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ϕ-PRIME SUBMODULES

NASER ZAMANI
Faculty of Science, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran
e-mail: naserzaka@yahoo.com, zamanin@uma.ac.ir

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Abstract. Let R be a commutative ring with non-zero identity and M be a unitary R-module. Let $\mathcal{S}(M)$ be the set of all submodules of M, and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. We say that a proper submodule P of M is a prime submodule relative to ϕ or ϕ-prime submodule if $a \in R$ and $x \in M$, with $ax \in P \setminus \phi(P)$ implies that $a \in (P :_R M)$ or $x \in P$. So if we take $\phi(N) = \emptyset$ for each $N \in \mathcal{S}(M)$, then a ϕ-prime submodule is exactly a prime submodule. Also if we consider $\phi(N) = \{0\}$ for each submodule N of M, then in this case a ϕ-prime submodule will be called a weak prime submodule. Some of the properties of this concept will be investigated. Some characterisations of ϕ-prime submodules will be given, and we show that under some assumptions prime submodules and ϕ_1-prime submodules coincide.

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1. Introduction. Throughout the paper R is a commutative ring with non-zero identity and M is a unitary R-module. Prime ideals play an essential role in ring theory. One of the natural generalisations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodules, (see for example [**2**–**6**]). These have led to more information on the structure of the R-module M. For an ideal I of R and a submodule N of M let $\sqrt{I}$ denote the radical of I, and $(N :_R M) = \{r \in R : rM \subseteq N\}$, which is clearly a submodule of M. We say that N is a radical submodule of M if $\sqrt{(N :_R M)} = (N :_R M)$. Then a proper submodule P of M is called a prime submodule if $r \in R$ and $x \in M$, with $rx \in P$ implies that $r \in (P :_R M)$ or $x \in P$. It is easy to see that P is a prime submodule of M if and only if $(P :_R M)$ is a prime ideal of R and the $R/(P :_R M)$-module M/P is torsion-free (the R-module X is said to be torsion-free if the annihilator of any non-zero element of X is zero). By restricting where rx lies we can generalise this definition. A submodule $P \neq M$ is said to be a *weak prime submodule* of M if $r \in R$ and $x \in M$, $0 \neq rx \in P$ gives that $r \in (P :_R M)$ or $x \in P$. We will say that $P \neq M$ is an almost prime submodule if $r \in R$ and $x \in M$, with $rx \in P \setminus (P :_R M)P$ implies that $r \in (P :_R M)$ or $x \in P$. So any prime submodule is a weak prime submodule and any weak prime submodule is an almost prime submodule. Let $\mathcal{S}(M)$ be the set of all submodules of M and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is said to be a ϕ-prime submodule if $r \in R$ and $x \in M$, with $rx \in P \setminus \phi(P)$ implies that $r \in (P :_R M)$ or $x \in P$. Since $P \setminus \phi(P) = P \setminus (P \cap \phi(P))$, so without loss of generality, throughout this paper we will consider $\phi(P) \subseteq P$. In the rest of the paper we use the following functions $\phi : \mathcal{S}(M) \to$

$\mathcal{S}(M) \cup \{\emptyset\}$.

$$\begin{aligned}
\phi_{\emptyset}(N) &= \emptyset, & \forall \quad N \in \mathcal{S}(M),\\
\phi_0(N) &= \{0\}, & \forall \quad N \in \mathcal{S}(M),\\
\phi_1(N) &= (N :_R M)N, & \forall \quad N \in \mathcal{S}(M),\\
\phi_2(N) &= (N :_R M)^2 N, & \forall \quad N \in \mathcal{S}(M),\\
\phi_{\omega}(N) &= \cap_{i=1}^{\infty}(N :_R M)^i N, & \forall \quad N \in \mathcal{S}(M).
\end{aligned}$$

Then it is clear that $\phi_{\emptyset}$- and ϕ_0-prime submodules are prime and weak prime submodules respectively. Evidently for any submodule and every positive integer n, we have the following implications:

$$\text{prime} \Rightarrow \phi_{\omega} - \text{prime} \Rightarrow \phi_n - \text{prime} \Rightarrow \phi_{n-1} - \text{prime}.$$

For functions $\phi, \psi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$, we write $\phi \leq \psi$ if $\phi(N) \subseteq \psi(N)$ for each $N \in \mathcal{S}(M)$. So whenever $\phi \leq \psi$, any ϕ-prime submodule is ψ-prime.

In this paper, among other results concerning the properties of ϕ-prime submodules, some characterisations of this notion will be investigated. Some of the results in this paper are inspired from [**1**].

2. Results. The following theorem asserts that under some conditions ϕ-prime submodules are prime.

THEOREM 2.1. *Let R be a commutative ring and M be an R-module. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function and P be a ϕ-prime submodule of M such that $(P :_R M)P \nsubseteq \phi(P)$. Then P is a prime submodule of M.*

Proof. Let $a \in R$ and $x \in M$ be such that $ax \in P$. If $ax \notin \phi(P)$, then since P is ϕ-prime, we have $a \in (P :_R M)$ or $x \in P$.

So let $ax \in \phi(P)$. In this case we may assume that $aP \subseteq \phi(P)$. For, let $aP \nsubseteq \phi(P)$. Then there exists $p \in P$ such that $ap \notin \phi(P)$, so that $a(x + p) \in P \setminus \phi(P)$. Therefore, $a \in (P :_R M)$ or $x + p \in P$ and hence $a \in (P :_R M)$ or $x \in P$. Second, we may assume that $(P :_R M)x \subseteq \phi(P)$. If this is not the case, there exists $u \in (P :_R M)$ such that $ux \notin \phi(P)$ and so $(a + u)x \in P \setminus \phi(P)$. Since P is a ϕ-prime submodule, we have $a + u \in (P :_R M)$ or $x \in P$. So $a \in (P :_R M)$ or $x \in P$. Now since $(P :_R M)P \nsubseteq \phi(P)$, there exist $r \in (P :_R M)$ and $p \in P$ such that $rp \notin \phi(P)$. So $(a + r)(x + p) \in P \setminus \phi(P)$, and hence $a + r \in (P :_R M)$ or $x + p \in P$. Therefore, $a \in (P :_R M)$ or $x \in P$ and the proof is complete. □

COROLLARY 2.2. *Let P be a weak prime submodule of M such that $(P :_R M)P \neq 0$. Then P is a prime submodule of M.*

Proof. In the above theorem set $\phi = \phi_0$. □

COROLLARY 2.3. *Let P be a ϕ-prime submodule of M such that $\phi(P) \subseteq (P :_R M)^2 P$. Then for each $a \in R$ and $x \in M$, $ax \in P \setminus \cap_{i=1}^{\infty}(P :_R M)^i P$ implies that $a \in (P :_R M)$ or $x \in P$. In other words P is ϕ_{ω}-prime.*

Proof. If P is a prime submodule of M, then the result is clear. So suppose that P is not a prime submodule of M. Then by Theorem 2.1 we have $(P :_R M)P \subseteq$

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$\phi(P) \subseteq (P :_R M)^2 P \subseteq (P :_R M)P$, that is, $\phi(P) = (P :_R M)P = (P :_R M)^2 P$. Hence, $\phi(P) = (P :_R M)^i P$ for all $i \geq 1$ and the result follows. □

COROLLARY 2.4. *Let M be an R-module and P be a ϕ-prime submodule of M. Then $(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)}$ or $\sqrt{(\phi(P) :_R M)} \subseteq (P :_R M)$. If $(P :_R M) \subsetneq \sqrt{(\phi(P) :_R M)}$, then P is not a prime submodule of M; while if $\sqrt{(\phi(P) :_R M)} \subsetneq (P :_R M)$, then P is a prime submodule of M. If $\phi(P)$ is a radical submodule of M, either $(P :_R M) = (\phi(P) :_R M)$ or P is a prime submodule of M.*

Proof. If P is not a prime submodule of M, then by Theorem 2.1, we have $(P :_R M)P \subseteq \phi(P)$. Hence $\sqrt{(P :_R M)^2} \subseteq \sqrt{((P :_R M)P :_R M)} \subseteq \sqrt{(\phi(P) :_R M)}$. So $(P :_R M) \subseteq \sqrt{(\phi(P) :_R M)}$. If P is a prime submodule of M, then $\sqrt{(\phi(P) :_R M)} \subseteq \sqrt{(P :_R M)} = (P :_R M)$ (note that we may assume that $\phi(P) \subseteq P$), and all the claims of the corollary follow. □

REMARK A. Suppose that P is a ϕ-prime submodule of M such that $\phi(P) \subseteq (P :_R M)P$ (resp. $\phi(P) \subseteq (P :_R M)^2 P$) and that P is not a prime submodule. Then by Theorem 2.1, we have $\phi(P) = (P :_R M)P$ (resp. $\phi(P) = (P :_R M)^2 P$). In particular, if P is a weak prime (resp. ϕ_2-prime) submodule but not a prime submodule then $(P :_R M)P = 0$ (resp. $(P :_R M)P = (P :_R M)^2 P$).

Let R_1 and R_2 be two commutative rings with identity. Let M_1 and M_2 be R_1-module and R_2-module respectively and let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is of the form $N = N_1 \times N_2$ for some submodule N_1 of M_1 and N_2 of M_2. Furthermore, $N = N_1 \times N_2$ is a prime submodule of M if and only if $N = P_1 \times M_2$ or $N = M_1 \times P_2$ for some prime submodule P_1 of M_1 and P_2 of M_2. (In fact, to see the non-trivial direction, let $N_1 \times N_2$ be a prime submodule of $M_1 \times M_2$. Then either N_1 must be a prime submodule of M_1 or N_2 must be a prime submodule of M_2. Now $(N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2) = (N_1 \times N_2 :_R M_1 \times M_2)$ is a prime ideal of $R = R_1 \times R_2$. So either $(N_1 :_{R_1} M_1) = R_1$ or $(N_2 :_{R_2} M_2) = R_2$, which means that either $N_1 = M_1$ or $N_2 = M_2$ and the claim follows.) If P_1 is a weak prime submodule of M_1, then $P_1 \times M_2$ need not be a weak prime submodule of M. Indeed $P_1 \times M_2$ is a weak prime submodule of M if and only if $P_1 \times M_2$ is a prime submodule of $M_1 \times M_2$. To see the non-trivial direction, let $P_1 \times M_2$ be a weak prime submodule of $M_1 \times M_2$. Let $r_1 \in R_1$ and $x_1 \in M_1$, with $r_1 x_1 \in P_1$. Let $0 \neq x_2 \in M_2$. Then $(r_1, 1)(x_1, x_2) = (r_1 x_1, x_2) \in P_1 \times M_2 \setminus \{(0, 0)\}$. By assumption, this gives that $(r_1, 1) \in (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2) = (P_1 :_{R_1} M_1) \times R_2$ or $(x_1, x_2) \in P_1 \times M_2$, that is, $r_1 \in (P_1 :_{R_1} M_1)$ or $x_1 \in P_1$. Therefore, P_1 is a prime submodule of M_1 and hence $P_1 \times M_2$ is a prime submodule of $M_1 \times M_2$.

However, if P_1 is a weak prime submodule of M_1, then $P_1 \times M_2$ is a ϕ-prime submodule if $\{0\} \times M_2 \subseteq \phi(P_1 \times M_2)$.

To see this, we have $P_1 \times M_2 \setminus \phi(P_1 \times M_2) \subseteq P_1 \times M_2 \setminus \{0\} \times M_2 = (P_1 \setminus \{0\}) \times M_2$. Now let $(r_1, r_2)(x_1, x_2) = (r_1 x_1, r_2 x_2) \in P_1 \times M_2 \setminus \phi(P_1 \times M_2)$. Then $r_1 x_1 \in P_1 \setminus \{0\}$ and by the assumption on P_1 we have $r_1 \in (P_1 :_{R_1} M_1)$ or $x_1 \in P_1$. This gives that $(r_1, r_2) \in (P_1 :_T M_1) \times R_2 = (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ or $(x_1, x_2) \in P_1 \times M_2$. Therefore, $P_1 \times M_2$ is a ϕ-prime submodule of $M_1 \times M_2$.

COROLLARY 2.5. *Let R_1 and R_2 be two commutative rings, and let M_1 and M_2 be R_1-module and R_2-module respectively. Let $M = M_1 \times M_2$ and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. Suppose that P_1 is a weak prime submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(P_1 \times M_2)$. Then $P_1 \times M_2$ is a ϕ-prime submodule of $M_1 \times M_2$.*

Proposition 2.6. *With the same notations as in Corollary 2.5, let ϕ be a function such that $\phi_\omega \leq \phi$. Then for any weak prime submodule P_1 of M_1, $P_1 \times M_2$ is a ϕ-prime submodule of $M_1 \times M_2$.*

Proof. If P_1 is a prime submodule of M_1, then $P_1 \times M_2$ is prime and so a ϕ-prime submodule of $M_1 \times M_2$. Suppose that P_1 is not a prime submodule of M_1. Then by Remark A, we have $(P_1 :_{R_1} M_1)P_1 = 0$. This gives that

$$(P_1 \times M_2 :_{R_1\times R_2} M_1 \times M_2)^i(P_1 \times M_2) = [(P_1 :_{R_1} M_1)^i P_1] \times M_2 = 0 \times M_2,$$

for all $i \geq 1$ and hence we have

$$0 \times M_2 = \cap_{i=1}^{\infty}(P_1 \times M_2 :_{R_1\times R_2} M_1 \times M_2)^i(P_1 \times M_2) = \phi_\omega(P_1 \times M_2) \subseteq \phi(P_1 \times M_2),$$

and the result follows by the above corollary. □

Theorem 2.7. *Let M be an R-module and $0 \neq x \in M$ such that $Rx \neq M$ and $(0 :_R x) = 0$. If Rx is not a prime submodule of M, then Rx is not a ϕ_1-prime submodule of M.*

Proof. Since Rx is not a prime submodule of M, there exist $a \in R$ and $y \in M$ such that $a \notin (Rx :_R M)$, $y \notin Rx$, but $ay \in Rx$. If $ay \notin (Rx :_R M)x$, then by our definition Rx is not a ϕ_1-prime submodule. So let $ay \in (Rx :_R M)x$. We have $y + x \notin Rx$ and $a(y + x) \in Rx$. If $a(y + x) \notin (Rx :_R M)x$, then again by our definition Rx is not a ϕ_1-prime submodule. So let $a(y + x) \in (Rx :_R M)x$, then $ax \in (Rx :_R M)x$, which gives that $ax = rx$ for some $r \in (Rx :_R M)$. Since $(0 :_R x) = 0$, it gives that $a = r \in (Rx :_R M)$, which contradicts with our assumption. □

Corollary 2.8. *Let x be a non-zero element of an R-module M such that $(0 :_R x) = 0$ and that $Rx \neq M$. Then Rx is a prime submodule of M if and only if Rx is a ϕ_1-prime submodule of M.*

Proposition 2.9. *Let P be a ϕ_1-prime submodule of M. Then the following holds:*
(i) *If a is a zero divisor in M/P, then $aP \subseteq (P :_R M)P$.*
(ii) *Let J be an ideal of R such that $(P :_R M) \subseteq J$ and $J \subseteq Z_R(M/P)$, then $JP = (P :_R M)P$.*

Proof. (i) By assumption, there exists $x \in M \setminus P$ such that $ax \in P$. If $a \in (P :_R M)$ then clearly $aP \subseteq (P :_R M)P$. So let $a \notin (P :_R M)$. Since P is a ϕ_1-prime submodule of M, we must have $ax \in (P :_R M)P$. Now for any $y \in P$, $y + x \notin P$ and $a(y + x) \in P$. Hence as P is a ϕ_1-prime submodule $a(y + x) \in (P :_R M)P$, which gives that $ay \in (P :_R M)P$. So $aP \subseteq (P :_R M)P$ and the result follows.

(ii) This follows from (i). □

Theorem 2.10. *Let M be an R-module and let a be an element of R such that $aM \neq M$. Suppose $(0 :_M a) \subseteq aM$. Then aM is a ϕ_1-prime submodule of M if and only if it is a prime submodule of M.*

Proof. The direction $\Leftarrow$ is clear. So we prove $\Rightarrow$. Let $b \in R$ and $x \in M$ such that $bx \in aM$. We show that $b \in (aM :_R M)$ or $x \in aM$. If $bx \notin (aM :_R M)aM$, then $b \in (aM :_R M)$ or $x \in aM$, since aM is a ϕ_1-prime submodule. So suppose $bx \in (aM :_R M)aM$. Now $(b + a)x \in aM$. If $(b + a)x \notin (aM :_R M)aM$, then, since aM is a ϕ_1-prime submodule, $b + a \in (aM :_R M)$ or $x \in aM$, which gives that $b \in (aM :_R M)$ or $x \in aM$. So assume that $(b + a)x \in (aM :_R M)aM$. Then $bx \in (aM :_R M)aM$ gives

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that $ax \in (aM :_R M)aM$. Hence there exists $y \in (aM :_R M)M$ such that $ax = ay$ and so $x - y \in (0 :_M a)$. This gives that $x \in (aM :_R M)M + (0 :_M a) \subseteq aM + (0 :_M a) \subseteq aM$, and the result follows. □

In the next theorem we give several characterisations of ϕ-prime submodules.

THEOREM 2.11. *Let P be a proper submodule of M and let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. Then the following are equivalent:*

(i) *P is a ϕ-prime submodule of M;*

(ii) *for $x \in M \setminus P$, $(P :_R x) = (P :_R M) \cup (\phi(P) :_R x)$;*

(iii) *for $x \in M \setminus P$, $(P :_R x) = (P :_R M)$ or $(P :_R x) = (\phi(P) :_R x)$;*

(iv) *for any ideal I of R and any submodule L of M, if $IL \subseteq P$ and $IL \nsubseteq \phi(P)$, then $I \subseteq (P :_R M)$ or $L \subseteq P$.*

Proof. (i) ⇒ (ii). Let $x \in M \setminus P$ and $a \in (P :_R x) \setminus (\phi(P) :_R x)$. Then $ax \in P \setminus \phi(P)$. Since P is a ϕ-prime submodule of M, so $a \in (P :_R M)$. As we may assume that $\phi(P) \subseteq P$, the other inclusion always holds.

(ii) ⇒ (iii). If a subgroup is the union of two subgroups, it is equal to one of them.

(iii) ⇒ (iv). Let I be an ideal of R and L be a submodule of M such that $IL \subseteq P$. Suppose $I \nsubseteq (P :_R M)$ and $L \nsubseteq P$. We show that $IL \subseteq \phi(P)$. Let $a \in I$ and $x \in L$. First let $a \notin (P :_R M)$. Then, since $ax \in P$, we have $(P :_R x) \neq (P :_R M)$. Hence by our assumption, $(P :_R x) = (\phi(P) :_R x)$. So $ax \in \phi(P)$. Now assume that $a \in I \cap (P :_R M)$. Let $u \in I \setminus (P :_R M)$. Then $a + u \in I \setminus (P :_R M)$. So by the first case, for each $x \in L$ we have $ux \in \phi(P)$ and $(a + u)x \in \phi(P)$. This gives that $ax \in \phi(P)$. Thus in any case $ax \in \phi(P)$. Therefore, $IL \subseteq \phi(P)$.

(iv) ⇒ (i). Let $ax \in P \setminus \phi(P)$. By considering the ideal (a) and the submodule (x), the result follows. □

Let S be a multiplicatively close subset of R. Then by [**7**, 9.11 (v)] each submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M. Also, it is well known that there is a one-to-one correspondence between the set of all prime submodules P of M with $(P :_R M) \cap S = \emptyset$ and the set of all prime submodules of $S^{-1}M$, given by $P \to S^{-1}P$ (see [**6**, Theorem 3.4]). Furthermore, it is easy to see that if P is a weak prime submodule of M with $S^{-1}P \neq S^{-1}M$, then $S^{-1}P$ is a weak prime submodule of $S^{-1}M$. This fact remains true for ϕ_1-prime submodules P of M with $S^{-1}P \neq S^{-1}M$. In the next theorem we want to generalise this fact for ϕ-prime submodules. In the following, for a submodule N of M we put $N(S) = \{x \in M : \exists s \in S, sx \in N\}$. Then $N(S)$ is a submodule of M containing N and $S^{-1}(N(S)) = S^{-1}N$. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. We define $(S^{-1}\phi) : \mathcal{S}(S^{-1}M) \to \mathcal{S}(S^{-1}M) \cup \{\emptyset\}$ by $(S^{-1}\phi)(S^{-1}N) = S^{-1}(\phi(N(S)))$ if $\phi(N(S)) \neq \emptyset$ and $(S^{-1}\phi)(S^{-1}N) = \emptyset$ if $\phi(N(S)) = \emptyset$. Since dealing with prime submodules P we can always assume that $\phi(P) \subseteq P$, there is no loss of generality in assuming that $\phi(N) \subseteq N$, and hence $(S^{-1}\phi)(S^{-1}N) \subseteq S^{-1}N$. Also we note that $(S^{-1}\phi_\emptyset) = \phi_\emptyset$, $(S^{-1}\phi_0) = \phi_0$, and whenever M is finitely generated $(S^{-1}\phi_i) = \phi_i$ for $i = 1, 2$. In the next theorem we show that if $S^{-1}(\phi(N)) \subseteq (S^{-1}\phi)(S^{-1}N)$, then ϕ-primeness of P together with $S^{-1}P \neq S^{-1}M$ imply that $S^{-1}P$ is $(S^{-1}\phi)$-prime.

For a submodule L of M, we define $\phi_L : \mathcal{S}(M/L) \to \mathcal{S}(M/L) \cup \{\emptyset\}$ by $\phi_L(N/L) = (\phi(N) + L)/L$ for $N \supseteq L$ and $\emptyset$ for $\phi(N) = \emptyset$.

THEOREM 2.12. *Let M be an R-module and let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$. Let P be a ϕ-prime submodule of M.*

(i) *If $L \subseteq P$ is a submodule of M, then P/L is a ϕ_L-prime submodule of M/L.*

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(ii) *Suppose that S is a multiplicatively closed subset of R such that $S^{-1}P \neq S^{-1}M$ and $S^{-1}(\phi(P)) \subseteq (S^{-1}\phi)(S^{-1}P)$. Then $S^{-1}P$ is an $(S^{-1}\phi)$-prime submodule of $S^{-1}M$. Furthermore, if $S^{-1}P \neq S^{-1}((\phi(P))$, then $P(S) = P$.*

Proof. (i) Let $a \in R$ and $\bar{x} \in M/L$ with $a\bar{x} \in P/L \setminus \phi_L(P/L)$, where $\bar{x} = x + L$, for some $x \in M$. By the definition of ϕ_L, this gives that $ax \in P \setminus (\phi(P) + L)$. So we have $ax \in P \setminus \phi(P)$, which gives that $a \in (P :_R M)$ or $x \in P$. Therefore, $a \in (P/L :_R M/L)$ or $x \in P$ and so P/L is a ϕ_L-prime submodule.

(ii) Let $a/s \in S^{-1}R$ and $x/t \in S^{-1}M$ with $ax/st \in S^{-1}P \setminus (S^{-1}\phi)(S^{-1}P)$. Then by our assumption, $ax/st \in S^{-1}P \setminus S^{-1}(\phi(P))$. Therefore, there exists $u \in S$ such that $uax \in P \setminus \phi(P)$ (note that for each $v \in S$, $vax \notin \phi(P)$). Since P is ϕ-prime and $(P :_R M) \cap S = \emptyset$, it gives that $ax \in P \setminus \phi(P)$ and so $a \in (P :_R M)$ or $x \in P$. Therefore, $a/s \in S^{-1}(P :_R M) \subseteq (S^{-1}P :_{S^{-1}R} S^{-1}M)$ or $x/t \in S^{-1}P$. Hence $S^{-1}P$ is an $(S^{-1}\phi)$-prime submodule of $S^{-1}M$.

To prove the last part of the theorem, let $x \in P(S)$. Then there exists $s \in S$ such that $sx \in P$. If $sx \notin \phi(P)$, then $x \in P$. If $sx \in \phi(P)$, then $x \in \phi(P)(S)$. So $P(S) = P \cup (\phi(P)(S))$. Hence $P(S) = P$ or $P(S) = (\phi(P)(S))$. If the second holds, then we must have $S^{-1}P = S^{-1}P(S) = S^{-1}(\phi(P)(S)) = S^{-1}(\phi(P))$, which is not the case. So $P(S) = P$ and the proof is complete. □

Let $S^{-1}P$ be an $(S^{-1}\phi)$-prime submodule of $S^{-1}M$. Then evidently $(P :_R M) \cap S = \emptyset$. In general we do not know under what conditions P is a ϕ-prime submodule of M. Even in the case $\phi = \phi_0, \phi_1$ and ϕ_2 we could not answer this question.

As we mentioned previously, for two commutative rings R_1 and R_2 and two modules M_1 and M_2 over R_1 and R_2 respectively, the prime submodules of the $R = R_1 \times R_2$ module $M = M_1 \times M_2$ are in the form $P_1 \times M_2$ or $M_1 \times P_2$, where P_1 is a prime submodule of M_1 and P_2 is a prime submodule of M_2. This is not true for correspondence ϕ-prime submodules in general. For example, if P_1 is a ϕ_0-prime submodule of M_1, then $P_1 \times M_2$ is not necessarily a ϕ_0-prime submodule of $M_1 \times M_2$. To be more specific let $R_1 = R_2 = M_1 = M_2 = \mathbb{Z}_6$, and suppose $P_1 = \{0\}$. Then evidently P_1 is a ϕ_0-prime submodule of M_1. However, $(2, 1)(3, 1) \in P_1 \times M_2$ and $(3, 1) \notin P_1 \times M_2$. Also as $(2, 1)(2, 1) \notin P_1 \times M_2$, $(2, 1)M \nsubseteq P_1 \times M_2$. However, in this direction we have the following result.

THEOREM 2.13. *Let the notation be as in the above paragraph. Let $\psi_i : \mathcal{S}(M_i) \to \mathcal{S}(M_i) \cup \{\emptyset\}$. Let $\phi = \psi_1 \times \psi_2$. Then each of the following types is a ϕ-prime submodule of $M_1 \times M_2$:*

(i) *$N_1 \times N_2$ where N_i is a proper submodule of M_i, with $\psi_i(N_i) = N_i$.*
(ii) *$P_1 \times M_2$ where P_1 is a prime submodule of M_1.*
(iii) *$P_1 \times M_2$ where P_1 is a ψ_1-prime submodule of M_1 and $\psi_2(M_2) = M_2$.*
(iv) *$M_1 \times P_2$ where P_2 is a prime submodule of M_2.*
(v) *$M_1 \times P_2$ where P_2 is a ψ_2-prime submodule of M_2 and $\psi_1(M_1) = M_1$.*

Proof. (i) This is clear, since $N_1 \times N_2 \setminus \phi(N_1 \times N_2) = \emptyset$.

(ii) If P_1 is a prime submodule of M_1, then $P_1 \times M_2$ as a prime submodule of $M_1 \times M_2$ is ϕ-prime.

(iii) Let P_1 be a ψ_1-prime submodule of M_1 and $\psi_2(M_2) = M_2$. Let $(r_1, r_2) \in R$ and $(x_1, x_2) \in M$ be such that $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in P_1 \times M_2 \setminus \phi(P_1 \times M_2) = P_1 \times M_2 \setminus \psi_1(P_1) \times \psi_2(M_2) = P_1 \times M_2 \setminus \psi_1(P_1) \times M_2 = (P_1 \setminus \psi_1(P_1)) \times M_2$. So $r_1 \in$

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$(P_1 :_{R_1} M_1)$ or $x_1 \in P_1$. Therefore, $(r_1, r_2) \in (P_1 :_{R_1} M_1) \times R_2 = (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ or $(x_1, x_2) \in P_1 \times M_2)$. So $P_1 \times M_2$ is a ϕ-prime submodule of $M_1 \times M_2$.

Parts (iv) and (v) are proved similarly as (ii) and (iii) respectively. □

A question that arises here is whether any prime submodule of M has one of the above forms. As it has been shown in [**1**, Theorem 16], this is true for the ideal and the ring cases. But we were not able to prove similar results for the module case.

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REFERENCES

1. D. D. Anderson and M. Bataineh, Generalizatins of prime ideals, *Comm. Algebra* **36** (2008), 686–696.

2. J. Dauns, Prime modules, *J. Reine Angew. Math.* **298** (1978), 156–181.

3. C. P. Lu, Prime submodules, *Comm. Math. Univ. Sancti Pauli* **33** (1984), 61–69.

4. R. L. McCasland and M. E. Moore, Prime submodules, *Comm. Algebra* **20** (1992), 1803–1817.

5. R. L. McCasland and P. F. Smith, Prime submodules of Noetherian modules, *Rocky Mountain J. Math.* **23** (1993), 1041–1062.

6. M. E. Moore and S. J. Smith, Prime and radical submodules of modules over commutative rings, *Comm. Algebra* **30** (2002), 5037–5064.

7. R. Sharp, *Steps in commutative algebra* (Cambridge University Press, Cambridge, 2000).