gamma} \in \{-\gamma, \gamma\}, \quad \varepsilon = \gamma/\bar{\gamma}, \end{aligned}$$

where $x - y = c - a$, $2x + y = b - a$, $x + 2y = b - c$, $\varepsilon = 1$.

(c) *If M is of type* (III), *then the only non-vanishing components of the curvature tensor* (*with respect to main basis from Theorem* 4.1) *are*:

$$R_{1221} = R_{4334} = -R_{1331} = -R_{4224} = -R_{1441} = -R_{3223} = \alpha,$$
$$R_{2114} = R_{2334} = -R_{3114} = R_{3224} = k, \quad R_{1223} = R_{1443} = R_{1332} = -R_{1442} = \bar{k},$$

where $\bar{k} \in \{k, -k\}$, $\varepsilon = k/\bar{k}$.

In all cases the main basis with $\varepsilon = 1$ *exists.*

PROOF. The proof follows by long and straightforward calculations using the Osserman condition, which implies the constancy of $\operatorname{tr} \mathscr{K}_X$, $\operatorname{tr} \mathscr{K}_X^2$ and $\operatorname{tr} \mathscr{K}_X^3$ on the unit vector X. □

6. Geometry of manifolds when $\mathscr{K}_X$ is diagonalizable

This section is devoted to the study of Osserman manifolds with the diagonalizable Jacobi operator. In the first step we use the formula for the covariant differentiation of the curvature tensor R as well as the second Bianchi identity and the properties of the Ricci tensor ρ to prove that it is a locally symmetric manifold. As a consequence, in the second step we consider in more details the existence of certain complex and para-complex structures. Moreover, in this section it is confirmed that the modified Osserman conjecture for pseudo-Riemannian manifolds holds under the additional assumption that the Jacobi operator is diagonalizable.

Throughout this section we use the basis e_1, e_2, e_3, e_4 of the tangent space T_pM given by (3.1)–(3.3) and its dual basis denoted by $\theta^1, \theta^2, \theta^3, \theta^4$. We denote by ω_i^j, Ω_i^j the connection 1-forms and the curvature 2-forms respectively.

6.1. The components of the covariant derivative of the curvature tensor when $\mathscr{K}_X$ is diagonalizable. The main purpose of this subsection is to prove that $\nabla R = 0$ if $\mathscr{K}_X$ is diagonalizable.

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We combine (2.3) with the components of the curvature tensor for type (Ia) (Theorem 5.1 (a)) to compute all curvature 2-forms Ω_i^j

$$\begin{aligned}
&\Omega_3^2 = \Omega_2^3 = y\theta^1 \wedge \theta^4 + c\theta^2 \wedge \theta^3, &&\Omega_1^4 = \Omega_4^1 = y\theta^2 \wedge \theta^3 + c\theta^1 \wedge \theta^4,\\
(6.1)\quad &\Omega_4^2 = \Omega_2^4 = (x+y)\theta^1 \wedge\theta^3 + b\theta^2\wedge\theta^4, &&\Omega_3^1 = \Omega_1^3 = (x+y)\theta^2\wedge\theta^4 + \theta^1\wedge\theta^3,\\
&\Omega_4^3 = -\Omega_3^4 = -x\theta^1\wedge\theta^2 + a\theta^3\wedge\theta^4, &&\Omega_1^2 = -\Omega_2^1 = -x\theta^3\wedge\theta^4 + a\theta^1\wedge\theta^2.
\end{aligned}$$

The following proposition deals with the symmetry properties of the connection 1-forms which allow us to prove $\nabla R = 0$ and the integrability of an almost complex and a para-complex structure.

PROPOSITION 6.1. *The eigenvalues a, b, c from Theorem* 5.1 (a) *cannot be all different.*

PROOF. We combine (6.1) and the components of the curvature tensor with the structural equations and their differentials,

$$(6.2)\qquad d\Omega_i^j = \sum_s (\Omega_s^j \wedge \omega_i^s - \omega_s^j \wedge \Omega_i^s),$$

for Ω_2^1, Ω_3^1 and Ω_4^1 to obtain

$$\begin{aligned}
&(a-c)B \wedge (\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3) + (b-a)C \wedge (\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4) = 0,\\
(6.3)\quad &(b-c)A \wedge (\theta^2 \wedge \theta^3 - \theta^1 \wedge \theta^4) + (b-a)C \wedge (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) = 0,\\
&(b-c)A \wedge (\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4) + (c-a)B \wedge (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) = 0,
\end{aligned}$$

where

$$\begin{aligned}
&A = \omega_1^2 - \omega_3^4 = A_1\theta^1 + A_2\theta^2 + A_3\theta^3 + A_4\theta^4,\\
(6.4)\quad &B = \omega_1^3 + \omega_2^4 = B_1\theta^1 + B_2\theta^2 + B_3\theta^3 + B_4\theta^4,\\
&C = \omega_1^4 - \omega_2^3 = C_1\theta^1 + C_2\theta^2 + C_3\theta^3 + C_4\theta^4.
\end{aligned}$$

Let us assume that a, b, c are different. One can see that the equations (6.3) form a linear system of equations. Solving this system we get

$$\begin{aligned}
&B_1 = sA_4, \quad B_2 = -sA_3, \quad B_3 = -sA_2, \quad B_4 = sA_1,\\
(6.5)\quad &C_1 = tA_3, \quad C_2 = tA_4, \quad C_3 = tA_1, \quad C_4 = tA_2,\\
&\text{for} \quad s = \frac{b-c}{a-c} \neq 0, \quad t = \frac{b-c}{a-b} \neq 0.
\end{aligned}$$

Now, we introduce 1-forms φ^1, φ^2, φ^3 and φ^4 as follows:

$$(6.6)\qquad A = \varphi^2, \quad B = s\varphi^3, \quad C = t\varphi^4,$$

$$(6.7)\qquad \varphi^1 = A_2\theta^1 - A_1\theta^2 - A_4\theta^3 + A_3\theta^4.$$

Using the structural equations and the second Bianchi identity, it follows from (6.6) that

$$(6.8)\qquad \begin{gathered} d\varphi^2 = \Omega_1^2 - \Omega_3^4 - st\varphi^3 \wedge \varphi^4, \qquad d\varphi^3 = \Omega_1^4 + \Omega_2^4 - t\varphi^2 \wedge \varphi^4, \\ d\varphi^4 = \Omega_1^4 - \Omega_2^3 - s\varphi^3 \wedge \varphi^2. \end{gathered}$$

First notice that $\varphi^1, \varphi^2, \varphi^3$ and φ^4 form an orthogonal basis of T_p^*M. Now we compute the components of the covariant derivatives of the forms A, B and C, and then we use (6.6)–(6.8) to find the divergence of φ^1

$$(6.9)\qquad \begin{aligned} \operatorname{div}\varphi^1 &= -A_{2;1} + A_{1;2} - A_{4;3} + A_{3;4}, & \operatorname{div}\varphi^1 &= 2(x-a) + st\Pi, \\ s\operatorname{div}\varphi^1 &= 2(b+x+y) + t\Pi, & t\operatorname{div}\varphi^1 &= 2(y-c) + s\Pi, \end{aligned}$$

where we put $\Pi = -A_1^2 - A_2^2 + A_3^2 + A_4^2 = \|A\|^2$. From the last three equalities of (6.9) we get:

$$(6.10)\qquad 3\operatorname{div}\varphi^1 = \frac{\Pi}{(a-c)(a-b)}[(b-c)^2 + (a-c)^2 + (a-b)^2].$$

Now we have two possibilities either $\Pi \neq 0$ or $\Pi = 0$. If $\Pi \neq 0$ we will express $\operatorname{div}\varphi^1$ in another way. Let η_1, η_2, η_3 and η_4 denote the corresponding dual basis of φ^1, φ^2, φ^3 and φ^4 in T_pM. Then

$$(6.11)\qquad \operatorname{div}\varphi^1 = \Lambda_1^2(\eta_2) + \Lambda_1^3(\eta_3) + \Lambda_1^4(\eta_4),$$

where Λ_j^i are the connection forms with respect to the base $\{\eta_i\}$. Notice that

$$(6.12)\qquad \begin{aligned} \varphi^1 \wedge \varphi^2 - \varphi^3 \wedge \varphi^4 &= -\Pi(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4), \\ \varphi^1 \wedge \varphi^3 - \varphi^2 \wedge \varphi^4 &= \Pi(\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4), \\ \varphi^1 \wedge \varphi^4 + \varphi^2 \wedge \varphi^3 &= -\Pi(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3). \end{aligned}$$

Consequently, the differentials $d\varphi^2, d\varphi^3$ and $d\varphi^4$ can be expressed in terms of φ^i, since, $d\varphi^i = -\Lambda_1^i(\eta_j)\varphi^j \wedge \varphi^1 + \dots$, where $\dots$ denotes forms not containing $\varphi^j \wedge \varphi^1$. We finally find

$$(6.13)\qquad \Lambda_1^2(\eta_2) = \frac{x-a}{\Pi}, \quad \Lambda_1^3(\eta_3) = \frac{b+x+y}{s\Pi}, \quad \Lambda_1^4(\eta_4) = \frac{y-c}{t\Pi}.$$

By direct computations, from (6.11) and (6.13), we see that $\operatorname{div}\varphi^1 = 0$. This is a contradiction with (6.10) and $\Pi \neq 0$.

The second case, $\Pi = 0$, leads directly to the contradiction with the initial assumptions. Now we apply the relations (6.9) to obtain

$$a - x = (b+x+y)/(-s) = (c-y)/t,$$

and consequently $a = b = c = 0$. This is a contradiction with the assumption that a, b and c are all different. □

COROLLARY 6.2.

(a) *If* $a = b \neq c$, *then* $\omega_1^3 + \omega_2^4 = 0$ *and* $\omega_1^2 - \omega_3^4 = 0$.
(b) *If* $a = c \neq b$, *then* $\omega_2^3 - \omega_1^4 = 0$ *and* $\omega_1^2 - \omega_3^4 = 0$.
(c) *If* $b = c \neq a$, *then* $\omega_2^3 - \omega_1^4 = 0$ *and* $\omega_1^3 + \omega_2^4 = 0$.

PROOF. (a) follows directly from the first two equations of (6.3). (b) and (c) can be obtained in a similar way. □

REMARK. Let us mention, that for example if $a = b$, then $c = 4b = 4a$, and similarly, if $b = c$, then $a = 4b = 4c$.

LEMMA 6.3. *M is a locally symmetric space, that is,* $\nabla R = 0$.

PROOF. We combine the components of curvature tensor and the symmetries of the connection one-forms and the curvature two-forms to see that

(6.14) $$R_{ijkl;h} = 0 \quad \text{if } i, j, k, l \text{ are different,}$$

(6.15) $$R_{ijji;h} = 0,$$

(6.16) $$\sum_h R_{jikj;h}\theta^h = -\sum_s R_{sikj}\omega_j^s - \sum_s R_{jiks}\omega_j^s - \mathscr{S}_{kj}\omega_i^k - \mathscr{S}_{ij}\omega_k^i.$$

One can use (6.15) and the Bianchi identity as well as Theorem 5.1 (a) to check

(6.17) $$\begin{aligned} \sum_h R_{1242;h}\theta^h &= \sum_h R_{1231;h}\theta^h = (b-a)(\omega_2^3 - \omega_1^4), \\ \sum_h R_{2132;h}\theta^h &= \sum_h R_{1241;h}\theta^h = (c-a)(\omega_2^4 + \omega_1^3), \\ \sum_h R_{1323;h}\theta^h &= \sum_h R_{1341;h}\theta^h = (b-c)(\omega_1^2 - \omega_3^4). \end{aligned}$$

Let $a = b \neq c$. Then Corollary 6.2 (a) and the relations (6.17) imply $R_{1242;h} = R_{2131;h} = R_{2132;h} = R_{1241;h} = R_{3123;h} = R_{1341;h} = 0$. These relations together with (6.7) and (6.9) give $\nabla R = 0$.

Similarly we can prove that $\nabla R = 0$ if $a = c \neq b$ and $b = c \neq a$. If $a = b = c$, then (6.16) imply $R_{1242;h} = R_{2131;h} = R_{2132;h} = R_{1241;h} = R_{3123;h} = R_{1341;h} = 0$. These relations together with (6.6) imply $\nabla R = 0$. □

6.2. An existence and the integrability of an almost complex and a para-complex structure. In this subsection we construct an almost complex structure and a para-complex structure on a contractable neighbourhood U_p of arbitrary point p of a manifold M satisfying respectively the condition $a = b \neq c$ or $a = c \neq b$ and $b = c \neq a$. Since $a = b \neq c$ and $a = c \neq b$ are equivalent conditions we study only the first one. We prove also that these structures are integrable.

Case I. We consider $a = b \neq c$. We notice that the line bundle over the timelike unit sphere $S^- U_p$ is trivial. We will use this fact for the line bundle obtained by the eigenspaces of $\mathscr{K}_X$ at the corresponding points. Using the fixed global section of this bundle for arbitrary E_1 we define $F(E_1) = E_4$ as the value of the section at the corresponding point. We complete now the definition of the endomorphism F by using a fixed basis E_1, E_2, E_3, E_4 such that

$$F(E_1) = E_4, \quad F(E_2) = -E_3, \quad F(E_4) = E_1, \quad F(E_3) = -E_2. \tag{6.18}$$

One can check that $F^2 = \mathrm{id}$, $\langle F\cdot, F\cdot\rangle = -\langle\cdot,\cdot\rangle$, and consequently F is a para-complex structure on M. For more details about a para-complex structure see for example [14]. We note that the basis E_1, E_2, E_3, E_4 depends on the choice of the vector E_1. So it is interesting that endomorphism F does not depend on the choice of E_1.

LEMMA 6.4. *If E_1, E_2, E_3, E_4 and $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ are bases of T_pM such that their curvature tensors are both given as in Theorem* 5.1 (a) *for $a = b \neq c$, then they define the same para-complex structure.*

PROOF. Let F be given by (6.18) and let $\tilde{F}$ be the corresponding endomorphism with respect to the base $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$. It is enough to check that

$$\mathscr{K}_{\tilde{E}_1}(F\tilde{E}_i) = aF\tilde{E}_i, \quad \mathscr{K}_{\tilde{E}_1}(F\tilde{E}_4) = aF\tilde{E}_4, \tag{6.19}$$

for $i = 2, 3$. If we put $\tilde{E}_j = \alpha_j E_j$, using the known components of the curvature tensor we directly verify formula (6.19). □

Case II. If $b = c \neq a$, we define similarly a field of endomorphisms J on TU_p such that

$$J(E_1) = E_2, \quad J(E_2) = -E_1, \quad J(E_3) = -E_4, \quad J(E_4) = E_3. \tag{6.20}$$

One can check that J satisfies, $J^2 = -\mathrm{id}$, $\langle J\cdot, J\cdot\rangle = \langle\cdot,\cdot\rangle$, and consequently J is an almost complex structure. We also have that the almost complex structure J is independent of the choice of E_1 (similar to Lemma 6.4).

LEMMA 6.5. *The almost complex structure J and the para-complex structure F are integrable.*

PROOF. To prove the integrability of the para-complex structure F we need to prove that $\nabla F = 0$. We have, for example,

$$\begin{aligned} \nabla F(E_1) - F(\nabla E_1) &= \nabla E_4 - F(\omega_1^i E_i) = \omega_4^i E_i - \omega_1^i (F E_i) \\ &= (\omega_4^1 - \omega_1^4)E_1 + (\omega_4^2 + \omega_1^3)E_2 + (\omega_1^2 + \omega_4^3)E_3 = 0. \end{aligned} \tag{6.21}$$

Similarly, one can check

$$\nabla F(E_i) - F(\nabla E_i) = 0, \quad i = 2, 3, 4. \tag{6.22}$$

We use now (6.21) to prove that $\nabla F = 0$. The integrability of the almost complex structure J in Case II can be proved similarly. □

Note that if the manifold M is orientable we can define respectively para-Kähler and Kähler structures F and J, globally on M.

REMARK. The results of this section show that Osserman conjecture holds when the Jacobi operator is diagonalizable. Under the given assumption that $\mathcal{K}_X$ is diagonalizable in this section it was shown that (M, g) is in one of the described in the Main Theorem (c), (1)–(3). If $a = b = c$, then M has constant sectional curvature.

7. The case when $\mathcal{K}_X$ has a complex eigenvalue

The main purpose of this section is to prove that there does not exist a timelike or spacelike Osserman manifold of type (Ib), that is, the characteristic polynomial of $\mathcal{K}_X$ has no complex zero. Because of Theorem 5.1 (b) it is enough to consider formulae with $\varepsilon = 1$. First we compute the curvature 2-forms Ω_i^j using (2.4) and Theorem 5.1 (b)

$$\begin{aligned}
\Omega_2^1 &= -\Omega_1^2 = -a\theta^1 \wedge \theta^2 + \gamma\theta^1 \wedge \theta^3 + \gamma\theta^2 \wedge \theta^4 + x\theta^3 \wedge \theta^4, \\
\Omega_3^1 &= \Omega_1^3 = \gamma\theta^1 \wedge \theta^2 + b\theta^1 \wedge \theta^3 + (x+y)\theta^2 \wedge \theta^4 + \gamma\theta^3 \wedge \theta^4, \\
\Omega_4^1 &= \Omega_1^4 = c\theta^1 \wedge \theta^4 + y\theta^2 \wedge \theta^3, \\
\Omega_3^2 &= \Omega_2^3 = y\theta^1 \wedge \theta^4 + c\theta^2 \wedge \theta^3, \\
\Omega_4^2 &= \Omega_2^4 = \gamma\theta^1 \wedge \theta^2 + (x+y)\theta^1 \wedge \theta^3 + b\theta^2 \wedge \theta^4 + \gamma\theta^3 \wedge \theta^4, \\
\Omega_4^3 &= -\Omega_3^4 = -x\theta^1 \wedge \theta^2 - \gamma\theta^1 \wedge \theta^3 - \gamma\theta^2 \wedge \theta^4 + a\theta^3 \wedge \theta^4.
\end{aligned} \tag{7.1}$$

Let us mention that the curvature 2-forms Ω_i^j for $\mathcal{K}_X$ with a double real zero of its minimal polynomial are also given by these formulae.

We use the same notations A, B, C for the corresponding 1-forms as in Subsection 6.1. Then we use the analogous procedure for curvature 2-forms Ω_2^1, Ω_3^1, Ω_4^1 as in Subsection 6.1. We introduce now 1-forms $\tilde{A}$, C, $\tilde{B}$ as follows

$$\begin{aligned}
\tilde{A} &= (c-a)B + \gamma A = 2\gamma(-C_3\theta^1 - C_4\theta^2 - C_1\theta^3 - C_2\theta^4) = 2\gamma\varphi^3, \\
C &= C_1\theta^1 + C_2\theta^2 + C_3\theta^3 + C_4\theta^4 = \varphi^2, \\
\tilde{B} &= \gamma B + (a-c)A = 2\gamma(C_2\theta^1 - C_1\theta^2 - C_4\theta^3 + C_3\theta^4) = 2\gamma\varphi^4.
\end{aligned} \tag{7.2}$$

Using the second Bianchi identity one can check

$$\begin{aligned} dC &= (1/p)\tilde{A} \wedge \tilde{B} + \Omega_1^4 - \Omega_2^3, \\ d\tilde{A} &= sC \wedge \tilde{B} + tC \wedge \tilde{A} + (c-a)(\Omega_1^3 + \Omega_2^4) + \gamma(\Omega_1^2 - \Omega_3^4), \\ d\tilde{B} &= sC \wedge \tilde{A} - tC \wedge \tilde{B} + \gamma(\Omega_1^3 + \Omega_2^4) + (a-c)(\Omega_1^2 - \Omega_3^4), \end{aligned} \tag{7.3}$$

where

$$\gamma \neq 0, \quad p = \gamma^2 + (a-c)^2 \neq 0, \quad s = \frac{p - 2(a-c)^2}{p}, \quad t = \frac{2\gamma(c-a)}{p}. \tag{7.4}$$

Let us introduce the following notations

$$\Pi = -C_1^2 - C_2^2 + C_3^2 + C_4^2, \tag{7.5}$$

$$\Sigma = -C_{1;1} - C_{2;2} + C_{3;3} + C_{4;4}, \tag{7.6}$$

$$\rho = -C_{3;2} + C_{4;1} - C_{1;4} + C_{2;3}. \tag{7.7}$$

LEMMA 7.1. (a) $\Sigma = 0$.

(b) Π *is a nonzero constant.*

(c) a, c, γ *satisfy the following equation*

$$18\gamma^4 - 3(c-a)(7c+2a)\gamma^2 + (c-a)(c-4a) = 0.$$

PROOF. If we express the differentials dC, $d\tilde{A}$ and $d\tilde{B}$ using the covariant differentiation we get

$$\rho = -(4\gamma^2/p)\Pi + 2(c-y), \tag{7.8}$$

$$\rho = -s\Pi + a - 2c + x, \tag{7.9}$$

$$\rho = -s\Pi + a - 2c - x - y, \tag{7.10}$$

$$\Sigma = t\Pi + (c-a)(a+x+y)/\gamma + 2\gamma, \tag{7.11}$$

$$\Sigma = -t\Pi - (c-a)(a-x)/\gamma - 2\gamma, \tag{7.12}$$

$$0 = -C_{1;2} + C_{2;1} - C_{3;4} + C_{4;3}, \tag{7.13}$$

$$0 = -C_{1;3} + C_{3;1} - C_{1;4} + C_{4;1}, \tag{7.14}$$

where $x = (c-a)/3$, $y = 2(a-c)/3$.

(a) If we add (7.11) to (7.12), we get $\Sigma = 0$.

(b) Now, if we assume $\Pi = 0$ then (7.8) gives $\rho = 2(c-y) = 2/3(5c-2a)$. From this relation and (7.9) we get $c = 2/5a$. Now, if we use (7.11) and $\Sigma = 0$, $\Pi = 0$ we find $2\gamma = (c-a)/\gamma(a-x)$, and therefore $2\gamma^2 = -18/25a^2$. This is a contradiction with $\gamma \neq 0$. We use (7.9) and (7.10) to get

$$\Pi = \frac{p}{p+2\gamma^2}(5c-2a). \tag{7.15}$$

(c) One can combine (7.8) and (7.11) to finish the proof. □

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From now on, we assume $\Pi \neq 0$. We now introduce a new orthogonal basis $\{\varphi^i, i = 1, 2, 3, 4\}$ of T_p^*M, where $\varphi^2, \varphi^3, \varphi^4$ are given by (7.2). Then we have

LEMMA 7.2.
(a) $\varphi^1 = C_4\theta^1 - C_3\theta^2 - C_2\theta^3 + C_1\theta^4$,
(b) $\operatorname{div}\varphi^1 = \operatorname{div}\varphi^2 = \operatorname{div}\varphi^3 = \operatorname{div}\varphi^4 = 0$.

PROOF. (a) It follows directly from the orthogonality conditions. (b) By a straightforward computation we find

$$\begin{aligned} \varphi^1 \wedge \varphi^2 - \varphi^3 \wedge \varphi^4 &= \Pi(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3), \\ \varphi^1 \wedge \varphi^3 - \varphi^2 \wedge \varphi^4 &= \Pi(-\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4), \\ \varphi^1 \wedge \varphi^4 - \varphi^2 \wedge \varphi^3 &= \Pi(-\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4), \end{aligned} \tag{7.16}$$

as well as

$$-\rho = \operatorname{div}\varphi^1 = \Lambda_1^4(\eta_4) + \Lambda_1^3(\eta_3) + \Lambda_1^2(\eta_2), \tag{7.17}$$

where $\{\eta_i\}$ is the dual basis of $\{\varphi^i\}$, and Λ_j^i are the corresponding connections 1-forms. Furthermore, we combine (7.8)–(7.10) to find that $3\operatorname{div}\varphi^1 = 2\Pi/p(4\gamma^2 - p)$. The structural equations $d\varphi^k = -\sum_j \Lambda_j^k \wedge \varphi^j$, for $k = 2, 3, 4$, imply

$$\Lambda_1^4(\eta_4) = \frac{a - 2c - x - y}{2\Pi}, \quad \Lambda_1^3(\eta_3) = \frac{a + x - 2c}{2\Pi}, \quad \Lambda_1^2(\eta_2) = \frac{c - y}{\Pi}. \tag{7.18}$$

Consequently, the right side of (7.17) vanishes and hence we have

$$\operatorname{div}\varphi^1 = 0, \quad \text{that is,} \quad 3\gamma^2 = (a - c)^2. \tag{7.19}$$

We see that $c \neq a$, because in the contrary, Lemma 7.1 (c) would imply $\gamma = 0$. If we now use (7.19) and (7.11) we find $c = -2a$. Therefore, one can use (7.4) and (7.15) to get

$$p = 12a^2, \quad s = -1/2, \quad t = -\gamma/(2a), \quad x = -a, \quad y = 2a \quad \text{and} \quad \Pi = -8a. \tag{7.20}$$

To find the divergence of the forms φ^2, φ^3 and φ^4, it suffices to calculate $d\varphi^1$. Combining (7.13) and (7.14) with the covariant differentiation, we get

$$\begin{aligned} d\varphi^1 = {} & (-C_{4;2} - C_{3;1})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + (-C_{4;3} - C_{2;1})(\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4) \\ & + (-C_{4;4} + C_{1;1})(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3). \end{aligned} \tag{7.21}$$

We then use (7.2), (7.19), (7.21), (7.13) and (7.14), to obtain that the divergence of the forms φ^1, and φ^i in general, vanishes. □

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The next step is the calculations of all covariant derivatives, $C_{i;j}$. More precisely, we have the following result.

LEMMA 7.3. *The covariant derivatives* $C_{i;j}$ *are given by the following formulae*:

$$
(7.22)\qquad
\begin{aligned}
C_{1;1} &= C_{4;4} = -C_1C_4/2 + (\gamma/4a)(C_2^2 + C_3^2),\\
C_{2;2} &= C_{3;3} = C_1C_4/2 + (\gamma/4a)(C_1^2 + C_4^2),\\
-C_{3;4} &= C_{1;2} = -C_1C_3/4 - 3C_2C_4/4 - (\gamma/4a)(C_1C_2 - C_3C_4),\\
-C_{4;3} &= C_{2;1} = 3C_1C_3/4 + C_2C_4/4 - (\gamma/4a)(C_1C_2 - C_3C_4),\\
C_{1;3} &= -C_{2;4} = -C_1C_2/4 - 3C_3C_4/4 - (\gamma/4a)(C_1C_3 - C_2C_4),\\
C_{3;1} &= -C_{4;2} = 3C_1C_2/4 + C_3C_4/4 - (\gamma/4a)(C_1C_3 - C_2C_4),\\
C_{2;3} &= -2a + (2C_1^2 + C_2^2 + C_3^2 - 2C_4^2)/4 + (\gamma/2a)C_1C_4,\\
C_{3;2} &= 2a + (-2C_1^2 + C_2^2 + C_3^2 + 2C_4^2)/4 + (\gamma/2a)C_1C_4,\\
C_{1;4} &= -4a + (2C_1^2 + C_2^2 - C_3^2 - 4C_4^2)/4 + (\gamma/2a)C_2C_3,\\
C_{4;1} &= 4a + (-4C_1^2 - C_2^2 + C_3^2 + 2C_4^2)/4 + (\gamma/2a)C_2C_3.
\end{aligned}
$$

PROOF. First we have

$$
(7.23)\qquad
\begin{aligned}
\operatorname{div}\varphi^2 &= \Lambda_2^4(\eta_4) + \Lambda_2^3(\eta_3) + \Lambda_2^1(\eta_1),\\
\operatorname{div}\varphi^3 &= \Lambda_3^4(\eta_4) + \Lambda_3^2(\eta_2) + \Lambda_3^1(\eta_1),\\
\operatorname{div}\varphi^4 &= \Lambda_4^3(\eta_3) + \Lambda_4^2(\eta_2) + \Lambda_4^1(\eta_1).
\end{aligned}
$$

It is easy to see that the sum of the first two members of the each equation from (7.23) vanishes, and then the third member also vanishes. Let

$$
(7.24)\qquad \lambda = C_{4;2} + C_{3;1},\quad \mu = C_{4;3} + C_{2;1},\quad \delta = C_{1;1} - C_{4;4}.
$$

Hence, we combine (7.21), (7.23) with Lemma 7.2 (b) to find the following homogeneous system

$$
(7.25)\qquad
\begin{aligned}
&2(C_1C_3 + C_2C_4)\lambda - 2(C_1C_2 + C_3C_4)\mu - (C_1^2 - C_2^2 + C_3^2 - C_4^2)\delta = 0,\\
&(C_1^2 + C_2^2 + C_3^2 + C_4^2)\lambda - 2(C_1C_4 + C_2C_3)\mu - 2(C_1C_3 - C_2C_4)\delta = 0,\\
&2(C_2C_3 - C_1C_4)\lambda + (C_1^2 - C_2^2 - C_3^2 + C_4^2)\mu - 2(C_1C_2 - C_3C_4)\delta = 0.
\end{aligned}
$$

But this homogeneous system has the determinant equal to $\Pi^3 \neq 0$, and consequently $\lambda = \mu = \delta = 0$. Hence, it follows

$$
(7.26)\qquad
\begin{aligned}
&C_{1;1} = C_{4;4}, \quad C_{3;1} = -C_{4;2}, \quad C_{4;3} = -C_{2;1},\\
&C_{2;2} = C_{3;3}, \quad C_{1;3} = -C_{2;4}, \quad C_{3;4} = -C_{1;2}.
\end{aligned}
$$

Finally, using the relations (7.26), (7.20) and the covariant differentiation of the forms C, $\tilde{A}$ and $\tilde{B}$ and the relations (7.3) we get the relations (7.22). □

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Using results of the previous lemmas we can prove the main result in this section.

THEOREM 7.4. *There does not exist a timelike or spacelike Osserman manifold of type* (Ib), *that is, the characteristic polynomial of the Jacobi operator cannot have a complex root.*

PROOF. We use the relations (7.22) to find

$$\begin{aligned} &d\varphi^1 = 0, \qquad d\varphi^2 = (\varphi^1\wedge\varphi^2 + \varphi^3\wedge\varphi^4)/2,\\ &d\varphi^3 = -(\varphi^1\wedge\varphi^3 + \varphi^2\wedge\varphi^4)/4 - (\gamma/4a)(\varphi^1\wedge\varphi^4 + \varphi^2\wedge\varphi^3),\\ &d\varphi^4 = (\gamma/4a)(\varphi^1\wedge\varphi^3 + \varphi^2\wedge\varphi^4) - (\varphi^1\wedge\varphi^4 + \varphi^2\wedge\varphi^3)/4, \end{aligned} \tag{7.27}$$

and consequently we have

$$\begin{aligned} &d(\varphi^1\wedge\varphi^4 - \varphi^2\wedge\varphi^3) = 0, && d(\theta^1\wedge\theta^3 + \theta^2\wedge\theta^4) = 0,\\ &d(\varphi^1\wedge\varphi^2 - \varphi^3\wedge\varphi^4) = 0, && d(\theta^1\wedge\theta^4 - \theta^2\wedge\theta^3) = 0,\\ &d(\varphi^1\wedge\varphi^3 - \varphi^2\wedge\varphi^4) = 0, && d(\theta^1\wedge\theta^2 + \theta^3\wedge\theta^4) = 0. \end{aligned} \tag{7.28}$$

Finally, we study the Christoffel symbols Γ^i_{jk} in the basis $\{\theta^i\}$ to finish the proof. Because $c = -2a$, we have

$$\begin{aligned} 2\gamma(-C_3\theta^1 - C_4\theta^2 - C_1\theta^3 - C_2\theta^4) &= -3a(\omega_1^3 + \omega_2^4) + \gamma(\omega_1^2 - \omega_3^4),\\ 2\gamma(C_2\theta^1 - C_1\theta^2 - C_4\theta^3 + C_3\theta^4) &= \gamma(\omega_1^3 + \omega_2^4) + 3a(\omega_1^2 - \omega_3^4). \end{aligned} \tag{7.29}$$

First we use the symmetry properties of the connection 1-forms ω^i_j and (7.28) to have

$$\begin{aligned} &(\Gamma^2_{11} - \Gamma^1_{33}) + (\Gamma^4_{23} - \Gamma^4_{32}) = 0, && (-\Gamma^1_{12} - \Gamma^2_{44}) + (\Gamma^3_{14} - \Gamma^3_{41}) = 0,\\ &(-\Gamma^1_{13} + \Gamma^3_{44}) + (\Gamma^2_{41} - \Gamma^2_{14}) = 0, && (-\Gamma^2_{24} - \Gamma^3_{34}) + (\Gamma^1_{32} - \Gamma^1_{23}) = 0. \end{aligned} \tag{7.30}$$

We combine the relations $\omega^i_j = \sum_k \Gamma^i_{kj}\theta^k$ with (7.29) to obtain

$$\begin{aligned} &\Gamma^3_{11} + \Gamma^4_{12} = (\gamma/2a)C_3 + C_2/2, && \Gamma^2_{41} - \Gamma^4_{43} = (\gamma/2a)C_3 - C_2/2,\\ &\Gamma^2_{11} - \Gamma^4_{13} = (\gamma/2a)C_2 - C_3/2, && \Gamma^3_{41} + \Gamma^4_{42} = (\gamma/2a)C_2 + C_3/2,\\ &\Gamma^3_{21} + \Gamma^4_{22} = (\gamma/2a)C_4 - C_1/2, && \Gamma^2_{31} - \Gamma^4_{33} = -(\gamma/2a)C_4 - C_1/2,\\ &\Gamma^2_{21} - \Gamma^4_{23} = -(\gamma/2a)C_1 - C_4/2, && \Gamma^3_{31} + \Gamma^4_{32} = (\gamma/2a)C_1 - C_4/2. \end{aligned} \tag{7.31}$$

One can use the relations (7.31) and symmetries of ω^i_j to get that the right sides of the equations in the system (7.30) are equal $-C_4$, $-C_3$, $-C_2$ and $-C_1$ respectively. It means that $C_1 = C_2 = C_3 = C_4 = 0$, and consequently $\Pi = 0$, which is a contradiction. □

8. The case when $\mathcal{K}_X$ is not diagonalizable and has all real eigenvalues

This case is of special interest because in this class it was found the first example of a nonflat Osserman manifold which is not locally rank-one symmetric. The main result in the section is the following.

THEOREM 8.1. *Let M be a manifold of type* (II) *or* (III). *Then its Jacobi operator has a triple root α.*

Before proceeding with the proof let us mention that these manifolds satisfy some interesting properties

(a) M is foliated by two-dimensional totally geodesic isotropic submanifolds;

(b) there exists an isotropic frame f_1, f_2, f_3, f_4 such that the components of the curvature tensor are determined by

$$R_{1441} = R_{2332} = R_{1243} = R_{1342} = \alpha \quad \text{and} \quad R_{4334} = 2 \tag{8.1}$$

that is,

$$R_{1441} = R_{2332} = R_{1243} = R_{1342} = -\alpha \quad \text{and} \quad R_{1332} = R_{1314} = \sqrt{2}, \tag{8.2}$$

if M is of type (II) or (III) respectively. For details see [6] and [7]. It is important to notice that all known examples are Ricci flat, that is, $\alpha = 0$.

The rest of the section is devoted to the proof of Theorem 8.1. First we introduce some notions and notations which we use in this section. Let us mention that the proof of Theorem 8.1 will be given in a sequence of lemmas and propositions.

Firstly, we consider in this section a manifold of type (II), that is, the corresponding Jacobi operator $\mathcal{K}_X$ with a double zero of its minimal polynomial. It means that there exist a main basis such that the matrix of $\mathcal{K}_X$ is given by (4.3). Therefore we need to study different relations between parameters α and β. The main goal is to prove that if $\beta = 4\alpha$, then $\alpha = \beta = 0$ and the scalar curvature τ vanishes. The manifolds satisfying these conditions admit a field of null parallel planes and consequently they can be endowed with a Walker metric (8.8).

In this section, from now on, we assume that $\varepsilon = +1$ (see Theorem 5.1 (b)), and so

$$a = \alpha - 1/2, \quad b = -\alpha - 1/2, \quad c = -\beta, \quad \gamma = 1/2. \tag{8.3}$$

Let 1-forms A, B and C be as in (6.4) and $\tilde{A}$ and $\tilde{B}$ as in (7.2) from the previous sections. These 1-forms are the following linear combinations of the basic forms θ^i, that is,

$$A = \sum A_i\theta^i, \quad B = \sum B_i\theta^i, \quad C = \sum C_i\theta^i, \tag{8.4}$$

$$\tilde{A} = \sum \tilde{A}_i\theta^i, \quad \tilde{B} = \sum \tilde{B}_i\theta^i. \tag{8.5}$$

These coefficients are not independent. We only prove those relations which are important for our proof of Proposition 8.6. More precisely, we prove the following lemma.

LEMMA 8.2.
(a) $\tilde{A} = -\tilde{B}$.
(b) *The coefficients* $\tilde{A}_i$, $i = 1, \dots, 4$, *satisfy the relations* $\tilde{A}_1 = \tilde{A}_4$ *and* $\tilde{A}_2 = -\tilde{A}_3$.

PROOF. The proof is similar to the previous ones, where now the second Bianchi identity is employed. □

COROLLARY 8.3. *If* $\alpha \neq \beta$, *then* $\beta = 4\alpha$.

PROOF. If we substitute (8.3) in the equality $\tilde{A} = -\tilde{B}$, we get:

$$(\alpha - \beta)(A - B) = 0. \tag{8.6}$$

From (8.6), in the case $\alpha \neq \beta$, we have $A = B$. If we differentiate this equation we finally get

$$\Omega_1^3 + \Omega_2^4 = \Omega_1^2 - \Omega_3^4. \tag{8.7}$$

The component in (8.7) of $\theta^1 \wedge \theta^2$ leads to $1 = a - x$. From (8.3) it follows that the last equality gives $\beta = 4\alpha$. □

Now we will show that $\beta = 4\alpha$ only when $\alpha = \beta = 0$. First we start with the following geometric fact.

PROPOSITION 8.4. *The plane generated by the vectors* $e_1 - e_4$ *and* $e_2 + e_3$ *is a parallel null plane.*

PROOF. One can check by straightforward computations that $e_1 - e_4$ and $e_2 + e_3$ are null vectors. Moreover, the plane generated by them coincides with its orthogonal complement and consequently this plane is a null one.

To prove that this plane is parallel it is necessary to prove $\nabla_X(e_1 - e_4)$ and $\nabla_X(e_2 + e_3)$ belong also to this plane, where X is an arbitrary smooth vector field. We use the equality $A = B$, where $A = \omega_1^2 - \omega_3^4$, $B = \omega_1^3 + \omega_2^4$ to see

$$\nabla_X(e_1 - e_4) = -\omega_4^1(X)(e_1 - e_4) + (\omega_1^2(X) - \omega_4^2(X))(e_2 + e_3),$$
$$\nabla_X(e_2 + e_3) = (\omega_2^1(X) + \omega_3^1(X))(e_1 - e_4) + \omega_3^2(X)(e_2 + e_3).$$ □

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COROLLARY 8.5. *If a 4-dimensional manifold admits a parallel 2-dimensional null plane then one can endow it with a coordinate system* (u_i) *such that the metric is given by the following matrix*

$$\begin{bmatrix} f & s & 1 & 0 \\ s & g & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{8.8}$$

where f, s, g *are arbitrary* $\mathbb{C}^\infty$ *functions.*

PROOF. The proof follows from [28, Theorem 1] and Proposition 8.4. □

PROPOSITION 8.6. *If a manifold* M *satisfies the timelike Osserman condition and can be endowed with the Walker metric* (8.8), *then the scalar curvature* τ *vanishes.*

PROOF. Let (u_i) be a coordinate system on a manifold M such that the metric is given by the matrix in (8.8). Since the manifold satisfies the Osserman condition, it follows that M is an Einstein manifold and the following Ricci curvatures are

$$\rho_{14} = \rho_{23} = 0, \quad \rho_{13} = \rho_{14} = -\tau/2, \tag{8.9}$$

or equivalently, using the notation $\partial_i f = \partial f/\partial u_i$, $\partial_i \partial_j f = \partial^2 f/\partial u_i \partial u_j$,

$$\partial_4\partial_4 s + \partial_3\partial_4 f = 0, \tag{8.10}$$

$$\partial_3\partial_3 s + \partial_3\partial_4 g = 0, \tag{8.11}$$

$$\partial_3\partial_3 f + \partial_3\partial_4 s = \partial_4\partial_4 g + \partial_3\partial_3 f + 2\partial_3\partial_4 s, \tag{8.12}$$

$$\partial_3\partial_4 s + \partial_4\partial_4 g = \partial_4\partial_4 g + \partial_3\partial_3 f + 2\partial_3\partial_4 s. \tag{8.13}$$

Suppose $\tau \neq 0$. The relations (8.12) and (8.13) imply $\partial_3\partial_3 f = \partial_4\partial_4 g$. If we choose $X = a_1\partial_1 + a_3\partial_3 + a_4\partial_4$ and $a_3 = -(1 + a_1^2 f)/2a_1$, we see that $\langle X, X\rangle = -1$.

Let us study now the polynomial

$$\mathscr{P}(a_1, a_4) = a_1^4 \operatorname{tr}\mathscr{K}_X^2 = a_1^4 \sum g^{ij} \langle \mathscr{K}_X \partial_i, \mathscr{K}_X \partial_j \rangle. \tag{8.14}$$

The Osserman condition implies that $\operatorname{tr}\mathscr{K}_X^2 = 2\alpha^2 + \beta^2 = 18\alpha^2$ for $\beta = 4\alpha$ and consequently all coefficients for $a_1^{\beta_1} a_4^{\beta_4}$ vanish except for a_1^4 and it is equal to $18\alpha^2$. One can check by direct computations that

$$\tau = -(\partial_3\partial_3 f + \partial_4\partial_4 g + 2\partial_3\partial_4 s). \tag{8.15}$$

We now use (8.14), (8.15) and the coefficient for $a_1^5 a_4$ to find

$$\partial_3\partial_4 f\, \partial_3\partial_3 g + \partial_3\partial_4 s \partial_3\partial_3 s + \partial_4\partial_4 s \partial_3\partial_3 g + 2\partial_4\partial_4 g \partial_3\partial_4 g + 3\partial_3\partial_4 g \partial_3\partial_4 s = 0.$$

By using (8.10)–(8.13) one can check that the above relation gives $2\partial_3\partial_4 f\,(\partial_3\partial_4 s + \partial_3\partial_3 f) = 0$. If $\partial_3\partial_4 s + \partial_3\partial_3 f = 0$, then because of (8.12), it follows that $\tau = 0$, that is, $\alpha = \beta = 0$, and this is a contradiction. Consequently,

$$\partial_3\partial_4 f = 0. \tag{8.16}$$

In the same way, if we put $Y = a_2\partial_2 + a_3\partial_3 + a_4\partial_4$ and $a_4 = -(1 + a_2^2 g^2)/2a_2$ we have $\langle Y, Y\rangle = -1$. Similarly as before, using the coefficient for $a_2 a_3^5$ of $\mathrm{tr}\mathscr{K}_X^2$ we get

$$\partial_3\partial_4 g = 0. \tag{8.17}$$

We now use (8.10) and (8.11) to see

$$\partial_4\partial_4 s = \partial_3\partial_3 s = 0. \tag{8.18}$$

One can check, using (8.14), the coefficients for a_1^4 and $a_1^6 a_4^2$ are the following ones

$$(\partial_3\partial_4 s)^2 + 2(\partial_3\partial_3 f)^2 + \partial_4\partial_4 f\,\partial_3\partial_3 g + 4\partial_3\partial_4 f\,\partial_3\partial_3 s = 144\alpha^2, \tag{8.19}$$

and

$$\begin{aligned}\partial_4\partial_4 f\,\partial_4\partial_4 g + (\partial_4\partial_4 s)^2 + 2\partial_4\partial_4 s\,\partial_3\partial_4 f\\ + (\partial_3\partial_4 f)^2 + 2\partial_4\partial_4 f\,\partial_3\partial_4 s + \partial_4\partial_4 f\,\partial_3\partial_3 f = 0.\end{aligned} \tag{8.20}$$

We combine now the relations (8.16)–(8.18) with (8.20) to obtain

$$\partial_4\partial_4 f\,(\partial_4\partial_4 g + 2\partial_3\partial_4 s + \partial_3\partial_3 f) = 0. \tag{8.21}$$

The relation (8.21) implies $\partial_4\partial_4 f = 0$, as on the contrary $\tau = 0$. By the analogous procedure for the vector Y and $\mathrm{tr}\mathscr{K}_Y^2$, the coefficient for $a_2^6 a_3^2$ gives $\partial_3\partial_3 g = 0$. The solution of the corresponding partial differential equations can be written in the following form

$$\begin{aligned} f &= -(\mu + \tau/2)u_3^2/2 + u_3\psi + u_4\psi_1 + \psi_2,\\ g &= -(\mu + \tau/2)u_4^2/2 + u_3\varphi + u_4\varphi_1 + \varphi_2,\\ s &= \mu u_3 u_4 + u_3\nu + u_4\nu_1 + \nu_2.\end{aligned} \tag{8.22}$$

The functions $\mu, \psi, \psi_1, \psi_2, \varphi, \varphi_1, \varphi_2, \nu, \nu_1, \nu_2$, in (8.22) depend only on the variables u_1 and u_2. Consequently, (8.19) has a simpler form

$$(\partial_3\partial_4 s)^2 + 2(\partial_3\partial_3 f)^2 = 144\alpha^2. \tag{8.23}$$

We combine now (8.15) and (8.22) to find $\partial_3\partial_4 s = \mu$, $\partial_3\partial_3 f = -(\mu + \tau/2)$ and hence, having in mind $\tau = 24\alpha$, it follows from (8.23)

$$\mu^2 + 16\mu\alpha + 48\alpha^2 = 0. \tag{8.24}$$

The solutions of (8.24) are the following functions

$$\mu_1 = -12\alpha, \qquad \mu_2 = -4\alpha.$$

Therefore, the functions f, s, g are given by (8.22), where μ is a constant set to be $\mu = C\alpha$, $C \neq 0$.

It is easy to see that the equation $\rho_{11} = \tau/4f$ can be written in terms of the derivatives of functions f, s, g as follows

$$\begin{aligned} (8.25) \qquad & -\partial_4 f\, \partial_4 g + (\partial_4 s)^2 - g\partial_4\partial_4 f - \partial_4 s\partial_3 f + \partial_4 f\, \partial_3 s \\ & \quad - 2s\partial_3\partial_4 f + 2\partial_2\partial_4 f - 2\partial_1\partial_4 s + f\,\partial_3\partial_4 s = 0. \end{aligned}$$

We use now (8.22) and (8.25) to obtain

$$\begin{aligned} (8.26) \qquad & -\psi_1\varphi_1 + (\mu u_3 + \nu_1)^2 - \mu(\mu u_3 + \nu_1)(-(\mu + \tau/2)u_3 + \psi) \\ & \quad + \psi_1(\mu u_4 + \nu + 2\partial_2\psi_1 - 2\partial_1\nu_1 + f\,\mu) = 0. \end{aligned}$$

Equation (8.26) with respect to the variable u_3 is of polynomial type and hence the coefficient for u_3^2 vanishes, that is, $\mu^2 + \mu(\mu + \tau/2) = 0$, and from here follows that $\mu = 0$ and consequently $\tau = 0$. This is the contradiction with the assumption $\tau \neq 0$. □

COROLLARY 8.7. *There does not exist an Osserman timelike manifold whose characteristic polynomial of endomorphism $\mathcal{K}_X$ has a double real zero α, and real zero β such that $\beta = 4\alpha \neq 0$.*

PROOF. Proposition 8.6 implies the scalar curvature τ vanishes and consequently $\alpha = \beta = 0$. □

9. Examples

When the Jacobi operator of a timelike or spacelike Osserman manifold (M, g) is diagonalizable (type (Ia)), then their local characterization is of the same kind as in the Riemannian 4-dimensional manifolds (Section 1). This section is devoted to the examples that show that if the Jacobi operator is not diagonalizable (types (II) and (III)) then M may not even be a locally symmetric space.

In Subsection 9.1 we use the results of Wu ([30]) to show the existence of this 'exceptional example' and the explicit construction of the metric is given by Rakić ([23]). This example admits nice geometric structures. In Subsection 9.2 we describe timelike (spacelike) Osserman manifolds which are not even locally symmetric (but they are Ricci flat).

9.1. The existence of a locally rank-two symmetric Osserman manifold. The main purpose of this subsection is to study manifolds with an endomorphism $\mathcal{K}_X$

such that its matrix is given by (4.3) where the eigenvalues α and β vanish. We show that there exists a locally rank-two symmetric manifold $M = G/H$ in this class. Moreover, a manifold $M = G/H$ can be endowed with an integrable antiquaternionic structure and an integrable dual (neutral) structure.

We recall some basic facts from [30] to clarify our construction.

In general H is the generic symbol of holonomy groups and $\mathfrak{h}$, of holonomy algebras. We denote (the identity component of) the full group of isometries of an inner product space V by $PO(V)$ when we are not concerned with signature, and by $SO(p, d-p)$ when we are. The corresponding Lie algebras are then $\mathfrak{po}(V)$ and $\mathfrak{so}(p, d-p)$. $PO(V)$ is a subgroup of the automorphism group $Gl(V)$ of V which is usually identified with the group of nonsingular $d \times d$ matrices in the presence of a basis; $\mathfrak{po}(V)$ is then a Lie subalgebra of the full matrix algebra $\mathfrak{gl}(V)$, which is identified via the same basis with $\mathrm{Hom}(V, V)$.

DEFINITION 9.1. A connected Lie subgroup H^* of $GL(V)$ is called an *algebraic holonomy group* if and only if there exists curvature tensors $\{R^1, \dots, R^r\}$ on V such that the Lie algebra $\mathfrak{h}^*$ of H^* is exactly the linear span of the $R^i(x, y)$, all $x, y \in V$, $i = 1, \dots r$. *An algebraic Riemannian holonomy group* is an algebraic holonomy group such that the $\{R^1, \dots, R^r\}$ are all Riemannian curvature tensors. Thus, if an algebraic holonomy group is either Riemannian or Kählerian, it is a subgroup of $PO(V)$.

DEFINITION 9.2. A triple $\{V, R, H\}$ is called *a symmetric holonomy system* if and only if R is the curvature tensor on V and H is a connected Lie subgroup of $GL(V)$ such that,

$$\mathfrak{h} = \mathrm{span}\{R(x, y) : x, y \in V\}, \tag{9.1}$$

$$R(h(x), y) + R(x, h(y)) + [R(x, y), h] = 0, \quad \text{for all } h \in \mathfrak{h} \text{ and } x, y \in V. \tag{9.2}$$

It is called a *Riemannian symmetric holonomy system* if and only if R is furthermore a Riemannian curvature tensor.

Wu has proved in [30] that every such H can actually be realized as the holonomy group of an appropriate symmetric space. More precisely, he has proved the following corollary.

COROLLARY 9.3. *If $\{V, R, H\}$ is a Riemannian holonomy system, then there exists a simply connected Riemannian symmetric space whose tangent space at a point can be identified with V, whose curvature tensor is R, and whose holonomy group is H.*

The following fact is also well known.

PROPOSITION 9.4. *Let G be a simply-connected solvable Lie group and K an arbitrary connected Lie subgroup of G. Then G/K is diffeomorphic to a Euclidean space.*

We prove in this section the following theorem.

THEOREM 9.5. *There exists a homogeneous symmetric pseudo-Riemannian space M with a metric of signature $(2, 2)$ such that the matrix of the endomorphism $\mathcal{K}_X$ is given by (4.3), where $\alpha = \beta = 0$.*

PROOF. Recall that $R(X, Y) \in \mathfrak{so}(2, 2)$. Let us take the matrix m

$$m = \begin{bmatrix} 0 & -1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix} \tag{9.3}$$

and put

$$\begin{aligned} &R(E_1, E_4) = R(E_2, E_3) = 0, \\ &R(E_1, E_2) = R(E_3, E_1) = R(E_4, E_2) = R(E_3, E_4) = m. \end{aligned} \tag{9.4}$$

We denote by H the 1-dimensional connected Lie subgroup of $GL(V)$ such that its Lie algebra is generated by the endomorphism m. Then all conditions in Corollary 9.3 are fulfilled, that is, there exists a symmetric space M with its tangent space identified by $V = (\mathbb{R}^4, \langle\cdot,\cdot\rangle)$ of signature $(2, 2)$ and its curvature tensor R. We use Proposition 9.4 to see that this space is diffeomorphic with $\mathbb{R}^4$. More precisely, the proof of Corollary 9.3 implies that the algebra $\mathfrak{g} = \mathfrak{h} \oplus V$ is the Lie algebra of the group G. The Lie brackets $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ are defined by

$$\begin{aligned} [h_1, h_2] &= \text{bracket in } \mathfrak{h} && \text{if } h_1, h_2 \in \mathfrak{h}, \\ [h_1, x] &= h_1(x) && \text{if } h_1 \in \mathfrak{h},\ x \in V, \\ [x, y] &= R(x, y) && \text{if } x, y \in V. \end{aligned} \tag{9.5}$$

This algebra $\mathfrak{g}$ is solvable and hence Proposition 9.4 implies our homogeneous space M is diffeomorphic with $\mathbb{R}^4$. □

REMARK. Since the sectional curvature of the plane $E_1 \wedge E_4$ is vanishing, it is easy to verify that M is a rank-two symmetric space.

This method was used by Rakić [23] to determine explicitly an Osserman metric on $\mathbb{R}^4$ with the curvature prescribed by (9.3) and (9.4).

EXAMPLE 1. Let $M = \mathbb{R}^4$, (u_1, u_2, u_3, u_4) be the Cartesian coordinates and

$$\begin{aligned}6g = u_2^2 du_1 \otimes du_1 + u_1^2 du_2 \otimes du_2 - u_1 u_2 [du_1 \otimes du_2 + du_2 \otimes du_1] \\ - 3[du_1 \otimes du_4 + du_4 \otimes du_1 + du_2 \otimes du_3 + du_3 \otimes du_2].\end{aligned}$$

Then $(\mathbb{R}^4, g)$ is a timelike Osserman rank-two symmetric space whose curvature tensor is given by (9.3) and (9.4) (see [23, Theorem 1.1]).

The next theorem deals with the construction of an integrable para-quaternionic structure on a manifold M satisfying the conditions of Theorem 9.5.

THEOREM 9.6. *Let M be a manifold satisfying the conditions of Theorem 9.5. Then M can be endowed with an integrable para-quaternionic structure.*

PROOF. We define the endomorphisms i, j, k on elements of an orthonormal basis $E_1, \dots, E_4$ as follows

$$\begin{array}{lll} i(E_1) = E_4, & j(E_1) = E_3, & k(E_1) = -E_2, \\ i(E_2) = -E_3, & j(E_2) = E_4, & k(E_2) = E_1, \\ i(E_3) = -E_2, & j(E_3) = E_1, & k(E_3) = E_4, \\ i(E_4) = E_1, & j(E_4) = E_2, & k(E_4) = -E_3. \end{array} \tag{9.6}$$

Endomorphisms i, j, k are well defined, and it is as in the previous cases interesting, since they are independent of the choice of the first vector E_1. Let us remark the endomorphisms i, j, k satisfy the multiplication properties

$$i^2 = j^2 = -k^2 = 1, \quad ij = -ji = k. \tag{9.7}$$

This structure is known as a para-quaternionic structure. For some details see for example [24]. Let us denote

$$p = \omega_1^4 - \omega_3^2, \quad q = \omega_1^2 + \omega_4^3, \quad r = \omega_4^2 + \omega_3^1. \tag{9.8}$$

We now use Lemma 6.4 (i) to see

$$\nabla i = q(j - k), \quad \nabla j = -qi, \quad \nabla k = -qi, \tag{9.9}$$

which implies the integrability of our para-quaternionic structure. If the manifold M is orientable, we can define this para-quaternionic structure globally. □

Let us remark that $\nabla(j - k) = 0$ and $(j - k)^2 = 0$. It means $j - k$ is an integrable dual (neutral) structure (see [24]).

9.2. Examples of nonsymmetric manifolds with the nondiagonalizable $\mathscr{K}_X$. Manifolds satisfying Osserman condition have very interesting and rich geometry. Let us mention the following. Among manifolds with nondiagonalizable $\mathscr{K}_X$ one can find also some recurrent spaces, harmonic spaces and others.

EXAMPLE 2. Let $M = \mathbb{R}^4$, (u_1, u_2, u_3, u_4) be the usual coordinates, and

$$g = (u_1u_2)^2[du_1 \otimes du_1 + du_2 \otimes du_2] \\ + [du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2].$$

The characteristic polynomial of the Jacobi operator $\mathscr{K}_X$ is $p_\lambda(\mathscr{K}_X) = \det(\mathscr{K}_X - \lambda I) = \lambda^4$, for arbitrary nonnull vector X. Moreover, (M, g) satisfies the timelike and spacelike Osserman condition on the open subset $u_1u_2 \neq 0$ (type (II)). In this subset the manifold (M, g) is not locally symmetric. When $u_1u_2 = 0$, the minimal polynomial $m_\lambda(\mathscr{K}_X) = \lambda$, that is, the Jacobi operator is diagonalizable but not diagonalizable in the neighbourhood of the set $u_1u_2 = 0$. It is clear that $(\mathbb{R}^4, g)$ is not locally homogeneous, although the characteristic polynomial $p_\lambda(\mathscr{K}_X)$ is constant.

EXAMPLE 3. A manifold with the metric

$$g = u_2u_3du_1 \otimes du_1 - u_1u_4du_2 \otimes du_2 \\ + [du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2]$$

is simple harmonic. Moreover, this metric is neither symmetric nor recurrent (see [25, page 211]).

García-Rio, Vázquez-Abal and Vázquez-Lorenzo [16] have considered the following family of metrics on $\mathbb{R}^4$ of signature (2, 2), parameterized by some functions f_1 and f_2.

EXAMPLE 4. As in the previous example let

$$g_{(f_1,f_2)} = u_3f(u_1, u_2)du_1 \otimes du_1 + u_4f_2(u_1, u_2)du_2 \otimes du_2 + a[du_1 \otimes du_2 + du_2 \otimes du_1] \\ + b[du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2],$$

where $\partial f_1/\partial u_2 + \partial f_2/\partial/\partial u_1 = 0$. Then the characteristic polynomial of the Jacobi operator of $(M, g_{(f_1,f_2)})$ is $p_\lambda(\mathscr{K}_X) = \lambda^4$, that is, it is independent of the nonnull vector X, but with different minimal polynomials $m_\lambda(\mathscr{K}_X) = \lambda^2$ or $m_\lambda(\mathscr{K}_X) = \lambda^3$. There are examples when the minimal polynomials change degree from point to point. Functions f_1 and f_2 can be additionally chosen so that $(M, g_{(f_1,f_2)})$ is not locally symmetric ([16, Theorem 3]).

It is interesting to notice, as a consequence of these examples, that it is natural to define the Osserman condition using the Jordan form of the Jacobi operator in the case of pseudo-Riemannian manifold of arbitrary signature. In signature $(2, 2)$ this is equivalent to the independence of the minimal polynomial of the Jacobi operator on $p \in M$ and on the unit tangent timelike vector X.

Let us state now the following problems.

QUESTION 1. Do there exist 4-dimensional timelike (spacelike) Osserman manifolds of signature $(2, 2)$ whose minimal polynomial of the Jacobi operator has a multiple root $\alpha \neq 0$, and which is not Ricci flat?

Starting from the Osserman conjecture it is natural to consider the following problem.

QUESTION 2. Is a timelike (spacelike) Osserman manifold with the diagonalizable Jacobi operator either a locally rank-one symmetric space or a flat space?

The affirmative answer was given in [9] and [16] under some additional assumptions.

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University of Belgrade
Faculty of Mathematics
Studentski trg 16, p.p. 550
11000 Belgrade
Yugoslavia
e-mail: blazicn@matf.bg.ac.yu
e-mail: neda@matf.bg.ac.yu
e-mail: zrakic@matf.bg.ac.yu

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