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ON THE NUMBER OF DIVISORS OF $n^2 - 1$

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Abstract

We prove an asymptotic formula for the sum $\sum_{n\leq N} d(n^2-1)$, where $d(n)$ denotes the number of divisors of n. During the course of our proof, we also furnish an asymptotic formula for the sum $\sum_{d\leq N} g(d)$, where $g(d)$ denotes the number of solutions x in $\mathbb{Z}_d$ to the equation $x^2 \equiv 1 \pmod{d}$.

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1. Introduction

The main purpose of this note is to prove the following theorem.

THEOREM 1.1. *Let $d(n)$ denote the number of divisors of n. Then*

$$\sum_{n\leq N} d(n^2-1) \sim \frac{6}{\pi^2} N \log^2 N \quad \text{as } N \to \infty.$$

In consideration of the more general sum $\sum_{n\leq N} d(n^2+a)$, it was noted by Hooley [5] that, in the case where $a = -k^2$, we may factorise $n^2 + a$ as $(n-k)(n+k)$, and then the sum bears a close resemblance to

$$\sum_{n\leq N} d(n)\, d(n+2k),$$

which was first studied by Ingham [6]. As mentioned by Hooley, it is certainly possible in this case to compare these sums to show that

$$\sum_{n\leq N} d(n^2-k^2) \sim C(k) N \log^2 N$$

as $N \to \infty$ for some constant $C(k)$. Elsholtz *et al.* [4, Lemma 3.5] showed that $C(1) \leq 2$. Trudgian [8] reduced this to $C(1) \leq 12/\pi^2$, before Cipu [1] showed that $C(1) \leq 9/\pi^2$. Theorem 1.1 of this note gives the result that $C(1) = 6/\pi^2$.

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However, rather than work from Ingham's asymptotic formula, we give a proof that requires information on the number of solutions to the equation $x^2 \equiv 1 \pmod{d}$. Thus, before we prove Theorem 1.1, we first prove the following result which is of interest in its own right.

Theorem 1.2. *Let $g(d)$ denote the number of solutions to the equation $x^2 \equiv 1 \pmod{d}$ such that $1 \leq x \leq d$. Then*

$$\sum_{d<N} g(d) \sim \frac{6}{\pi^2} N \log N \quad \text{as } N \to \infty.$$

After proving our two theorems, we give some insight into how one might generalise this work.

It should also be noted that the sum in Theorem 1.1 plays a role in the theory of Diophantine m-tuples. We call a set of m distinct integers $\{a_1, \ldots, a_m\}$ a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. For example, the set $\{1, 3, 8, 120\}$ is a Diophantine quadruple. It has been shown by Dujella [3] that there are no Diophantine m-tuples for $m \geq 6$, and it has been conjectured that there are no Diophantine quintuples, though this has yet to be proven. The best result in this direction is that of Trudgian [8], who has recently shown that there are at most 2.3×10^{29} Diophantine quintuples. In this context, the sum appearing in Theorem 1.1 is useful, for it is equal to twice the number of Diophantine 2-tuples $\{a, b\}$ such that $ab + 1 \leq N^2$.

2. Proof of the main theorems

We start by manipulating the divisor sum in the usual way. We have that

$$\sum_{n\leq N} d(n^2-1) = \sum_{n\leq N}\Big(2\sum_{\substack{d\mid(n^2-1)\\ d<n}} 1\Big) = 2\sum_{d<N}\ \sum_{\substack{d<n\leq N\\ n^2\equiv 1 \ (\mathrm{mod}\ d)}} 1,$$

where the inner sum is now over the integers n in the interval $(d, N]$ such that n^2 is congruent to 1 modulo d. We let $g(d)$ denote the number of solutions to the equation $x^2 \equiv 1 \pmod{d}$, where $x \in \mathbb{Z}_d$. To estimate the inner sum, we first require the following lemma.

Lemma 2.1. *Let d be a positive integer. Writing $d = 2^a q$, where q is odd and $a \geq 0$, it follows that $g(d) = 2^{\omega(q)+s(a)}$, where $\omega(q)$ denotes the number of distinct prime factors of q and*

$$s(a) = \begin{cases} 0 & \text{if } a \leq 1,\\ 1 & \text{if } a = 2,\\ 2 & \text{if } a \geq 3.\end{cases}$$

Proof. This follows from Cipu [1, Lemma 4.1]. □

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Denote by $Q(x,d)$ the number of positive integers $n \le x$ such that $n^2 \equiv 1 \pmod{d}$. Lemma 2.1 allows us to estimate $Q(x,d)$, because in an interval of length d there will be $g(d)$ such numbers that satisfy the congruence. Therefore,

$$Q(x,d) = g(d)\frac{x}{d} + O(g(d)). \tag{2.1}$$

With this notation, we can write our original sum as

$$\sum_{n\le N} d(n^2-1) = 2\sum_{d<N}(Q(N,d) - Q(d,d)).$$

It follows now from (2.1) and the fact that $Q(d,d) = g(d)$ that

$$\sum_{n\le N} d(n^2-1) = 2N\sum_{d<N}\frac{g(d)}{d} + O\Big(\sum_{d<N} g(d)\Big). \tag{2.2}$$

The order of the error term can be bounded in the straightforward way by

$$\sum_{d<N} g(d) \ll \sum_{d<N} 2^{\omega(d)} \ll N\log N,$$

and so it remains to show that

$$\sum_{d<N}\frac{g(d)}{d} \sim \frac{3}{\pi^2}\log^2 N$$

as $N \to \infty$. To estimate this sum, we will use the following result, which can be found in Cojocaru and Murty [2, Theorem 2.4.1].

Lemma 2.2. *Let*

$$F(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

be a Dirichlet series with nonnegative coefficients converging for Re$(s) > 1$. Suppose that $F(s)$ extends analytically at all points on Re$(s) = 1$ apart from $s = 1$, and that at $s = 1$ we can write

$$F(s) = \frac{H(s)}{(s-1)^{1-\alpha}}$$

for some $\alpha \in \mathbb{R}$ and some $H(s)$ holomorphic and nonzero in the region Re$(s) \ge 1$. Then

$$\sum_{n\le x} a_n \sim \frac{cx}{(\log x)^{\alpha}}$$

with

$$c := \frac{H(1)}{\Gamma(1-\alpha)},$$

where Γ is the Gamma function.

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This result allows one to step from some 'well-behaved' Dirichlet series to an asymptotic formula for the partial sum of its coefficients. We will use this to prove Theorem 1.2, by exploiting the multiplicity of the function $g(d)$ to construct an appropriate Dirichlet series.

PROOF OF THEOREM 1.2. We will consider the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Note that, as $g(n)$ is multiplicative,

$$F(s) = \prod_p \left(1 + \frac{g(p)}{p^s} + \frac{g(p^2)}{p^{2s}} + \cdots\right).$$

More specifically, from Lemma 2.1 it follows that

$$F(s) = \left(1 + \frac{1}{2^s} + \frac{2}{4^s} + 4\left(\frac{1}{8^s} + \frac{1}{16^s} + \cdots\right)\right) \cdot \prod_{p \text{ odd}} \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots\right).$$

We now use the fact that

$$\frac{\zeta^2(s)}{\zeta(2s)} = \prod_p \frac{1 - p^{-2s}}{(1 - p^{-s})^2} = \prod_p \frac{1 + p^{-s}}{1 - p^{-s}} = \prod_p \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots\right),$$

where $\zeta(s)$ is the Riemann zeta-function (see [7] for more details). Thus

$$F(s) = \left(1 + \frac{1}{2^s} + \frac{2}{4^s} + \frac{4}{8^s - 4^s}\right)\left(\frac{1 - 2^{-s}}{1 + 2^{-s}}\right)\frac{\zeta^2(s)}{\zeta(2s)}.$$

By the properties of the Riemann zeta-function, $F(s)$ satisfies the conditions of Lemma 2.2 with $\alpha = -1$, so

$$\sum_{d<N} g(d) \sim cN \log N,$$

where

$$c := \lim_{s \to 1} (s-1)^2 F(s) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

This completes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.1. Now, it follows by partial summation that

$$\begin{aligned}\sum_{d<N} \frac{g(d)}{d} &= \frac{6}{\pi^2} \int_1^N \frac{\log t}{t}\, dt + o\left(\int_1^N \frac{\log t}{t}\, dt\right) \\ &= \frac{3}{\pi^2} \log^2 N + o(\log^2 N).\end{aligned}$$

Using the above estimate in (2.2) finishes the proof of Theorem 1.1. □

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3. Further notes

It would be interesting to see if one could extend this work so as to determine asymptotic estimates for the sums

$$\sum_{n\leq N} d(n^2-r^2) \quad \text{and} \quad \sum_{d<N} g_r(d),$$

where $g_r(d)$ denotes the number of solutions of the equation $x^2 \equiv r^2 \pmod{d}$ such that $1 \leq x \leq d$. If r is fixed, then note that if p is an odd prime and $k \geq 1$, the equation $x^2 \equiv r^2 \pmod{p^k}$ yields

$$p^k | (x-r)(x+r).$$

For a sufficiently large prime p, there will be exactly two solutions to the above, namely $x = r$ and $x = p^k - r$. Therefore, we have $g_r(p^k) = 2$ for all sufficiently large primes p, and thus one will inevitably require the factor $\zeta^2(s)/\zeta(2s)$ in the construction of an appropriate Dirichlet series. Thus, one can expect to obtain asymptotics of the form

$$\sum_{n\leq N} d(n^2-r^2) \sim \frac{A(r)}{\pi^2} N \log^2 N \quad \text{and} \quad \sum_{d<N} g_r(d) \sim \frac{B(r)}{\pi^2} N \log N,$$

where $A(r)$ and $B(r)$ are rational numbers dependent on r.

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