Proceedings of the Edinburgh Mathematical Society (2012) **55**, 543–549
DOI:10.1017/S001309151000101X

A NOTE ABOUT INVARIANT SKT STRUCTURES AND GENERALIZED KÄHLER STRUCTURES ON FLAG MANIFOLDS

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(Received 12 July 2010)

Abstract We prove that any invariant strong Kähler structure with torsion (SKT structure) on a flag manifold $M = G/K$ of a semi-simple compact Lie group G is Kähler. As an application we describe invariant generalized Kähler structures on M.

Keywords: SKT structures; (generalized) Kahler structures; flag manifold

2010 *Mathematics subject classification:* Primary 53D18

1. Introduction

A Hermitian manifold (M, g, J) admits a unique connection ∇^{B} (called the Bismut connection) which preserves the metric g and the complex structure J and has a skew-symmetric torsion tensor $c := g(\cdot, T^{\mathrm{B}}(\cdot,\cdot))$, where T^{B} is the torsion of ∇^{B}. The 3-form c can be expressed in terms of the Kähler form $\omega = g \circ J$ by

$$c = J\,\mathrm{d}\omega := \mathrm{d}\omega(J\cdot, J\cdot, J\cdot).$$

The manifold (M, g, J) is said to be strong Kähler with torsion (SKT) if the torsion 3-form c is closed or, equivalently, if $\partial\bar{\partial}\omega = 0$. SKT manifolds are a natural generalization of Kähler manifolds, and many results from Kähler geometry can be generalized to SKT geometry (see, for example, [**3**–**7**]).

SKT geometry is also closely related to generalized Kähler geometry, which was recently introduced by Hitchin [**12**] and previously appeared in physics as the geometry of the target space of $N = (2, 2)$ supersymmetric nonlinear sigma models (see, for example, [**8**, **13**]).

A generalized Kähler structure on a manifold M is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures such that the symmetric bilinear form $-(\mathcal{J}_1 \circ \mathcal{J}_2\cdot, \cdot)$ is positive definite, where $(\cdot, \cdot)$ is the standard scalar product of neutral signature of the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$. (For the definition and basic facts about generalized complex structures, see, for example, [**9**]). It was shown by Gualtieri [**9**, **10**] that a generalized Kähler structure on a manifold can be described in classical terms as a bi-Hermitian structure (g, J_+, J_-, b) in the sense of [**8**], i.e. a pair (g, J_+), (g, J_-) of SKT structures with common metric g and a 2-form b (called the b-field in the physics literature) such that

$$\mathrm{d}b = J_+\,\mathrm{d}\omega_+ = -J_-\,\mathrm{d}\omega_-, \tag{1.1}$$

where $\omega_\pm = g \circ J_\pm$ are Kähler forms.

Let G be a semi-simple compact Lie group and let $M = G/K$ be a flag manifold, i.e. an adjoint orbit of G. In this paper we describe invariant SKT structures and invariant generalized Kähler structures on M, as follows.

Theorem 1.1. *Any invariant SKT structure (g, J) on a flag manifold $M = G/K$ is Kähler, i.e. the Kähler form $\omega = g \circ J$ is closed.*

The description of invariant Kähler structures on flag manifolds is well known (see, for example, [**1**, **2**] and § 2).

Corollary 1.2. *Let (g, J_+, J_-, b) be an invariant bi-Hermitian structure in the sense of [**8**] on a flag manifold $M = G/K$ (which defines a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ via Gualtieri's correspondence). Then g is an invariant Kähler metric, J_+, J_- are two parallel invariant complex structures and b is any closed invariant 2-form. If the group G is simple, then $J_+ = J_-$ or $J_+ = -J_-$.*

The note is organized as follows. In § 2 we fix our conventions and we recall the basic facts on the geometry of flag manifolds and, in particular, the description of invariant Hermitian and Kähler structures [**1**, **2**]. With these preliminaries, Theorem 1.1 and Corollary 1.2 will be proved in § 3.

2. Preliminary material

2.1. Basic facts about flag manifolds

A flag manifold of a semi-simple compact Lie group G is an adjoint orbit $M = \mathrm{Ad}_G(h_0) \simeq G/K$ of an element h_0 of the Lie algebra of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the complex Lie algebras associated with the groups G and K respectively, and we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. We denote by R and R_0 the root systems of $\mathfrak{g}$ and $\mathfrak{k}$ with respect to $\mathfrak{h}$ and we set $R' := R \setminus R_0$. We write the root space decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \Bigl(\mathfrak{h} + \sum_{\alpha \in R_0} \mathfrak{g}_\alpha\Bigr) + \sum_{\alpha \in R'} \mathfrak{g}_\alpha,$$

and we identify the vector space $\mathfrak{m} = \sum_{\alpha \in R'} \mathfrak{g}_\alpha$ with the complexification of the tangent space $T_{h_0}M$.

Let $E_\alpha \in \mathfrak{g}_\alpha$ be root vectors of a Weyl basis. Thus,

$$\langle E_\alpha, E_{-\alpha}\rangle = 1 \quad \text{for all } \alpha \in R$$

https://doi.org/10.1017/S001309151000101X Published online by Cambridge University Press

(where $\langle X, Y\rangle := \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$ denotes the Killing form of $\mathfrak{g}$) and

$$N_{-\alpha,-\beta} = -N_{\alpha\beta} \quad \text{for all } \alpha, \beta \in R, \tag{2.1}$$

where $N_{\alpha\beta}$ are the structure constants defined by

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in R. \tag{2.2}$$

The Lie algebra of G is the fixed point set $\mathfrak{g}^\tau$ of the compact anti-involution τ, which preserves the Cartan subalgebra $\mathfrak{h}$ and sends E_α to $-E_{-\alpha}$ for any $\alpha \in R$. It is given by

$$\mathfrak{g}^\tau = \mathrm{i}\mathfrak{h}_{\mathbb{R}} + \sum_{\alpha\in R} \operatorname{span}\{E_\alpha - E_{-\alpha}, \mathrm{i}(E_\alpha + E_{-\alpha})\},$$

where $\mathfrak{h}_{\mathbb{R}} = \operatorname{span}\{H_\alpha := [E_\alpha, E_{-\alpha}],\ \alpha \in R\}$ is a real form of $\mathfrak{h}$. Note that

$$\beta(H_\alpha) = \beta([E_\alpha, E_{-\alpha}]) = \langle\beta, \alpha\rangle,$$

where $\langle\cdot,\cdot\rangle$ also denotes the scalar product on $\mathfrak{h}^*$ induced by the Killing form.

2.2. Invariant complex structures on $M = G/K$

We fix a system Π_0 of simple roots of R_0 and we extend it to a system $\Pi = \Pi_0 \cup \Pi'$ of simple roots of R. We denote by $R_0^+, R^+ = R_0^+ \cup R'_+$ the corresponding systems of positive roots. A decomposition

$$\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^- = \sum_{\alpha\in R'_+} \mathfrak{g}_\alpha + \sum_{\alpha\in R'_+} \mathfrak{g}_{-\alpha}$$

defines an Ad_K-invariant complex structure J on $T_{h_0}M = \mathfrak{m}^\tau$, such that

$$J|_{\mathfrak{m}^\pm} = \pm\mathrm{i}\,\mathrm{Id}\,.$$

We extend it to an invariant complex structure on M, also denoted by J. We will refer to R'_+ and Π' as the set of positive roots and the set of simple roots of J, respectively. It is known that any invariant complex structure on M can be obtained by this construction [**2**, **14**].

2.3. T-roots and isotropy decomposition

Let $\mathfrak{z} = \mathrm{i}\mathfrak{t} \subset \mathfrak{h}$ be the centre of the stability subalgebra $\mathfrak{k}^\tau$. The restriction of the roots from $R' \subset \mathfrak{h}^*$ to the subspace $\mathfrak{t}$ are called T-roots. Denote by

$$\kappa\colon R' \to R_T, \quad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

the natural projection onto the set R_T of T-roots. Note that $\alpha|_{\mathfrak{t}} = 0$ for any $\alpha \in R_0$. Any T-root ξ defines an Ad_K-invariant subspace

$$\mathfrak{m}_\xi := \sum_{\alpha\in R',\ \kappa(\alpha)=\xi} \mathfrak{g}_\alpha$$

of the complexified tangent space $\mathfrak{m}$ and

$$\mathfrak{m} = \sum_{\xi\in R_T} \mathfrak{m}_\xi$$

is a direct sum decomposition into non-equivalent irreducible Ad_K-submodules.

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2.4. Invariant metrics and Hermitian structures

We denote by $\omega_\alpha \in \mathfrak{g}^*$ the 1-forms dual to E_α, $\alpha \in R$, i.e.

$$\omega_\alpha(E_\beta) = \delta_{\alpha\beta}, \quad \omega_\alpha|_{\mathfrak{h}} = 0. \tag{2.3}$$

Any invariant Riemannian metric on M is defined by an Ad_K-invariant Euclidean metric g on $\mathfrak{m}^\tau$, whose complex linear extension has the form

$$g = -\frac{1}{2}\sum_{\alpha\in R'} g_\alpha \omega_\alpha \vee \omega_{-\alpha}, \tag{2.4}$$

where $\omega_\alpha \vee \omega_{-\alpha} = \omega_\alpha \otimes \omega_{-\alpha} + \omega_{-\alpha} \otimes \omega_\alpha$ is the symmetric product and g_ξ, $\xi \in R_T$, is a system of positive constants associated with the T-roots, $g_\xi = g_{-\xi}$ for any $\xi \in R_T$ and $g_\alpha := g_{\kappa(\alpha)}$. Note that the restriction of g to $\mathfrak{m}_\xi$ is proportional to the restriction of the Killing form, with coefficient of proportionality $-g_\xi$.

Any such metric g is Hermitian with respect to any invariant complex structure J and the corresponding Kähler form is given by

$$\omega = -\mathrm{i}\sum_{\alpha\in R'_+} g_\alpha \omega_\alpha \wedge \omega_{-\alpha}, \tag{2.5}$$

where R'_+ is the set of positive roots of J and in our conventions $\omega_\alpha \wedge \omega_{-\alpha} = \omega_\alpha \otimes \omega_{-\alpha} - \omega_{-\alpha} \otimes \omega_\alpha$.

2.5. Invariant Kähler structures

Any invariant symplectic form ω on M compatible with an invariant complex structure J as above (i.e. such that $g := -\omega \circ J$ is positive definite) is associated with a 1-form $\sigma \in \mathfrak{t}^*$ such that $\langle \sigma, \alpha_i\rangle > 0$ for any $\alpha_i \in \Pi'$ (the set of simple roots of J). As a form on $\mathfrak{m}$, it is given by

$$\omega = \omega_\sigma := -\mathrm{i}\sum_{\alpha\in R'_+} \langle \sigma, \alpha\rangle \omega_\alpha \wedge \omega_{-\alpha}.$$

The associated Kähler metric g has the coefficients $g_\alpha = g_{\kappa(\alpha)} = \langle \sigma, \alpha\rangle$, which, obviously, satisfy the following linearity property:

$$g_{\alpha+\beta} = g_\alpha + g_\beta \quad \text{for all } \alpha, \beta, \alpha+\beta \in R'_+. \tag{2.6}$$

In particular, if $\Pi' = \{\alpha_1, \dots, \alpha_m\}$ and

$$R' \ni \alpha \equiv k_1\alpha_1 + \cdots + k_m\alpha_m \ (\mathrm{mod}\, R_0),$$

then

$$g_\alpha = k_1 g_{\alpha_1} + \cdots + k_m g_{\alpha_m}.$$

To summarize, we obtain the following.

Proposition 2.1 (Alekseevsky and Perelomov [2]). *An invariant Hermitian structure (g, J) on M is Kähler if and only if the coefficients g_α associated with g by (2.4) satisfy the linearity property: if $\alpha, \beta, \alpha+\beta \in R'_+$, then $g_{\alpha+\beta} = g_\alpha + g_\beta$. Here R'_+ is the set of positive roots of J.*

2.6. The formula for the exterior derivative

An invariant k-form on $M = G/K$ can be considered as an Ad_K-invariant k-form ω on the Lie algebra $\mathfrak{g}$ such that $i_{\mathfrak{k}}(\omega) = 0$. We recall the standard Koszul formula for the exterior differential $\mathrm{d}\omega$:

$$(\mathrm{d}\omega)(X_0, \dots, X_k) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \tag{2.7}$$

for any $X_i \in \mathfrak{m} \subset \mathfrak{g}$. In (2.7) the hat means that the term is omitted.

3. Proof of our main results

We now prove Theorem 1.1 and Corollary 1.2. We preserve the notation from the previous sections. Let (g, J) be an invariant Hermitian structure on a flag manifold $M = G/K$. Let $g_\alpha = g_{k(\alpha)}$ the positive numbers associated with g and R'_+, Π' the set of positive (respectively, simple) roots of J, as before. Let $\omega = g \circ J$ be the Kähler form. To prove the theorem, we have to check that if the form $J\,\mathrm{d}\omega$ is closed, then the g_α satisfy the linearity property (2.6). We define the sign ε_α of a root $\alpha \in R' = R'_+ \cup (-R'_+)$ by $\varepsilon_\alpha = \pm 1$ if $\alpha \in \pm R'_+$. Note that ε_α depends only on $\kappa(\alpha)$. Now we calculate $\mathrm{d}\omega$ and $J\,\mathrm{d}\omega$ on basic vectors, as follows.

Lemma 3.1.

(i)

$$\mathrm{d}\omega(E_\alpha, E_\beta, E_\gamma) = 0 \quad \textit{if } \alpha + \beta + \gamma \neq 0 \tag{3.1}$$

and

$$\mathrm{d}\omega(E_\alpha, E_\beta, E_{-(\alpha+\beta)}) = -\mathrm{i} N_{\alpha\beta}(\varepsilon_\alpha g_\alpha + \varepsilon_\beta g_\beta - \varepsilon_{\alpha+\beta} g_{\alpha+\beta}). \tag{3.2}$$

(ii)

$$(J\,\mathrm{d}\omega)(E_\alpha, E_\beta, E_{-(\alpha+\beta)}) = N_{\alpha\beta}(\varepsilon_\beta \varepsilon_{\alpha+\beta} g_\alpha + \varepsilon_\alpha \varepsilon_{\alpha+\beta} g_\beta - \varepsilon_\alpha \varepsilon_\beta g_{\alpha+\beta}). \tag{3.3}$$

Proof. Relation (3.1) follows from (2.5) and (2.7). Relation (3.2) follows from (2.5), (2.7) and the following property of $N_{\alpha\beta}$ [**11**, Chapter 5]): if $\alpha, \beta, \gamma \in R$ are such that $\alpha + \beta + \gamma = 0$, then

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}. \tag{3.4}$$

Relation (3.3) follows from (3.2) and $JE_\alpha = \mathrm{i}\varepsilon_\alpha E_\alpha$ for any $\alpha \in R'$. □

Lemma 3.2. *Suppose that (g, J) is an SKT structure, i.e. $d(J\,\mathrm{d}\omega) = 0$. Then*

$$N^2_{\alpha\beta}(g_{\alpha+\beta} - g_\alpha - g_\beta) + \varepsilon_{\alpha-\beta} N^2_{\alpha,-\beta}(\varepsilon_{\alpha-\beta} g_{\alpha-\beta} - g_\alpha + g_\beta) = 0 \tag{3.5}$$

for any $\alpha, \beta \in R'_+$, where we assume that $\varepsilon_{\alpha-\beta} = 0$ if $\alpha - \beta \notin R'$.

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Proof. By a direct computation, we find

$$
\begin{aligned}
-\tfrac{1}{2}d(J\,\mathrm{d}\omega)&(E_\alpha, E_\beta, E_{-\alpha}, E_{-\beta}) \\
&= N_{\alpha\beta}^2(g_{\alpha+\beta} - g_\alpha - g_\beta) + \varepsilon_{\alpha-\beta}N_{\alpha,-\beta}^2(\varepsilon_{\alpha-\beta}g_{\alpha-\beta} - g_\alpha + g_\beta).
\end{aligned}
$$

This relation implies our claim. □

For any root

$$R'_+ \ni \alpha \equiv k_1\alpha_1 + \cdots + k_m\alpha_m \ (\mathrm{mod}\, R_0^+), \quad \alpha_i \in \Pi',$$

we define the length of α as $\ell(\alpha) = \sum_{i=1}^m k_i$. Note that $\ell(\alpha)$ depends only on the projection $\kappa(\alpha)$ of α onto $\mathfrak{t}^*$.

Proof of Theorem 1.1. By Proposition 2.1 we have to check that

$$g_{\alpha+\beta} = g_\alpha + g_\beta \tag{3.6}$$

for any $\alpha, \beta \in R'_+$ such that $\alpha+\beta \in R'_+$. We use induction on the length of $\gamma = \alpha+\beta \in R'_+$. Suppose first that $\gamma = \alpha + \beta \in R'_+$ has length 2. Then $\alpha - \beta \notin R'$; hence, $\varepsilon_{\alpha-\beta} = 0$. Identity (3.5) implies (3.6).

Suppose now that (3.6) holds for all $\gamma = \alpha+\beta \in R'_+$ with $l(\gamma) \leqslant k$. Let $\gamma \in R'_+$ with $\ell(\gamma) = k+1$ and suppose that $\gamma = \alpha + \beta$, where $\alpha, \beta \in R'_+$. We have to show that

$$g_\gamma = g_\alpha + g_\beta. \tag{3.7}$$

If $\alpha-\beta \notin R'$, our previous argument shows that (3.7) holds. Suppose now that $\alpha-\beta \in R'$. Without loss of generality, we may assume that $\alpha - \beta \in R'_+$. Then $\alpha = (\alpha-\beta)+\beta$ is a decomposition of the root α into a sum of two roots from R'_+. Since α has length less than or equal to k, our inductive assumption implies that $g_\alpha = g_{\alpha-\beta} + g_\beta$. Thus, the second term of the identity (3.5) vanishes and we obtain (3.7). This concludes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let (g, J_+, J_-, b) be a G-invariant bi-Hermitian structure in the sense of [**8**] on a flag manifold $M = G/K$. Then, by Theorem 1.1, $(g, J_\pm)$ are two Kähler structures and hence the b-field b is closed. The complex structures $J_\pm$ are parallel with respect to the Levi–Civita connection. If the group G is simple, the Kähler metric g is irreducible. The endomorphism $A = J_1 \circ J_2$ is symmetric with respect to g and parallel. An easy argument which uses the irreducibility of g shows that $J_1 = J_2$ or $J_1 = -J_2$. This concludes the proof of Corollary 1.2. □

Acknowledgements. D.V.A. thanks the University of Hamburg, where the final version of this paper was written, for hospitality and financial support. L.D. acknowledges financial support from a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, Project PN-II-ID-PCE-2011-3-0362.

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