# GROUP ALGEBRA MODULES. I

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**1. Introduction.** Some time ago, J. G. Wendel proved that the operators on the group algebra $L_1(G)$ which commute with convolution correspond in a natural way to the measure algebra $M(G)$ **(13)**. One might ask if Wendel's theorem can be restated in a more general setting. It is this question that is the point of departure for our present paper. Let $K$ be a Banach module over $L_1(G)$. Our interest is in operators from $L_1(G)$ into $K$, and from $K$ into $L_\infty(G)$, which commute with the module composition (where $L_\infty(G)$ is thought of as a module over $L_1(G)$ also). Such operators we call $(L_1(G), K)$- and $(K, L_\infty(G))$-homomorphisms, respectively. Investigations of various other kinds of module homomorphisms occur in A. Figà-Talamanca **(6)** and B. E. Johnson **(9; 10)**.

Section 2 contains the preliminary discussion. In § 3 we present a partial characterization of the $(L_1(G), K)$-homomorphisms, as a factor space of the second conjugate space of $K$. We conclude the section with explicit descriptions of the $(L_1(G), L_p(G))$-homomorphisms, $p \in [1, \infty]$, where $L_p(G)$ is a left $L_1(G)$-module under a generalized convolution, and where $L_p(G)$ is a right $L_1(G)$-module under a composition which is related to the construction of the Arens multiplication in the bidual of a Banach algebra. In § 4 we characterize the $(K, L_\infty(G))$-homomorphisms completely, in terms of the dual of $K$, and finally give explicit descriptions of the $(L_p(G), L_\infty(G))$-homomorphisms, $p \in (1, \infty]$, in the same manner as at the end of § 3.

**2. Preliminary discussion.** This section contains the preliminary material from which we shall draw throughout the rest of the paper. Henceforth let $G$ be an arbitrary locally compact topological group, abelian or not, with left Haar measure $m$. Let 1 be the identity element of $G$. For $p \in [1, \infty)$, we let, as is customary, $L_p(G)$ be the Banach space of $m$-measurable functions on $G$ whose $p$th powers are absolutely integrable. For any measurable function $f$ and any $s \in G$, let $({}_sf)t = f(st)$ and $(f_s)t = f(ts)$ for almost all $t \in G$. Corresponding to the Haar measure $m$, there exists a modular function $\Delta$ defined on $G$, with the property that for $s \in G$, $\Delta(s) = ||f_{s^{-1}}||/||f||$ for all non-zero $f \in L_1(G)$. For $U \subseteq G$, let $\xi_U$ be the characteristic function of $U$. Next, let $L_\infty(G)$ consist of the functions measurable and essentially bounded on $G$. An important subspace of $L_\infty(G)$ is $C_{ru}(G)$, which is the space of all bounded functions right uniformly continuous on $G$.

Received September 3, 1965.

If $E$ is a Banach space, let $E^*$ represent the topological conjugate space (dual) of $E$. The bidual of $E$ is then $E^{**}$, and it is well-known that $E$ can be embedded naturally as a subspace of $E^{**}$. Let the injection map be denoted by $\pi$. By identification, $L_1^*(G) = L_\infty(G)$, $L_1^{**}(G)$ is the space of bounded, finitely additive measures on $G$ absolutely continuous with respect to Haar measure, and for $p \in (1, \infty)$, $L_p^*(G) = L_q(G)$, where $1/p + /1q = 1$.

In this paper we shall use a multiplication introduced by Arens (**1**; **2**). For a Banach algebra $E$ with multiplication $\circ$, we describe the Arens multiplication $\circ$ in $E^{**}$ in the following way:

$$\begin{aligned} (x^* \circ x)y &= x^*(x \circ y), & x^* &\in E^*,\ x, y \in E, \\ (y^{**} \circ x^*)x &= y^{**}(x^* \circ x), & y^{**} &\in E^*,\ x^* \in E^*,\ x \in E, \\ (x^{**} \circ y^{**})x^* &= x^{**}(y^{**} \circ x^*), & x^{**}, y^{**} &\in E^{**},\ x^* \in E^*. \end{aligned}$$

As defined, $x^* \circ x \in E^*$ with $||x^* \circ x|| \leqslant ||x^*||\,||x||$, $y^{**} \circ x^* \in E^*$ with $||y^{**} \circ x^*|| \leqslant ||y^{**}||\,||x^*||$, and $x^{**} \circ y^{**} \in E^{**}$ with $||x^{**} \circ y^{**}|| \leqslant ||x^{**}||\,||y^{**}||$. Indeed, under $\circ$, $E^{**}$ is a Banach algebra which is at the same time an algebraic extension of $E$. Civin and Yood (**4**) pointed out that if $E$ has an approximate identity, then $R^{**}$ has a right identity under this Arens multiplication. In particular, $L_1(G)$ under convolution has this property. However, this right identity in $L_1^{**}(G)$ is not a left identity unless $G$ is discrete!

*Note.* Although we denote several compositions by $\circ$, it should not be confusing, for a simple glance to the right and to the left of the symbol $\circ$ shows us immediately where the bilinear operation acts.

A second multiplication on $E^{**}$, which we call the *transposed* Arens multiplication, is created thus:

$$\begin{aligned} (x \circ x^*)y &= x^*(y \circ x), & x^* &\in E^*,\ x, y \in E, \\ (x^* \circ x^{**})x &= x^{**}(x \circ x^*), & x^{**} &\in E^{**},\ x^* \in E^*,\ x \in E, \\ (x^{**} \circ y^{**})x^* &= y^{**}(x^* \circ x^{**}), & x^{**}, y^{**} &\in E^{**},\ x^* \in E^*. \end{aligned}$$

As before $E^{**}$ is a Banach algebra extension of $E$, and with the transposed Arens multiplication, $E^{**}$ has a left identity element whenever $E$ has an approximate identity. There is a connection between the two Arens multiplications. If $E$ is commutative, then the two multiplications on $E^{**}$ coincide precisely when $E^{**}$ is commutative with respect to (either) Arens multiplication.

We call $(K, \circ)$ a left $L_1(G)$-module if $K$ is a Banach space and if $\circ$ is a bilinear operation $\circ : L_1(G) \times K \to K$, with the following properties:

(a) $(f * g) \circ k = f \circ (g \circ k)$, $\qquad k \in K,\ f, g \in L_1(G)$,
(b) $||f \circ k|| \leqslant ||f||\,||k||$, $\qquad k \in K,\ f \in L_1(G)$,

where $*$ denotes convolution in $L_1(G)$. Normally we shall abbreviate $(K, \circ)$ to $K$. Right $L_1(G)$-modules are defined analogously.

Let $M(G)$ denote the Banach space of bounded, countably additive regular measures on $G$. Then $M(G)$ can be made into both a left and a right $L_1(G)$-module. We define the convolutions $*: L_1(G) \times M(G) \to L_1(G)$ and $*: M(G) \times L_1(G) \to L_1(G)$ (**8**, p. 290) by

$$(2.1) \qquad (\mu * f)t = \int_G f(s^{-1}t)d\mu(s), \qquad f \in L_1(G),\ \mu \in M(G),$$

$$(2.2) \qquad (f * \mu)t = \int_G \Delta(s^{-1})f(ts^{-1})d\mu(s), \quad f \in L_1(G),\ \mu \in M(G),$$

for $m$-almost all $t \in G$. By (**8**, Theorems 20,9. 20.12, and 20.13), $M(G)$ is both a left and a right $L_1(G)$-module. Even more, if $f \in L_1(G)$, $\mu \in M(G)$, then $f * \mu$ and $\mu * f$ are in $L_1(G)$. We next observe that since $M(G)$ is a factor space of $L_1^{**}(G)$, it inherits both the regular and the transposed Arens multiplications, $\circ$, with the result that $M(G)$ is a left $L_1(G)$-module under the two compositions $*$ and $\circ$. However, the Arens composition and the convolution which render $M(G)$ a left $L_1(G)$-module are identical. Indeed, let $\mu \in M(G)$, $f \in L_1(G)$, and let $h \in L_\infty(G)$. Then by an application of Fubini's theorem,

$$(2.3) \qquad (h \circ f)s = \Delta(s^{-1}) \int_G h(t)f(ts^{-1})dt \qquad \text{for almost all } s \in G,$$

whereupon

$$\begin{aligned}(\pi(f) \circ \mu)h = \mu(h \circ f) &= \int_G \Delta(s^{-1})[\textstyle\int_G h(t)f(ts^{-1})dt]d\mu(s)\\ &= \int_G h(t)[\textstyle\int_G \Delta(s^{-1})f(ts^{-1})d\mu(s)]dt\\ &= h(f * \mu),\end{aligned}$$

so that $\pi(f) \circ \mu = f * \mu$. We summarize in

2.1. LEMMA. *Under the two convolutions, $M(G)$ is a left and a right $L_1(G)$-module. The Arens composition (the transposed Arens composition) and the convolution making $M(G)$ a left (right) $L_1(G)$-module coincide.*

*Proof.* Except for the proof that the transposed Arens multiplication coincides with the convolution which makes $M(G)$ a right $L_1(G)$-module, everything has been proved above. The remainder of the proof follows the same pattern as the part presented.

One might like to alter the compositions (2.1) and (2.2) in the following way. Consider (2.1). Let $p \in [1, \infty]$. Instead of taking $\mu$ in $M(G)$, let $\mu \in L_1(G)$, and instead of having $f \in L_1(G)$, let $f \in L_p(G)$. It is verified in (**8**, Theorem 20.12) that with this alteration of (2.1), there is a composition $*: L_1(G) \times L_p(G) \to L_p(G)$ such that

$$(2.4) \qquad (f * k)s = \int_G f(st)k(t^{-1})dt = \int_G f(t)k(t^{-1}s)dt, \quad f \in L_1(G),\ k \in L_p(G),$$

for all $s \in G$. Furthermore, we have for this composition

2.2. LEMMA. *For $p \in [1, \infty]$, $L_p(G)$ is a left $L_1(G)$-module.*

https://doi.org/10.4153/CJM-1967-008-4 Published online by Cambridge University Press

*Proof.* The norm inequality follows from an application of Hölder's inequality. On the other hand, for $f, g \in L_1(G)$, $k \in L_p(G)$,

$$\begin{aligned}[(f * g) * k]s &= \int_G [\int_G f(tu)g(u^{-1})du]k(t^{-1}s)dt \\ &= \int_G [\int_G f(sw)g(w^{-1}v)dw]k(v^{-1})dv \\ &= [f * (g * k)]s,\end{aligned}$$

for almost all $s \in G$.

2.3. LEMMA. *For $p \in [1, \infty)$, $L_1(G) * L_p(G)$ is norm dense in $L_p(G)$. Furthermore, $L_1(G) * L_\infty(G)$ is norm dense in $C_{ru}(G)$.*

*Proof.* The first half of the lemma is proved in (**8**, Theorem 20.15). For the second half, let $k \in C_{ru}(G)$ and let $\epsilon > 0$. Then there is a symmetric neighbourhood $U$ of $1 \in G$ such that $m(U) < \infty$ and such that if $rs^{-1} \in U$, then $|k(r) - k(s)| < \epsilon$. Let $g = [1/m(U)]\xi_U$. Then for any $t \in G$,

$$\begin{aligned}|(g * k)t - k(t)| &= |\int_G g(s)k(s^{-1}t)ds - k(t)| \\ &\leqslant [1/m(U)]\int_U |[k(s^{-1}t) - k(t)]|ds < \epsilon\end{aligned}$$

so that $||g * k - k|| < \epsilon$.

Actually more than this can be proved. Indeed, $L_1(G) * L_p(G) = L_p(G)$ for $p \in [1, \infty)$, and $L_1(G) * L_\infty(G) = C_{ru}(G)$, as proved in (**7**). We shall use these facts later, indirectly.

If we next try to alter (2.2) in a similar fashion, we find that the composition of two elements need not result in an element of $L_p(G)$. Consequently, in order to render $L_p(G)$ a right $L_1(G)$-module, we need to use another approach. For $f \in L_1(G)$, let $f'$ be defined by the equation

(2.5) $$f'(s) = \Delta(s^{-1})f(s^{-1}) \qquad \text{for almost all } s \in G.$$

Then $f' \in L_1(G)$ and $||f|| = ||f'||$ (**8**, Theorem 20.2). Next, we define the composition $\circ : L_p(G) \times L_1(G) \to L_p(G)$, $p \in (1, \infty]$, by the formula

(2.6) $$k \circ f = f' * k, \qquad f \in L_1(G),\ k \in L_p(G).$$

2.4. LEMMA. *The map $f \to f'$ is an isometric algebraic anti-isomorphism of $L_1(G)$.*

*Proof.* Since $\Delta$ is a real-valued homomorphism, $(f')' = f$. The isometry follows from the above remark and we need only check that if $f, g \in L_1(G)$, then $(f * g)' = g' * f'$ almost everywhere in $G$. Let $s \in G$. Then (almost always)

$$\begin{aligned}(f * g)'(s) &= \Delta(s^{-1})\int_G f(u)g(u^{-1}s^{-1})du \\ &= \int_G g(u^{-1} s^{-1})\Delta(u^{-1} s^{-1})f(u)\Delta(u)du \\ &= (g' * f')(s).\end{aligned}$$

2.5. LEMMA. *$L_p(G)$ is a right $L_1(G)$-module under* $\circ$.

*Proof.* For $f \in L_1(G)$, $k \in L_p(G)$, we have

$$||k \circ f|| = ||f' * k|| \leqslant ||f'|| \, ||k|| = ||f|| \, ||k||.$$

If $g \in L_1(G)$, then by Lemma 2.4,

$$k \circ (f * g) = (f * g)' * k = (g' * f') * k = g' * (f' * k) = (k \circ f) \circ g.$$

We remark that via the first step in the construction of the Arens multiplication in $L_1^{**}(G)$, $L_1^*(G) = L_\infty(G)$ becomes a right $L_1(G)$-module. The composition defined above and this Arens composition on $L_\infty(G)$ are the same. For if $f \in L_1(G)$ and $k \in L_\infty(G)$, then

$$(k \circ f)s = (f' * k)s = \int_G \Delta(t^{-1}) f(t^{-1}) k(t^{-1}s) dt$$
$$= \Delta(s^{-1}) \int_G k(t) f(ts^{-1}) dt$$

for almost all $s \in G$, which is just the right-hand expression in (2.3).

Our final comments in this section concern $C_{ru}(G)$, the space of all right uniformly continuous, bounded functions on $G$. Since $C_{ru}(G)$ is a subspace of $L_\infty(G)$, we see that under the Arens composition, $C_{ru}(G)$ is a right $L_1(G)$-module. Furthermore, if $h \in C_{ru}^*(G)$ and $k \in C_{ru}(G)$, we define $h \circ k \in C_{ru}(G)$ by the equation $(h \circ k)f = h(k \circ f)$, for $f \in L_1(G)$, which makes sense since $L_\infty(G) \circ L_1(G) = L_1(G) * L_\infty(G) \subseteq C_{ru}(G)$ by Lemmas 2.3 and 2.4. On the other hand, Buck (**8**, p. 275) has defined a composition $* : C_{ru}^*(G) \times C_{ru}(G) \to C_{ru}(G)$ by the equation $(h * k)s = h({}_sk)$, for all $s \in G$.

2.6. LEMMA. *The Buck and the Arens compositions from $C_{ru}^*(G) \times C_{ru}(G)$ into $C_{ru}(G)$ are identical.*

*Proof.* Let $g \in L_1(G)$, and let $s \in G$. Then let $g_1 = \Delta(s^{-1}) g_{s^{-1}}$. Immediately, we obtain $g * ({}_{s^{-1}}f) = g_1 * f$, for almost all $s \in G$. In addition, if $k \in L_\infty(G)$, then for each $f \in L_1(G)$,

$$[{}_s(k \circ g)]f = (\int_G f(t)[{}_s(k \circ g)]t dt$$
$$= \int_G f(t) \Delta(t^{-1}) [\int_G k(u) \Delta(s^{-1}) g(ut^{-1} s^{-1}) du] dt$$
$$= (k \circ g_1) f,$$

so that ${}_s(k \circ g) = k \circ g_1$. Now let $h \in C_{ru}^*(G)$ and let $(e_\lambda)_{\lambda \in \Lambda}$ be a bounded approximate identity in $L_1(G)$. Then

$$[h * (k \circ g)]s = h({}_s(k \circ g)) = h(k \circ g_1) = \lim_\lambda h(k \circ (g_1 * e_\lambda))$$
$$= \lim_\lambda h(k \circ (g *_{s^{-1}} e_\lambda)) = \lim_\lambda [h \circ (k \circ g)]({}_{s^{-1}} e_\lambda)$$
$$= {}_s[h \circ (k \circ g)] = (h \circ (k \circ g))s,$$

and since $s$ is arbitrary, $h * (k \circ g) = h \circ (k \circ g)$. By the comments following Lemma 2.3, $L_\infty(G) \circ L_1(G) = C_{ru}(G)$, whence the result.

https://doi.org/10.4153/CJM-1967-008-4 Published online by Cambridge University Press

It is apparent that if $h_1, h_2 \in C_{ru}^*(G)$, then we can define $h_1 \circ h_2$ by $(h_1 \circ h_2)k = h_1(h_2 \circ k)$ for $k \in C_{ru}(G)$, with the result that $C_{ru}^*(G)$ becomes a Banach algebra under such a multiplication. Then Lemma 2.6 shows us that the Arens and the Buck compositions give identical multiplications on $C_{ru}^*(G)$.

**3. Characterizations of $(L_1(G), K)$-homomorphisms.** In the present section we shall discuss module homomorphisms from $L_1(G)$ to an arbitrary $L_1(G)$-module $K$. First, however, we prove the preliminary

3.1. LEMMA. *With respect to the Arens (or the transposed Arens) multiplication, the following statements are equivalent:*

(i) *$G$ is compact.*
(ii) $\pi(L_1(G)) \circ L_1^{**}(G) \subseteq \pi(L_1(G))$.
(iii) $L_1^{**}(G) \circ \pi(L_1(G)) \subseteq \pi(L_1(G))$.

*Proof.* For the Arens multiplication, the proof that (i) implies both (ii) and (iii) is a straightforward generalization of (**3**, Theorem 2.4). On the other hand, assume that $G$ is not compact. Let $(D_\lambda)_{\lambda \in \Lambda}$ be the net of compact sets of $G$, ordered by inclusion. For each $\lambda \in \Lambda$, let $k_\lambda = \xi_{D\lambda}$, and let $k = \xi_G$, so that $k_\lambda \to k$ in the weak-$(L_1(G))$ topology on $L_1^*(G)$. Let $f \in L_1(G)$ such that $\int_G f(t)dt \neq 0$. Then $k_\lambda \circ f \to k \circ f$ in the weak-$(L_1(G))$ topology. In addition, $k_\lambda \circ f \in C_\infty(G)$, for each $\lambda \in \Lambda$, whereas for each $s \in G$,

$$(k \circ f)s = \Delta(s^{-1})\int_G k(t)f(ts^{-1})dt = \int_G f(t)dt \neq 0,$$

so that $k \circ f \notin C_\infty(G)$. Now define $\mu \in L_1^{**}(G)$ so that

$$\mu|_{C_\infty(G)} = 0 \qquad \text{and} \qquad \mu(k \circ f) = 1.$$

Then

$$(\pi(f) \circ \mu)k_\lambda = \mu(k_\lambda \circ f) \not\to \mu(k \circ f) = (\pi(f) \circ \mu)k,$$

which means that $\pi(f) \circ \mu \notin \pi(L_1(G))$; (**11**, Theorem 1.76). Therefore

$$\pi(L_1(G)) \circ L_1^{**}(G) \not\subseteq \pi(L_1(G)),$$

so (ii) implies (i). To show that if $G$ is not compact, then

$$L_1^{**}(G) \circ \pi(L_1(G)) \not\subseteq \pi(L_1(G)),$$

we need only remark that in the above proof,

$$(\pi(f) \circ k)s = \int_G f(t)dt \neq 0;$$

thus the proof may be used in the obvious modified form. Hence (iii) implies (i). For the transposed Arens multiplication the proofs are similar.

We now study $(L_1(G), K)$-homomorphisms.

3.2. *Definition.* Let $K_1$, $K_2$ be left $L_1(G)$-modules. A map $R : K_1 \to K_2$ is called a left $(K_1, K_2)$-homomorphism if $R$ is a continuous, linear operator and if $R(f \circ k) = f \circ R(k)$ for each $k \in K_1$, $f \in L_1(G)$. Likewise, if $K_1$, $K_2$ are right $L_1(G)$-modules, then the map $R : K_1 \to K_2$ is a right $(K_1, K_2)$-homomorphism if $R$ is a continuous, linear operator and $R(k \circ f) = R(k) \circ f$ for each $k \in K_1, f \in L_1(G)$. The collection of $(K_1, K_2)$-homomorphisms is denoted by $\mathfrak{R}(K_1, K_2)$. They are left or right if $K_1$ and $K_2$ are left or right $L_1(G)$-modules, respectively.

In this section we shall let $K_1 = L_1(G)$, and we shall abbreviate $K_2$ to $K$. Through Theorem 3.6, let $K$ be a left $L_1(G)$-module. First we extend the module composition in $K$ in analogy with the construction of the Arens multiplication. For $f \in L_1(G)$, $k \in K$, and $k^* \in K^*$, define $(k^* \circ f)$ by the equation

$$(k^* \circ f)k = k^*(f \circ k), \qquad k \in K.$$

A routine check verifies that $k^* \circ f \in K^*$, and that the new operation is bilinear and jointly continuous and renders $K^*$ a right $L_1(G)$-module. Next, for $f \in L_1(G)$, $k^* \in K^*$, and $k^{**} \in K^{**}$, we define $(f \circ k^{**})$ by the equation

$$(f \circ k^{**})k^* = k^{**}(k^* \circ f), \qquad k^* \in K^*.$$

Again $f \circ k^{**}$ is linear and continuous, so $f \circ k^{**} \in K^{**}$.

Let the Arens compositions relating $I_1^{**}(G)$ and $L_1^*(G)$ and $L_1(G)$ be denoted by $\nabla$.

3.3. Lemma. *Let $R \in \mathfrak{R}(L_1(G), K)$. Then the adjoint $R^* : K^* \to L_1^*(G)$ satisfies the relation $R^*(k^* \circ f) = R^*(k^*) \nabla f$, for $f \in L_1(G)$, $k^* \in K^*$.*

*Proof.* For $g \in L_1(G)$, we have

$$[R^*(k^* \circ f)]g = (k^* \circ f)(R(g)) = k^*(f \circ R(g)) = k^*(R(f * g))$$
$$= (R^*(k^*))(f * g) = (R^*(k^*) \nabla f)g.$$

3.4. Lemma. *Let $R \in \mathfrak{R}(L_1(G), K)$. Let $e^{**}$ be a cluster point in $L_1^{**}(G)$ of an approximate identity $(e_\lambda)_{\lambda \in \Lambda}$ in $L_1(G)$. Then for each $f \in L_1(G)$, $R$ is given by*

$$\pi(R(f)) = f \circ R^{**}(e^{**}).$$

*Proof.* Since $R$ is continuous, its second adjoint $R^{**}$ is weak-$(L_1(G), K^*)$ continuous, which means that $R^{**}(\pi(e_\lambda)) \to R^{**}(e^{**})$ in the weak-$(K^*)$ topology of $K^{**}$. By thacing through the module compositions, we find that this implies that

$$[f \circ R^{**}(\pi(e_\lambda))]k^* \to (f \circ R^{**}(e^{**}))k^* \qquad \text{for any } k^* \in K^*.$$

Thus

$$\pi[R(f)] = \lim_\lambda \pi[R(f * e_\lambda)] = \lim_\lambda \pi[f \circ R(e_\lambda)] = \lim_\lambda [f \circ R^{**}(\pi(e_\lambda))]$$
$$= f \circ R^{**}(e^{**}).$$

Recently it has been proved (7) that any $L_1(G)$-module $K^*$ has the property that $K^* \circ L_1(G)$ is a linear space. Now if $R \in \mathfrak{R}(L_1(G), K)$ and if $e^{**}$

https://doi.org/10.4153/CJM-1967-008-4 Published online by Cambridge University Press

is a cluster point in $L_1^{**}(G)$ for an approximate identity in $L_1(G)$. we define $\rho_R$ by

$$\rho_R = R^{**}(e^{**})|_{(K^* \circ L_1(G))}.$$

Indeed $\rho_R$ is then fully defined on $K^* \circ L_1(G)$, inherits linearity and continuity from $R^{**}$, and in fact has its norm equal to the norm of $R^{**}(e^{**})$ restricted to $K^* \circ L_1(G)$. Thus $\rho_R \in (K^* \circ L_1(G))^*$. The fact that $K^* \circ L_1(G)$ is a full linear space has not been used. We could have defined $\rho_R$ on $K^* \circ L_1(G)$ and extended it linearly. However, the observation served to simplify the notation.

We mention that if $f \in L_1(G)$ and $k^* \in K^*$, then

$$k^*(R(f)) = (f \circ R^{**}(e^{**}))k^* = (R^{**}e^{**}))(k^* \circ f) = \rho_R(k^* \circ f),$$

so that there is a canonical equation joining $R$ and $\rho_R$ : $k^*(R(f)) = \rho_R(k^* \circ f)$.

3.5. THEOREM. *The map* $\rho : R \to \rho_R$ *is a linear isometric map of* $\mathfrak{R}(L_1(G), K)$ *into* $(K^* \circ L_1(G))^*$.

*Proof*. First we must show that the map is independent of the choice of $e^{**}$. Let $e_1^{**}$ and $e_2^{**}$ be cluster points of an approximate identity for $L_1(G)$. This means that $e_1^{**} = e_2^{**}$ when each is restricted to $L_\infty(G) \nabla L_1(G) = C_{ru}(G)$. Consequently, for any $f \in L_1(G)$ and $k^* \in K^*$,

$$(R^{**}(e_1^{**}))(k^* \circ f) = e_1^{**}(R^*(k^* \circ f)) = e_1^{**}\{[R^*(k^*)] \nabla f\}$$
$$= e_2^{**}\{[R^*(k^*)] \nabla f\} = (R^{**}(e_2^{**}))(k^* \circ f),$$

so that $\rho_R$ does not depend on the choice of $e^{**}$. As the adjoint of a linear operator is linear, the map $R \to \rho_R$ is linear. Finally, to show that the map is an isometry, we first observe that $\rho_R$ is a restriction of $R^{**}(e^{**})$, so $||\rho_R|| \leqslant ||R||$. On the other hand, let $f \in L_1(G)$ with $||f|| \leqslant 1$. Then

$$||R(f)|| = ||f \circ R^{**}(e^{**})|| = \sup_{||k^*|| \leqslant 1} |(f \circ R^{**}(e^{**}))k^*|$$
$$\leqslant \sup_{||k^* \circ f|| \leqslant 1} |\rho_R(k^* \circ f)| = ||\rho_R||.$$

Therefore $||\rho_R|| = ||R||$.

We proceed to give a criterion for the map $\rho$ to be onto $(K^* \circ L_1(G))^*$. For $p \in (K^* \circ L_1(G))^*$, we define $R'_p$ by the formula

$$(R'_p(k^*))f = p(k^* \circ f), \qquad f \in L_1(G),\ k^* \in K^*.$$

Then $R'_p : K^* \to L_\infty(G)$ is a linear continuous operator since $\circ$ is bilinear and since $p \in (K^* \circ L_1(G))^*$. For our given $p$, we should like to find a linear continuous operator $R_p : L_1(G) \to K$ with the property that $k^*(R_p(f)) = p(k^* \circ f)$ for all $f \in L_1(G)$, $k^* \in K^*$. But the existence of such an $R_p$ would mean that for all $f$ amd $k^*$, $(R'_p(k^*))f = k^*(R_p(f))$, which says precisely that $R'_p$ is the adjoint operator of $R$ . However, a necessary and sufficient condition for $R'_p$ to be an adjoint operator is for $R'_p$ to be weak-$(K, L_1(G))$ continuous. So suppose that $R'_p$ is indeed weak-$(K, L_1(G))$ continuous, with the result that $(R'_p(k^*))f = k^*(R_p(f))$. We shall show that $R_p \in \mathfrak{R}(L_1(G), K)$; that for any $f, g \in L_1(G)$, we have $R_p(f * g) = f \circ R_p(g)$. The module operations on $K$ yield the relation $(k^* \circ f) \circ g = k^* \circ (f * g)$, for all $k^* \in K^*$. This, together with the relation joining $R'_p$ and $p$, yields $R'_p(k^* \circ f) = (R'_p(k^*)) \nabla f$. Hence for each $k^* \in K^*$,

$$k^*[R_p(g * f)] = (R'_p(k^*))(f * g) = [R'_p(k^*) \nabla f]g = [R'_p(k^* \circ f)]g$$
$$= k^*[f \circ (R_p(g))].$$

Consequently $R_p \in \mathfrak{R}(L_1(G), K)$. Besides, $k^*(R_p(g)) = p(k^* \circ f)$, so that $\rho(R_p) = p$.

Thus we have found a criterion for the map $\rho$ to be onto: for each $p \in (K^* \circ L_1(G))^*$, the operator $R'_p$ must be weakly-$(K, L_1(G))$ continuous. We now display a more pliable criterion.

3.6. THEOREM. *The map $\rho$ is an isometry from $\mathfrak{R}(L_1(G), K)$ onto $(K^* \circ L_1(G))^*$ if and only if $L_1(G) \circ K^{**} \subseteq \pi(K)$.*

*Proof.* On the one hand, assume that $L_1(G) \circ K^{**} \subseteq \pi(K)$, and let $p \in (K^* \circ L_1(G))^*$. Let $k_\lambda^* \to k_0^*$ in $K^*$ in the weak-$(K)$ topology and let $f$ be an arbitrary element of $L_1(G)$. Define $(f \circ p)$ by the relation

$$(f \circ p)k^* = p(k^* \circ f), \qquad k^* \in K^*.$$

Then clearly $f \circ p \in K^{**}$. We must prove that $(R'_p)(k_\lambda^*))f \to (R'_p(k_0^*))f$, or what is the same, $(f \circ p)k^*_\lambda \to (f \circ p)k_0^*$. By virtue of the Hahn-Banach theorem, extend $p$ to $k^{**} \in K^{**}$. Then

$$(f \circ k^{**})k^* = k^{**}(k^* \circ f) = p(k^* \circ f) = (f \circ p)k^* \qquad \text{for any } k^* \in K^*,$$

whence $f \circ k^{**} = f \circ p \in K^{**}$. However, by hypothesis, $L_1(G) \circ K^{**} \subseteq \pi(K)$, so that $f \circ p \in \pi(K)$, and therefore $(f \circ p)k_\lambda^* \to (f \circ p)k_0^*$; therefore $R'_p$ is weakly-$(K, L_1(G))$ continuous. On the other hand, assume that

$$L_1(G) \circ K^{**} \not\subseteq \pi(K).$$

Let $f \in L_1(G)$ and $k^{**} \in K^{**}$ such that $f \circ k^{**} \notin \pi(K)$. Let $p$ be the restriction of $k^{**}$ to $(K^* \circ L_1(G))$. Then as in the first part of the proof, we find that $f \circ p = f \circ k^{**}$. Since $f \circ k^{**} \notin \pi(K)$, there exists a net $(k_\lambda^*)_{\lambda \in \Lambda}$ in $K^*$ converging in the weak-$(K)$ topology to $k_0^*$ in $K^*$ for which

$$(f \circ k^{**})k_\lambda^* \not\to (f \circ k^{**})k_0^*.$$

But this just means that

$$(R'(k_\lambda^*))f = (f \circ p)k_\lambda^* \not\to (f \circ p)k_0^*) = (R'_p)(k_0^*))f,$$

with the result that $R'_p$ is not weakly-$(K, L_1(G))$ continuous.

Next consider the right $L_1(G)$-module $K$ and let $R : L_1(G) \to K$ be a right $(L_1(H, K)$-homomorphism, so that $R(f * g) = R(f) \circ g$, $f, g \in L_1(G)$. We define $(f \circ k^*)$ and $(k^{**} \circ f)$ by the equations

$$(f \circ k^*)k = k^*(k \circ f), \qquad f \in L_1(G), k^* \in K^*, k \in K,$$

$$(k^{**} \circ f)k^* = k^{**}(f \circ k^*), \qquad f \in L_1(G), k^{**} \notin K^{**}, k^* \in K^*.$$

For the bilinear operations joining $L_1^{**}(G)$, $L_1^*(G)$, and $L_1(G)$, we choose the transposed Arens multiplication. We define $\rho_R$ by the equation

$$\rho_R = R^{**}(e^{**})|_{(L_1(G) \circ K^*)}$$

where $e^{**}$ is again a cluster point in $L_1^{**}(G)$ for an approximate identity in $L_1(G)$. (Once again **(7)** shows that $L_1(G) \circ K^*$ is a linear space.) We obtain the equation $k^*(R(f)) = \rho_R(f \circ k^*)$, resulting in

3.7. THEOREM. *The map $\rho$ is an isometry from $\mathfrak{R}(L_1(G), K)$ into $(L_1(G) \circ K^*)^*$. It is onto $(L_1(G) \circ K^*)^*$ if and only if $K^{**} \circ L_1(G) \subseteq \pi(K)$.*

*Proof.* The proof imitates those of Theorems 3.5 and 3.6, but with the alternative definitions described above.

Let us analyse Theorems 3.6 and 3.7. In the first place, if $G$ is compact, then $L_1(G) \circ L_1^{**}(G) \subseteq \pi(L_1(G))$ and $L_1^{**}(G) \circ L_1(G) \subseteq \pi(L_1(G))$, and also

$$(L_1^*(G) \circ L_1(G))^* = (L_1(G) \circ L_1^*(G))^* = M(G),$$

essentially by Lemmas 2.3 and 3.1. Therefore if $G$ is compact, then $\mathfrak{R}(L_1(G))$, $L_1(G))$ corresponds isometrically to $M(G)$. On the other hand, if $G$ is not compact, we gain no definitive information.

In the second place, if $p \in (1, \infty)$, then $L_p(G)$ is reflexive, and

$$(L_1(G) \circ L_p^*(G))^* = (L_p^*(G) \circ L_1(G))^* = L_p(G),$$

so that $\mathfrak{R}(L_1(G), L_p(G))$ corresponds isometrically to the space $L_p(G)$, and this holds for arbitrary $G$.

Finally we encounter $p = \infty$. If we can show that

$$L_1(G) \circ L_\infty^{**}(G) \subseteq \pi(L_\infty(G))$$

and $L_\infty^{**}(G) \circ L_1(G) \subseteq \pi(L_\infty(G))$ whenever $G$ is compact, then since

$$(L_\infty^*(G) \circ L_1(G))^* = (L_1(G) \circ L_\infty^*(G))^* = L_\infty(G)$$

if $G$ is compact (by Lemma 3.1), we have the result that $\mathfrak{R}(L_1(G), L_\infty(G))$ corresponds isometrically to $L_\infty(G)$—provided that $G$ is compact. Once again, if $G$ is not compact, we have no complete solution from Theorems 3.6 and 3.7. Now we show that $L_1(G) \circ L_\infty^{**}(G) \subseteq \pi(L_\infty(G))$ if and only if $G$ is compact, but we omit the proof that $L_\infty^{**}(G) \circ L_1(G) \subseteq \pi(L_\infty(G))$ if and only if $G$ is compact, because it is analogous. Assume that $G$ is compact, and let $f \in L_1(G)$ and $k^{**} \in L_\infty^{**}(G)$. Let $\pi(j)$ be the restriction of $k^{**}$ to $\pi(L_1(G))$, so that $j \in L_\infty(G)$. For any $k^* \in L_\infty^*(G)$, we have $k^* \circ f \in \pi(L_1(G))$, by Lemma 3.1, so that

$$(f \circ k^{**})k^* = k^{**}(k^* \circ f) = [\pi(j)](k^* \circ f) = (\pi(f) \circ \pi(j))k^*.$$

But $\pi(f) \circ \pi(j) \in \pi(L_\infty(G))$, so that $f \circ k^{**} \in \pi(L_\infty(G))$, and

$$L_1(G) \circ L_\infty^{**}(G) \subseteq \pi(L_\infty(G)).$$

Conversely, if $G$ is not compact, then by Lemma 3.1,

$$L_1^{**}(G) \circ L_1(G) \not\subseteq \pi(L_1(G)),$$

so that there exist $f \in L_1(G)$ and $k^* \in L_1^{**}(G)$ with the property that $k^* \circ f \notin \pi(L_1(G))$. Define $k^{**} \in L_\infty^{**}(G)$ such that

$$k^{**}|_{\pi(L_1(G))} = 0 \quad \text{and} \quad (f \circ k^{**})k^* = k^{**}(k^* \circ f) \neq 0.$$

Then $f \circ k^{**}$ is not completely defined on $\pi(L_1(G))$, so $f \circ k^{**} \notin \pi(L_\infty(G))$, completing our proof.

We conjecture that there is an attractive representation for $\mathfrak{R}(L_1(G), K)$, regardless of the $L_1(G)$-module $K$. Although we have not found one, we can add to the knowledge obtained from Theorems 3.6 and 3.7 and the comments following them. We shall presently describe $\mathfrak{R}(L_1(G), K)$ when $K = L_p(G)$, $p \in [1, \infty]$, without regard to the compactness of $G$. Before we state and prove our theorems we give the following definition and lemma.

3.8. *Definition.* Let $p_1, p_2 \in [1, \infty]$. We call a linear, continuous operator $R : L_{p_1}(G) \to L_{p_2}(G)$ left translation-invariant if

$$_s(R(g)) = R(_sg) \qquad \text{for each } g \in L_{p_1}(G) \text{ and all } s \in G.$$

Likewise, such an operator is right translation-invariant if

$$(R(g))_s = R(g_s) \qquad \text{for each } g \in L_{p_1}(G) \text{ and all } s \in G.$$

3.9. Lemma. *Let $R : L_{p_1}(G) \to L_{p_2}(G)$ be a linear continuous operator, where $p_1 \in [1, \infty]$ and $p_2 \in (1, \infty]$. Then $R$ is a left $(L_{p_1}(G), L_{p_2}(G)$-homomorphism if and only if $R$ is left translation-invariant. Furthermore, $R$ is a right $(L_1(G), L_1(G))$-homomorphism if and only if $R$ is right translation-invariant.*

*Proof.* Let $f \in L_1(G)$, $g \in L_{p_1}(G)$, and $h \in L_q(G)$, where $1/p_2 + 1/q = 1$ and if $p_2 = \infty$, we take $q = 1$. Then

$$\begin{aligned} h[R(f * g)] &= \int_G R(f * g)(s)h(s)ds \\ &= \int_G f(t)[\int_G (R*(h))(s)g(t^{-1}s)ds]dt \\ &= \int_G f(t)[\int_G h(s)R(_{t^{-1}}g)(s)ds]dt, \end{aligned}$$

whereas

$$\begin{aligned} h(f * R(g)) &= \int_G h(s)(f * R(g))(s)ds \\ &= \int_G f(t)\{\int_G h(s)(_{t^{-1}}[R(g)](s)ds\}dt. \end{aligned}$$

We readily verify that indeed $R(f * g) = f * R(g)$ if $R$ is left translation-invariant. Next, let $R$ be a left $(L_{1p}(G), L_{p_2}(G))$-homomorphism, $(e_\lambda)_{\lambda \in \Lambda}$ an approximate identity, $f \in L_1(G)$ and $t \in G$. Then $R({}_tf) = \lim_\lambda R({}_te_\lambda * f) = = \lim_\lambda({}_te_\lambda * (Rf) = {}_t(Rf)$, so $R$ is translation-invariant. The statement for right homomorphisms is similarly proved.

We mention that the asymmetry involving $L_p(G)$, $p \in (1, \infty)$, as a left and a right $L_1(G)$-module precludes any relation between the right translation-invariant operators and right homomorphisms. Also, we remark that we shall conclude from Theorem 4.4 that any left $(L_\infty(G), L_\infty(G))$-homomorphisms is at the same time a left translation-invariant operator. Whether or not the converse is true we do not know.

For $\mu \in M(G)$, define $\mu_1$ in the following way. For any $A \subseteq G$ such that $A^{-1}$ is $\mu$-measurable, let $\mu_1(A) = \mu(A^{-1})$. Then $\mu_1 \in M(G)$, and the formula which relates $\mu$ and $\mu_1$ is

$$(\mu_1 * f)s = \int_G f(ts)d\mu(t), \qquad f \in L_1(G), \qquad m\text{-almost all } s \in G.$$

For a linear map $R : L_1(G) \to L_1(G)$, let $R_1$ be defined by the equation

$$R_1(f) = [R(f')]', \qquad f \in L_1(G).$$

Note that $(R_1)_1 = R$. In addition, if $R$ is a left $(L_1(G), L_1(G))$-homomorphism, then $R_1$ is also a right $(L_1(G), L_1(G))$-homomorphism. For let $f, g \in L_1(G)$. Then

$$R_1(f * g) = [R(f * g)']' = [R(g' * f')]' = [g' * R(f')]'$$
$$= [R(f')]' * g = [R_1(f)] * g,$$

with the help of Lemma 2.4. Now we are ready for

3.10. THEOREM. *There is an isometric isomorphism between $\mathfrak{R}(L_1(G), L_1(G))$ and $M(G)$. For a linear, continuous operator $R: L_1(G) \to L_1(G)$, the following statements are equivalent:*

(i) $R(f * g) = f * R(g)$, $\quad f, g \in L_1(G)$.
(ii) $R(f) = f * \mu$, $\quad f \in L_1(G)$, *for some* $\mu \in M(G)$.
(iii) *$R$ is left translation-invariant.*
(iv) $R_1(f * g) = R_1(f) * g$, $\quad f, g \in L_1(G)$.
(v) $R_1(f) = \mu_1 * f$, $\quad f \in L_1(G)$, *for some* $\mu_1 \in M(G)$.
(vi) *$R_1$ is a right translation-invariant.*

*Proof.* The isometry and the equivalence of (i), (ii), and (iii) follow from (**13**, Theorems 1 and 4). The proof that (iv), (v), and (vi) are equivalent follows a similar course, and we omit it. To conclude, we show that if $f \in L_1(G)$, then $R(f) = f * \mu$ if and only if $R_1(f) = \mu_1 * f$. Indeed,

$$[\mu_1 * f']'(s) = \Delta(s^{-1})\int_G f'(ts^{-1})d\mu(t)$$
$$= \Delta(s^{-1}) \int_G \Delta(t^{-1})f(st^{-1})d\mu(t)$$
$$= (f * \mu)s$$

for almost all $s \notin G$, so that $(\mu_1 * f') = (f * \mu)'$, which means that $R(f) = f * \mu$ if and only if $R_1(f) = \mu_1 * f$, $f \in L_1(G)$; thus (i) is equivalent to (iv).

For a linear map $R : L_1(G) \to L_p(G)$, where $p \in (1, \infty]$, let $R_1$ be defined by the equation $R_1(f) = R(f')$, $f \in L_1(G)$. Note that the $R_1$ just defined differs from the one defined before Theorem 3.10. This is due in part to the lack of a corresponding anti-isomorphism which $L_1(G)$ has. In any case, the $R_1$ now defined is linear and continuous whenever $R$ is, and $(R_1)_1 = R$. Furthermore, if $R$ is a left $(L_1(G), L_p(G))$-homomorphism, then an easy check verifies that $R_1$ is a right $(L_1(G), L_p(G))$-homomorphism. It turns out that a consequence of the lack of symmetry in the definition of the convolution $L_1(G) * L_p(G)$, where $p \in (1, \infty]$, is that we cannot identify the right homomorphisms with either kind—right or left—of translation-invariant operators.

3.11. THEOREM. *There is an isometric isomorphism between* $\mathfrak{R}(L_1(G), L_p(G))$ *and* $L_p(G)$, *for* $p \in (1, \infty]$. *For a linear, continuous operator* $R : L_1(G) \to L_p(G)$, *the following statements are equivalent:*

(i) $R(f * g) = f * R(g)$, $f, g \in L_1(G)$.
(ii) $R(f) = f * h$, $f \in L_1(G)$, for *some* $h \in L_p(G)$.
(iii) *$R$ is left translation-invariant.*
(iv) $R_1(f * g) = R_1(f) \circ g$, $f, g \in L_1(G)$.
(v) $R_1(f) = h \circ f$, $f \in L_1(G)$, *for some* $h \in L_p(G)$.

*Proof.* First we deal with $p \in (1, \infty)$. Theorems 3.6 and 3.7 yield the isometry, and the formula of (ii) for a left $(L_1(G), L_p(G))$-homomorphism $R$ comes directly from the formula joining $R$ and $\rho_R$ just preceding Theorem 3.5. As in Theorem 3.10, (ii) implies (iii), by a simple computation. By virtue of Lemma 3.9, (iii) implies (i), so that (i), (ii), and (iii) are equivalent. Similarly, (iv) and (v) are equivalent. Finally, (i) and (iv) are equivalent by the comments preceding this theorem. Now let $p = \infty$, and let $R$ be a left $(L_1(G), L_\infty(G))$-homomorphism. With the aid of Lemma 2.3, and an approximate identity in $L_1(G)$, we conclude that the image of $L_1(G)$ under $R$ is contained in $C_{ru}(G)$. Next, if $k \in C_{ru}(G)$, then $k$ is continuous, so $k(1)$ is defined. Then the map $h_1 : L_1(G) \to$ complexes, defined by $h_1(f) = [R(f)]1$, $f \in L_1(G)$, yields $h_1 \in L_\infty(G)$. For almost all $t \in G$, define $h(t) = h_1(t^{-1})$, so that $h \in L_\infty(G)$ (**8**, p. 295). Then $h_1(f) = (f * h)1$, and Lemma 3.9 shows us that for almost all $t \in G$,

$$(R(f))t = [{}_t(R(f))]1 = [(R_t f))]1 = ({}_t f * h)1 = (f * h)_t.$$

Consequently $R(f) = f * h$, for $f \in L_1(G)$. Inasmuch as $L_\infty(G)$ is a left $L_1(G)$-module, we have $||h|| \leqslant ||R||$. On the other hand, the proof of Lemma 2.3 yields $||h|| \geqslant ||R||$, whence the isometry. The definition of $R_1$ gives the equivalence of (i) and (iv), and the remainder of the proof takes the same course as that for $p \in (1, \infty)$.

We conclude this section with a new, simplified proof of a multiplier theorem (**12**, Theorem 3.8.1). First, though, we need a little additional notation. Let

$G$ be abelian, and let $\Gamma$ denote the character group of $G$. Let $A(\Gamma)$ be the collection of Fourier transforms of the elements of $L_1(G)$, and $B(\Gamma)$ the collection of Fourier transforms of elements in $M(G)$. For $f \in L_1(G)$ and $\mu \in M(G)$, let $f_1$ and $\mu_1$ be the Fourier transforms, respectively. Now we can state and prove the following

3.12. THEOREM **(12)**. *Let $h$ be a complex-valued function defined on $\Gamma$. Assume that $f \in L_1(G)$ implies $f_1 h \in B(\Gamma)$. Then $h \in B(\Gamma)$.*

*Proof.* We first note that $L_1(G)$ is a (two-sided) ideal in $M(G)$. This, together with the hypothesis and the fact that each element of $L_1(G)$ can be factored **(5**, Theorem 1**)**, means that if $f \in L_1(G)$, then $f_1 h \in A(\Gamma)$. Now define $R : L_1(G) \to L_1(G)$ by the rule $R(f) = g$, where $g_1 = f_1 h$, for all $f \in L_1(G)$. Then $R$ is linear, and the closed graph theorem yields the continuity of $R$. For $f, g \in L_1(G)$, we have

$$[R(f * g)]_1 = (f * g)_1 h = f_1 g_1 h = f_1(R(g))_1 = [f * (R(g))]_1,$$

whence $R(f * g) = f * R(g)$ and $R$ is an $(L_1(G), L_1(G))$-homomorphism. Consequently Theorem 3.10 applies, and there exists an $\mu \in M(G)$ such that $R(f) = f * \mu$, for all $f \in L_1(G)$. Therefore, by the semi-simplicity of $L_1(G)$, $\mu_1 = h$, so that $h \in B(\Gamma)$.

**4. Characterizations of $(K, L_\infty(G))$-homomorphisms.** We turn our attention now to $(K, L_\infty(G))$-homomorphisms, where $K$ is again an $L_1(G)$-module. At first we shall ask that $K$ be a right $L_1(G)$-module under the composition $\circ$, and we take $L_\infty(G)$ to be a right $L_1(G)$-module, under the Arens composition. Once again let the Arens compositions relating $L_1{}^{**}(G)$ and $L_1{}^*(G)$ and $L_1(G)$ be denoted by $\nabla$.

Let $R : K \to L_\infty(G)$ be a right $(K, L_\infty(G))$-homomorphism, and let $R^* : L_\infty{}^*(G) \to K^*$ be the adjoint of $R$. If $m \in L_\infty{}^*(G)$, $k \in K$, and $f \in L_1(G)$, then

$$(R^*(m))(k \circ f) = m[R(k \circ f)] = m[R(k) \nabla f] = [m \nabla R(k)]f.$$

If $e^{**}$ is a right identity in $L_\infty{}^*(G)$, then for each $k \in K$, $e^{**} \nabla R(k) = R(k)$ **(4**, Lemma 3.8**)**. Consequently

$$(R^*(e^{**}))(k \circ f) = [e^{**} \nabla R(k)]f = (R(k))f.$$

We therefore define $\rho_R$ by the equation

$$\rho_R = R^*(e^{**})|_{K \circ L_1(G)},$$

with the result that $\rho_R \in (K \circ L_1(G))^*$, and with the equation relating $R$ and $\rho_R : \rho_R(k \circ f) = (R(k))f$, $f \in L_1(G)$, $k \in K$. We should just remark that, as in § 3, **(7)** assures us that $K \circ L_1(G)$ is a linear space, so that $\rho_R$ is completely defined. We next state

4.1. THEOREM. *The map* $\rho : R \to \rho_R$ *gives an isometric isomorphism between* $\mathfrak{R}(K, L_\infty(G))$ *and* $(K \circ L_1(G))^*$.

*Proof.* We must first prove that for each $R$, $\rho_R$ is independent of the choice of $e^{**}$. But $(R^*(e^{**}))(k \circ f) = [R(k)]f$, so that indeed $\rho_R$ is independent of the particular $e^{**}$ selected. Next we prove that $\rho$ is an isometry. Using Lemma 2.3 and an approximate identity $(e_\lambda)_{\lambda \in \Lambda}$ in $L_1(G)$, we obtain

$$||R|| = \sup_{||k|| \leqslant 1} \sup_{||f|| \leqslant 1} |(R(k))f| \leqslant ||\rho_R|| \leqslant \sup_{||k \circ f|| \leqslant 1} \sup_{\lambda} |\rho_R(k \circ (f * e_\lambda))|$$

$$= \sup_{||k \circ f|| \leqslant 1} \sup_{\lambda} |(R(k \circ f))e_\lambda| \leqslant ||R||.$$

The linearity of $\rho$ follows from the linearity of the adjoint operation for operators. Finally we demonstrate that $\rho$ is onto. Take $p \in (K \circ L_1(G))^*$ and define $R_p : K \to L_\infty(G)$ by the relation

$$(R_p(k))f = p(k \circ f), \qquad f \in L_1(G), k \in K.$$

Then $R$ is surely uniquely defined, and is a linear, continuous operator. In addition, for $f, g \in L_1(G)$, $k \in K$, we have

$$[R_p(k \circ f)]g = p((k \circ f) \circ g) = [R_p(k)](f * g) = [R_p(k) \nabla f]g,$$

which means that $R_p \in \mathfrak{R}(K, L_\infty(G))$, and its image under $\rho$ is, after all, $p$. Thus $\rho$ is onto.

Next we characterize the left $(K, L_\infty(G))$-homomorphisms, where $K$ and $L_\infty(G)$ are both left $L_1(G)$-modules. There are two ways of giving such a characterization. The first way is simple. We transform $K$ and $L_\infty(G)$ into right $L_1(G)$-modules by the equations

$$f \circ k = k \circ f', \qquad f \in L_1(G),\ k \in K,$$

$$f * h = h \nabla f', \qquad f \in L_1(G),\ h \in L_\infty(G).$$

Then the maps $R : K \to L_\infty(G)$ for which $R(k \circ f) = (\mathrm{R}k) \nabla f$, for all $f \in L_1(G)$ and $k \in K$, are exactly the maps $R : K \to L_\infty(G)$ for which $R(f \circ k) = f * R(k)$ for all $f \in L_1(G)$ and $k \in K$. Therefore we can apply Theorem 4.1 and formula (2.6). The result is that $\rho : R \to \rho_R$ gives an isometric isomorphism between $\mathfrak{R}(K, L_\infty(G))$ and $(L_1(G) \circ K)^*$.

The second way of characterizing the left $(K, L_\infty(G))$-homomorphisms utilizes the transposed Arens composition uniting $L_1^*(G)$ and $L_1(G)$ and rendering $L_1^*(G)$ a left $L_1(G)$-module. A procedure mirroring that in Theorem **4.1** applies. To get the process started, we let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity in $L_1(G)$, and $e^{**}$ a cluster point in $L_1^{**}(G)$. For $f \in L_1(G)$ and $k \in K$ we obtain the relation

$$(f \circ k)[R^*(e^{**})] = [R(k)]f.$$

This permits us to define $\rho_R \in (L_1(G) \circ K)^*$ by the relation

$$\rho_R = R^*(e^{**})|_{L_1(G)\circ K}.$$

Note that $\rho_R(f \circ k) = [R(k)]f, f \in L_1(G), k \in K$. The remainder of the work is a repetition of that in Theorem 4.1. We summarize in the following theorem.

4.2. THEOREM. *The map $\rho : R \to \rho_R$ is an isometry from $\mathfrak{R}(K, L_\infty(G))$ onto $(L_1(G) \circ K)^*$.*

In the rest of this section we discuss modules $K$, where $K = L_p(G)$, $p \in [1, \infty]$. For $h \in L_p^*(G)$ and $k \in L_p(G)$, we define the function $h \circ k$ by the equation

$$(h \circ k)f = h(k \circ f), \qquad f \in L_1(G),$$

so that $h \circ k \in L_\infty(G)$. The result concerning $\mathfrak{R}(L_1(G), L_\infty(G))$ has been presented in Theorem 3.11, and has an essentially different character from the results on $\mathfrak{R}(L_p(G), L_\infty(G)), p \in (1, \infty]$. This is due once again to the asymmetry in the operations making $L_p(H), p \in (1, \infty]$ into left and right $L_1(G)$-modules. In fact, let $p \in (1, \infty]$, and let $f \in L_1(G), k \in L_p(G)$. Then for an operator $R$, it turns out that $R(f * k) = f * R(k)$ if and only if

$$R(k \circ f') = R(k) \nabla f'.$$

However, by formula (2.6), $f * k = k \circ f'$. Thus $R$ is a left $(L_p(G), L_\infty(G))$-homomorphism if and only if $R$ is a right $(L_p(G), L_\infty(G))$-homomorphism. Thus we can refer simply to $(L_p(G), L_\infty(G))$-homomorphisms, and then prove

4.3. THEOREM. *Let $p \in (1, \infty)$. There is an isometric isomorphism between $\mathfrak{R}(L_p(G), L_\infty(G))$ and $L_p^*(G)$. For a linear continuous operator $R : L_p(G) \to L_\infty(G)$, the following statements are equivalent:*

(i) $R(k \circ f) = R(k) \nabla f, \qquad f \in L_1(G),\ k \in L_p(G).$

(ii) $R(k) = h \circ k, \qquad k \in L_p(G)$ *for some* $h \in L_p^*(G)$.

(iii) $R(f * k) = f * R(k), \qquad f \in L_1(G),\ k \in L_p(G).$

(iv) *$R$ is left translation-invariant.*

*Proof.* The isometry follows from Theorem 4.1 and Lemma 2.3. If $R \in \mathfrak{R}(L_p(G), L_\infty(G))$, then for each $k \in L_p(G)$,

$$(R(k))f = \rho_R(k \circ f) = (\rho_R \circ k)f;$$

thus if we define $h \in L_p^*(G)$ to be $\rho_R$, then $R(k) = h \circ k, k \in L_p(G)$, and (i) implies (ii). Next, (ii) implies (i) because $L_p(G)$ and $L_\infty(G)$ are right $L_1(G)$-modules. The comments preceding the statement of this theorem show that (i) and (iii) are equivalent. Lemma 3.9 and (iv) together yield (iii). On the other hand, a simple computation shows that if (ii) holds and if $k \in L_p(G)$ and $s \in G$, then

$$(h \circ k)s = \int_G h(t)k(st)dt,$$

so that

$$_s[R(k)] = {}_s(h \circ k) = h \circ ({}_sk) = R({}_sk),$$

which means that $R$ is left translation-invariant, and (ii) implies (iv), thereby completing the proof.

4.4. THEOREM. *There is an isometric isomorphism between* $\mathfrak{R}(L_\infty(G), L_\infty(G))$ *and* $C_{ru}^*(G)$. *For a linear continuous operator* $R : L_\infty(G) \to L_\infty(G)$, *the following statements are equivalent:*

(i) $R(k \circ f) = R(k) \nabla f, \quad f \in L_1(G),\ k \in L_\infty(G).$

(ii) $R(k) = h \circ k, \quad k \in L_\infty(G)$, *for some* $h \in C_{ru}^*(G)$.

(iii) $R(f * k) = f * R(k), \quad f \in L_1(G),\ k \in L_\infty(G).$

*Furthermore, any* $(L_\infty(G), L_\infty(G))$*-homomorphism is left translation-invariant.*

*Proof.* If $k \in L_\infty(G)$ and $h \in C_{ru}^*(G)$, then $(h \circ k)f = h(k \nabla f), f \in L_1(G)$, similar to the definition of the composition on $C_{ru}^*(G) \circ C_{ru}(G)$ preceding Lemma 2.6. In addition, $L_\infty(G) \nabla L_1(G)$ is dense in $C_{ru}(G)$, by Lemmas 2.3 and 2.4. Thus the isometry follows from Theorem 4.1. If $R$ is an $(L_\infty(G), L_\infty(G))$-homomorphism, let $h = \rho_R$ as in Theorem 4.1. Then by the definition of $h \circ k$ and by the equation $(R(k))f = h(k \nabla f)$, $f \in L_1(G)$, we conclude that $R(k) = h \circ k$, $k \in L_\infty(G)$. Thus (i) implies (ii). However, since $L_\infty(G)$ is a right $L_1(G)$-module, (ii) implies (i). The comments preceding Theorem 4.3 therefore complete the proof of the equivalence of (i), (ii), and (iii). To show that any homomorphism $R$ is left translation-invariant, we use (ii) and Lemma 2.6. Let $k \in C_{ru}(G)$ and let $h \in C_{ru}^*(G)$ correspond to $R$. Then for $s, t \in G$,

$$_s[R(k)]t = {}_s(h \circ k)t = (h \circ k)st = h({}_{st}k) = (h \circ {}_sk)t = [R({}_sk)]t,$$

so that $_s[R(k)] = R({}_sk)$ and $R$ is left translation-invariant.

If $G$ is compact, it is easy to show that an $(L_\infty(G), L_\infty(G))$-homomorphism is left translation-invariant, and conversely. As we mentioned directly after Lemma 5.9, we do not know if this is true when $G$ is an arbitrary locally compact group.

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