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# FACTORIALS AND THE RAMANUJAN FUNCTION

JHON J. BRAVO

*Departamento de Matemáticas, Universidad del Cauca,*
*Calle 5 No 4-70, Popayán, Colombia*
*e-mail: jbravo@unicauca.edu.co*

and FLORIAN LUCA

*School of Mathematics, University of the Witwatersrand,*
*Private Bag 3, Wits 2050, Johannesburg,*
*South Africa*
*e-mail: Florian.Luca@wits.ac.za*

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**Abstract.** In 2006, F. Luca and I. E. Shparlinski (*Proc. Indian Acad. Sci. (Math. Sci.)* **116**(1) (2006), 1–8) proved that there are only finitely many pairs $(n, m)$ of positive integers which satisfy the Diophantine equation $|\tau(n!)| = m!$, where $\tau$ is the Ramanujan function. In this paper, we follow the same approach of Luca and Shparlinski (*Proc. Indian Acad. Sci. (Math. Sci.)* **116**(1) (2006), 1–8) to determine all solutions of the above equation. The proof of our main theorem uses linear forms in two logarithms and arithmetic properties of the Ramanujan function.

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**1. Introduction.** The Ramanujan tau function is the arithmetic function $\tau$ defined by the expansion

$$q\prod_{k=1}^{\infty}(1-q^k)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

which is valid for each complex number $q$ such that $|q| < 1$. The first three values of $\tau$ are $\tau(1) = 1$, $\tau(2) = -24$ and $\tau(3) = 252$. The Ramanujan function possesses many arithmetic properties. Below, we list some of them which will be used as we find convenient to do so.

- $\tau$ is an integer–valued multiplicative function, that is $\tau(ab) = \tau(a)\tau(b)$ for relatively prime positive integers $a$ and $b$.
- For any prime $p$ and an integer $r \geq 0$,

$$\tau(p^{r+2}) = \tau(p^{r+1})\tau(p) - p^{11}\tau(p^r).$$

- It follows, from the above property, that $\tau(p) \mid \tau(p^r)$ for all odd $r$. This can be proved easily by induction on the values of the odd parameter $r$. This fact plays an important role at the end of the paper.

- By the famous result of Deligne (see [7]), for any prime $p$ and a positive integer $n$, we have

$$|\tau(p)| \leq 2p^{11/2} \quad \text{and} \quad |\tau(n)| \leq d(n)n^{11/2},$$

where $d(n)$ is the number of divisors of $n$.

In 2000, Luca [5] found all the positive integer $n, m$ such that $f(n!) = m!$, where $f$ is any one of the multiplicative arithmetical functions $\varphi$, $\sigma$, $d$, which are the Euler function, the sum of divisors function and the number of divisors function, respectively. In 2006, Luca and Shparlinski [6] looked at this problem for the Ramanujan function and proved that there are only finitely many pairs of positive integers $(n, m)$ such that

$$|\tau(n!)| = m!. \tag{1}$$

In this note, we follow the same approach of [6] and use arithmetic properties of $\tau$ as well as an explicit lower bound for linear forms in two logarithms to determine all solutions of the above Diophantine equation (1). More precisely, our main result is the following.

THEOREM 1. *The only solutions of the Diophantine equation* (1) *in positive integers* $n$ *and* $m$ *are* $(n, m) \in \{(1, 1), (2, 4)\}$. *Namely,* $|\tau(1!)| = 1!$ *and* $|\tau(2!)| = 4!$.

The plan of the proof is to first find an upper bound on $n$, then one for $m$, which will be reduced by using standard facts about the Ramanujan function. We start with some preliminary lemmas.

**2. Preliminary lemmas.** One of the possible approaches for studying arithmetic properties of $\tau$ is to remark that the sequence $\mathbf{w} := (w_r)_{r=0}^{\infty}$ defined by $w_r = \tau(2^r)$ is a binary recurrent sequence of integers satisfying the recurrence

$$w_r = -24w_{r-1} - 2048w_{r-2} \quad \text{for all} \quad r \geq 2,$$

with the initial conditions $w_0 = 1$ and $w_1 = -24$. Thus

$$w_r = \frac{\alpha_1^{r+1} - \beta_1^{r+1}}{\alpha_1 - \beta_1} \quad \text{for} \quad r \geq 0,$$

where $\alpha_1 = -12 + 4i\sqrt{119}$ and $\beta_1 = \overline{\alpha_1}$ are the zeros of the characteristic polynomial of $\mathbf{w}$, namely $\lambda^2 + 24\lambda + 2048$. If we let $\alpha = -3/2 + i\sqrt{119}/2$ and $\beta = \overline{\alpha}$, then we have that $\alpha_1 = 8\alpha$ and $\beta_1 = 8\beta$. Consequently, the sequence $\mathbf{u} := (u_r)_{r=0}^{\infty}$ given by formula

$$u_r = \frac{\alpha^r - \beta^r}{\alpha - \beta} \quad \text{for} \quad r \geq 0, \tag{2}$$

is a binary recurrence sequence satisfying the relation

$$u_r = -3u_{r-1} - 32u_{r-2} \quad \text{for all} \quad r \geq 2,$$

with the initial conditions $u_0 = 0$ and $u_1 = 1$. From the above, it is easy to see that $w_r = 8^r u_{r+1}$ for all $r \geq 0$. We shall use this fact later.

To prove Theorem 1, we first need to find estimates for $\log|\alpha^s - \beta^s|$ with $s$ being any positive integer. In order to do so, we let $\Lambda := (\beta/\alpha)^s - 1$ so that $|\alpha^s - \beta^s| = |\alpha|^s|\Lambda|$.

First of all, observe that if $|\Lambda| > 1/2$, then $|\alpha^s - \beta^s| > |\alpha|^s/2$, and therefore

$$\log|\alpha^s - \beta^s| > s\log|\alpha| - \log 2. \tag{3}$$

Let us now suppose that $|\Lambda| \leq 1/2$. Then, the inequality $|\log(1+\Lambda)| \leq 2|\Lambda| \leq 1$ holds, where log refers to the principal branch of the logarithm function.

On the other hand, $\log(1+\Lambda) = s\log(\beta/\alpha) + 2\pi N_s i$, where $\Theta = \arg(\beta/\alpha) = 2.604842\ldots$ and $N_s$ is the integer that puts $s\Theta + 2\pi N_s$ in the interval $(-\pi, \pi]$. In fact, one can easily see that $N_s$ is given by

$$N_s = \left\lfloor \frac{1}{2} - \frac{s\Theta}{2\pi} \right\rfloor,$$

where, as usual, $\lfloor \cdot \rfloor$ denotes the greatest integer function. Notice that if $s \geq 2$, then $N_s < 0$. If we let $k = -N_s$, then we can rewrite the expression for $\log(1+\Lambda)$ as

$$\log(1+\Lambda) = s\log(\beta/\alpha) - 2k\log(-1). \tag{4}$$

We now get ready to find a lower bound on $|\log(1+\Lambda)|$ by using a lower bound for nonzero linear forms in two logarithms due to Laurent, Mignotte and Nesterenko [**4**]. We begin by recalling some standard terminology and notation. For an algebraic number $\eta$ we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

with $d$ being the degree of $\eta$ over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

being the minimal primitive polynomial over the integers having positive leading coefficient $a_0$ and $\eta$ as a root.

With the above notation, Laurent, Mignotte and Nesterenko (see Corollary 1 in [**4**]) proved the following theorem.

THEOREM 2. *Let $\gamma_1, \gamma_2$ be two non–zero algebraic numbers, and let $\log\gamma_1$ and $\log\gamma_2$ be any determinations of their logarithms. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$, and*

$$\Gamma = b_2 \log\gamma_2 - b_1 \log\gamma_1,$$

*where $b_1$ and $b_2$ are positive integers. Further, let $A_1, A_2$ be real numbers $> 1$ such that*

$$\log A_i \geq \max\left\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\}, \quad i = 1, 2.$$

*Then, assuming that $\gamma_1$ and $\gamma_2$ are multiplicatively independent, we have*

$$\log|\Gamma| > -30.9D^4\left(\max\left\{\log b', \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log A_1 \log A_2,$$

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*where*

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

In order to apply Theorem 2, we take $\gamma_1 := -1$, $\gamma_2 := \beta/\alpha$, $b_1 := 2k$ and $b_2 := s$. Hence,

$$\Gamma := b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Note that $\Gamma = \log(1 + \Lambda)$ appears in the left-hand side of the relation (4) and satisfies the inequality $|\Gamma| \leq 2|\Lambda|$. The algebraic number field containing $\gamma_1, \gamma_2$ is $\mathbb{Q}(i\sqrt{119})$, so we can take $D = 1$. We next observe that

$$\alpha\beta(x - \beta/\alpha)(x - \alpha/\beta) = \alpha\beta x^2 - ((\alpha + \beta)^2 - 2\alpha\beta)x + \alpha\beta,$$

is a polynomial with integer coefficients and so the above polynomial is a multiple of the minimal primitive polynomial of $\beta/\alpha$ over the integers. Therefore, we deduce that

$$h(\gamma_2) \leq \frac{1}{2} \log |\alpha\beta| = \log |\alpha|.$$

From the above, and taking into account that $\Theta < (8/5) \log |\alpha|$, it follows that we can take $A_1$ and $A_2$ such that $\log A_1 = \pi$ and $\log A_2 = (8/5) \log |\alpha|$. So

$$b' = \frac{5k}{4 \log |\alpha|} + \frac{s}{\pi}.$$

We need an upper bound on $b'$. Since $|\log(1 + \Lambda)| \leq 2|\Lambda| \leq 1$, we get that

$$|2\pi k i| = |s \log(\beta/\alpha) - \log(1 + \Lambda)| \leq s\Theta + 1,$$

giving

$$b' \leq \frac{5(1 + s\Theta)}{8\pi \log |\alpha|} + \frac{s}{\pi} < \frac{7s}{10},$$

which holds for all $s \geq 2$. Finally, the fact that $\gamma_1$ and $\gamma_2$ are multiplicatively independent follows from the fact that $\alpha/\beta$ is not a root of unity. Therefore, we can apply Theorem 2 to the linear form appearing in relation (4) and get that

$$\begin{aligned}\log |\Gamma| &\geq -30.9\,(\max\{\log(7s/10), 21, 1/2\})^2 \cdot \pi \cdot (8/5) \log |\alpha| \\ &> -22370 \log^2(1 + s) \log |\alpha|,\end{aligned}$$

where we have used the fact that $\max\{\log(7s/10), 21, 1/2\} < 12 \log(1 + s)$ for all $s \geq 5$, which is easily seen. Consequently, for $s \geq 5$, we obtain that

$$\begin{aligned}\log |\alpha^s - \beta^s| &= s \log |\alpha| + \log |\Lambda| \geq s \log |\alpha| + \log(|\Gamma|/2) \\ &> s \log |\alpha| - 22370 \log^2(1 + s) \log |\alpha| - \log 2 \\ &> s \log |\alpha| - 22400 \log^2(1 + s) \log |\alpha|. \qquad (5)\end{aligned}$$

But one checks easily that the above inequality (5) is also valid for $s = 1, 2, 3, 4$.

Finally, and going in the other direction, the inequality

$$\log|\alpha^s - \beta^s| = s\log|\alpha| + \log|\Lambda| \leq s\log|\alpha| + \log 2, \tag{6}$$

clearly holds for all $s \geq 1$. Let us summarize what we have proved so far as a lemma.

LEMMA 1. *There is a positive number $c$, which can be taken as 22400, such that for $s \geq 1$,*

$$|\log|\alpha^s - \beta^s| - s\log|\alpha|| < c\log^2(s+1)\log|\alpha|.$$

*Proof.* This lemma follows immediately from (3) and (5), together with the comment following it, and (6). □

For any integer $t \geq 1$, we denote the $t$th cyclotomic polynomial in $\alpha$ and $\beta$ by $\Phi_t(\alpha, \beta)$, so

$$\Phi_t(\alpha, \beta) = \prod_{\substack{k=1 \\ \gcd(k,t)=1}}^{t} (\alpha - \zeta_t^k \beta),$$

where $\zeta_t$ is a primitive $t^{th}$ root of unity. These polynomials are linked to Lucas sequences by the formula

$$\alpha^t - \beta^t = \prod_{d|t} \Phi_d(\alpha, \beta). \tag{7}$$

For any positive integer $t$ let $\mu(t)$ denote the Möbius function of $t$. Then, it follows from (7) that

$$\Phi_t(\alpha, \beta) = \prod_{d|t} (\alpha^{t/d} - \beta^{t/d})^{\mu(d)}. \tag{8}$$

We may now deduce, following the approach of [**8**, Lemma 4.1], our next result.

LEMMA 2. *For $t > 2 \times 10^{11}$,*

$$\left(\frac{2t\log\log t}{2c'(\log\log t)^2 + 5} - ct^{1/3}\log^2(1+t)\right)\log|\alpha| < \log|\Phi_t(\alpha, \beta)|,$$

*where $c = 22400$ and $c' = 1.781072417990198$.*

*Proof.* In view of Lemma 1 and the relation (8), and taking into account the known formula $\varphi(t) = \sum_{d|t}(t/d)\mu(d)$, which holds for all $t \geq 1$, we have that

$$\begin{aligned} |\log|\Phi_t(\alpha, \beta)| - \varphi(t)\log|\alpha|| &\leq \sum_{d|t} |\mu(d)| \left|\log|\alpha^{t/d} - \beta^{t/d}| - \frac{t}{d}\log|\alpha|\right| \\ &< \sum_{\substack{d|t \\ \mu(d)\neq 0}} c\log^2\left(1 + \frac{t}{d}\right)\log|\alpha| \\ &< 2^{\omega(t)} c\log^2(1+t)\log|\alpha|, \end{aligned}$$

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where $\omega(t)$ denotes the number of distinct prime divisors of $t$. In particular, we obtain the following inequality

$$\left(\varphi(t) - 2^{\omega(t)} c \log^2(1+t)\right) \log|\alpha| < \log|\Phi_t(\alpha, \beta)|. \tag{9}$$

Robin [3, Theorem 13] showed that

$$\omega(t) \leq \frac{\log t}{\log\log t - 1.1714} \quad \text{for all} \quad t \geq 26.$$

Using the above bound, we deduce that $2^{\omega(t)} < t^{1/3}$ holds for all $t > 2 \times 10^{11}$. Now the lemma follows immediately from (9) and by using the fact that

$$\frac{t}{\varphi(t)} < c' \log\log t + \frac{5}{2\log\log t},$$

(see [1, Theorem 15]) which is valid for all $t \geq 3$ except when $t = 223092870$. $\square$

**3. Absolute upper bounds.** Assume throughout that equation (1) holds. We will get some upper bounds on $n$ and $m$. To begin with, note that

$$m! = |\tau(n!)| \leq d(n!)(n!)^{11/2} < 2(n!)^6,$$

where we made use of the inequality $d(t) < 2\sqrt{t}$ which holds for all integers $t \geq 1$; hence, $m! < 2(n!)^6$. From this we deduce that $m < 6n$, since otherwise we would have that $m! \geq n!(2n!)^5 > 2(n!)^6$, where we have used the fact that

$$\frac{(tn+1) \times \cdots \times (t+1)n}{n!} = \binom{(t+1)n}{tn},$$

is an integer at least 2 for all $t = 1, \ldots, 5$.

We now consider the sequence $\mathbf{v} := (v_r)_{r=2}^{\infty}$ defined by $v_r = \Phi_r(\alpha, \beta)$. An important known fact is that $v_r \mid u_r$ for all $r \geq 2$, which is easily deduced from (2) and (7). If we write $v_r = A_r B_r$, where $A_r$ and $B_r > 0$ are integers, $B_r$ containing all primitive prime divisors of $u_r$, then it is known (see [2]) that every prime factor of $B_r$ is congruent to $\pm 1 \pmod{r}$.

Recall that, for a sequence $(t_n)_n$, a primitive prime divisor of a term $t_n$ is a prime $p$ that divides $t_n$, but does not divide $t_i$ for any $i$ with $1 \leq i < n$.

Moreover, we have the following remarkable property.

LEMMA 3. *In the notation above, $A_r$ always divides $r$.*

*Proof.* First, one can check by hand that the assertion of the lemma holds for $r = 2$. To see why the lemma holds for $r > 2$, let $p$ be a prime divisor of $A_r$. By [2, Proposition 2.3], $p$ does not divide $\alpha\beta = 32$ and $r = m_p p^k$, where $k \geq 0$ and $m_p$ is the order of appearance of $p$ in the sequence $\mathbf{u}$. Since $p$ is not a primitive divisor of $u_r$, then we have one of the following possibilities:

$$r = m_p p^k \quad \text{with} \quad k \geq 1, \quad \text{or} \quad r = m_p \quad \text{and} \quad p \mid (\alpha - \beta)^2 = -7 \cdot 17.$$

If $r = m_p p^k$ with $k \geq 1$, then $m_p \mid r/p$ implying that $p \mid u_{r/p}$. Since $p \nmid 32$, we have that $p > 2$ and so, by [**2**, Proposition 2.1(vi)], we deduce that $p \mid\mid u_r/u_{(r/p)}$. Since $r/p \mid r$ and $r > 2$, it follows now from the expression (17) of [**2**] that $v_r \mid u_r/u_{(r/p)}$. Hence, $p \mid\mid v_r$.

We now suppose that $m_p = r$ and $p \mid 7 \cdot 17$. In this case, in view of Corollary 2.2 and Proposition 2.1(viii) from [**2**], we get that $m_p = r = p$ and $p \mid\mid u_p = u_r$. Thus, $p \mid\mid A_r$.

From the above, we have that, in any case, $A_r \mid r$. □

Let $a(n)$ be the order at which the prime 2 appears in the prime factorization of $n!$. Observe that, if $n \geq 4$, then $n/2 < a(n) < n$. Also, it follows by (1), and because of the fact that $\tau$ is multiplicative, that $w_{a(n)} \mid m!$. In fact, the following properties of divisibility hold

$$B_{a(n)+1} \mid v_{a(n)+1} \mid u_{a(n)+1} \mid w_{a(n)} \mid m!.$$

We now argue exactly as in [**6**, Section 3]. Since $B_{a(n)+1} \mid m!$ and $m < 6n$, it follows that all prime factors $\ell$ of $B_{a(n)+1}$ satisfy $\ell < 6n$. Since $a(n) > n/2$, there are at most 26 primes $\ell < 6n$ with $\ell \equiv \pm 1 \pmod{a(n)+1}$. Furthermore, again since $B_{a(n)+1} \mid m!$ and $m < 6n$, and all prime factors $\ell$ of $B_{a(n)+1}$ satisfy $\ell \equiv \pm 1 \pmod{a(n)+1}$, it follows that $\ell^{14} \nmid B_{a(n)+1}$. Hence, $B_{a(n)+1} < (6n)^{338}$ $(338 = 26 \times 13)$, and so

$$\begin{aligned} \log|\Phi_{a(n)+1}(\alpha, \beta)| &= \log|v_{a(n)+1}| = \log|A_{a(n)+1}| + \log B_{a(n)+1} \\ &< \log(a(n)+1) + 338 \log(6n) \\ &< 339 \log(6n). \end{aligned} \tag{10}$$

Notice that if $n \geq 5 \times 10^{12}$, then $a(n) + 1 > n/2 + 1 > 2 \times 10^{11}$, and so we can apply Lemma 2 by taking $t = a(n) + 1$ and obtain that

$$\left( \frac{(n+2)\log\log(n/2+1)}{2c'(\log\log n)^2 + 5} - cn^{1/3}\log^2(n+1) \right) \log|\alpha| < \log|\Phi_{a(n)+1}(\alpha, \beta)|, \tag{11}$$

where $c$ and $c'$ are those given in Lemma 2, and where we have used additionally the fact that $n/2 < a(n) < n$. Consequently, the above inequality (10) combined with (11) yields

$$\left( \frac{(n+2)\log\log(n/2+1)}{2c'(\log\log n)^2 + 5} - cn^{1/3}\log^2(n+1) \right) \log|\alpha| < 339 \log(6n),$$

which gives, by using *Mathematica*, that $n < 5 \times 10^{12}$, which is a contradiction. Hence, $n < 5 \times 10^{12}$ and therefore $m < 3 \times 10^{13}$. Let us record what we have just proved.

LEMMA 4. *If* $(n, m)$ *is a solution in positive integers* $n$ *and* $m$ *of equation* (1), *then*

$$n < 5 \times 10^{12} \quad \textit{and} \quad m < 3 \times 10^{13}.$$

**4. Reducing the bounds.** After finding an upper bound on $n$ and $m$ the next step is to reduce them to a range in which the solutions of the equation (1) can be identified by using a computer. To do this, we use several times the following lemma, which plays a crucial role in this task.

LEMMA 5. *Let $m_0$ and $n_\star$ be positive integers, and let $p$ a prime number such that $P(\tau(p)) \geq m_0$, where $P(t)$ denotes the largest prime factor of $t$ if $t > 1$ and $P(1) = 1$.*

(*a*) *If $n_\star < p^2$ and $a = \lfloor n_\star/p \rfloor$ is an odd number, then there is no solution to the equation* (1) *in positive integers $n$ and $m$ with*

$$n \in \{ap, ap+1, \ldots, ap+p-1\} \quad \textit{and} \quad m < m_0.$$

(*b*) *There is no solution to the equation* (1) *in positive integers $n$ and $m$ with*

$$p \leq n < 2p \quad \textit{and} \quad m < m_0.$$

*Proof.* To prove ($a$), we write $n$ as $n = ap + b$ for some integer $b$ with $0 \leq b < p$. Then, it is known that $a$ is the order at which $p$ appears in the prime factorization of $n!$. Using this and the fact that $a$ is an odd number, as well as the fact that $\tau$ is a multiplicative function, we get that $\tau(p) \mid \tau(p^a) \mid m!$, giving that $P(\tau(p)) \mid m!$. Hence, $P(\tau(p)) \leq m < m_0$ which is impossible. Thus, equation (1) has no solutions in this range for $n$ and $m$. Part ($b$) of the lemma follows immediately from part ($a$) by taking $n_\star = p$. □

Let us now use Lemma 5 to reduce our bounds. In order to do so, we take $n_0 = 5 \times 10^{12}$ and $m_0 = 3 \times 10^{13}$, and we first put $n_\star = n_0/10$. With the help of *Mathematica* we search for a set $\mathcal{P}_1$ of 50 prime numbers, all of them greater than $\sqrt{n_0}$ and spaced a distance of at least 10,000, with the property that $P(\tau(p)) \geq m_0$ for all $p \in \mathcal{P}_1$. Some elements of the set $\mathcal{P}_1$ are

$$\mathcal{P}_1 = \{2246099, 2266129, 2276137, 2286139, \ldots, 2776733, 2786741, 2796751\}.$$

Next, we find a prime number $p_1 \in \mathcal{P}_1$ such that $a_1 = \lfloor n_\star/p_1 \rfloor$ is an odd number. It then follows from Lemma 5($a$) that there is no solution to the equation (1) in positive integers $n$ and $m$ with $n \in \{a_1p_1, a_1p_1 + 1, \ldots, a_1p_1 + p_1 - 1\}$ and $m < m_0$.

We now take $n_\star = a_1p_1 + p_1$ and find a prime number, say $p_2 \in \mathcal{P}_1$, which satisfies that $a_2 = \lfloor n_\star/p_2 \rfloor$ is an odd number. Using Lemma 5($a$) once more, we conclude that there is no solution to equation (1) with $n \in \{a_2p_2, a_2p_2 + 1, \ldots, a_2p_2 + p_2 - 1\}$ and $m < m_0$. Again, we take $n_\star = a_2p_2 + p_2$ and repeat the process as many times as possible in order to remove some intervals located on the right of $n_\star$. To do this, we use a simple code written in *Mathematica*, and we finally achieve $n_0$, that is, we reduce the upper bound on $n$ a factor of 10. Namely $n < n_0/10 = 5 \times 10^{11}$, and therefore $m < 3 \times 10^{12}$.

Now, we update the values of $n_0$ and $m_0$ and repeat the process to reduce the new upper bounds even more. In fact, taking $n_0 = 5 \times 10^{11}$, $m_0 = 3 \times 10^{12}$, and the same set of primes $\mathcal{P}_1$, we reduce the upper bounds on $n$ and $m$ a new factor of 10. This process was done three times more and we finally conclude that $n < n_0 = 5 \times 10^7$ and $m < m_0 = 3 \times 10^8$.

At this point, we were not successful in finding primes $p \in \mathcal{P}_1$ such that $\lfloor n_\star/p \rfloor$ is an odd number. Hence, we search for a new set $\mathcal{P}_2$ of 30 prime numbers, all of them bigger than $\sqrt{n_0}$ and spaced a distance of at least 1000, such that $P(\tau(p)) \geq m_0$ for all $p \in \mathcal{P}_2$. Below, we present some elements of the set $\mathcal{P}_2$.

$$\mathcal{P}_2 = \{7079, 9091, 10093, 11113, \ldots, 39451, 40459, 41467, 42473\}.$$

With this new list of primes we get that $n < n_0 = 5 \times 10^4$ and $m < m_0 = 3 \times 10^5$. Also, we generate a third set of primes to obtain that $n < n_0 = 5 \times 10^2$ and $m < m_0 = 3 \times 10^3$.

Since we have now more comfortable upper bounds for $n$ and $m$, we use Lemma 5(b) in our argument to reduce the bounds. Indeed, taking a prime $p$, $p < n_0 < 2p$, such that $P(\tau(p)) \geq m_0$, we reduce $n_0$ by almost half. After doing this several times, and update $n_0$ and $m_0$, we finally obtain the range $1 \leq n \leq 12$, $1 \leq m \leq 72$.

Finally, we used *Mathematica* and checked that the only solutions of the equation (1) in this range are those given by Theorem 1.

Theorem 1 is therefore proved.

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