Glasgow Math. J. **59** (2017) 503–511. © Glasgow Mathematical Journal Trust 2016.
doi:10.1017/S001708951600032X.

DISTRIBUTIVE LATTICES OF TILTING MODULES AND SUPPORT τ-TILTING MODULES OVER PATH ALGEBRAS

YICHAO YANG
Département de mathématiques, Université de Sherbrooke,
Sherbrooke, Québec, Canada, J1K 2R1
E-mail: yichao.yang@usherbrooke.ca

(Received 23 July 2015; accepted 2 January 2016; first published online 10 June 2016)

Abstract. In this paper, we study the poset of basic tilting kQ-modules when Q is a Dynkin quiver, and the poset of basic support τ-tilting kQ-modules when Q is a connected acyclic quiver respectively. It is shown that the first poset is a distributive lattice if and only if Q is of types $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$ with a non-linear orientation and the second poset is a distributive lattice if and only if Q is of type $\mathbb{A}_1$.

2010 *Mathematics Subject Classification.* 16G20, 16G70, 05E10.

1. Introduction. Let Q be a finite connected acyclic quiver and kQ be the path algebra of Q over an algebraically closed field k. Denote by mod-kQ the category of finite dimensional right kQ-modules, by ind-kQ the category of indecomposable modules in mod-kQ and by Γ(mod kQ) the Auslander–Reiten quiver of kQ. For $M \in$ mod-kQ, we denote by add M (respectively, Fac M, Sub M) the category of all direct summands (respectively, factor modules, submodules) of finite direct sums of copies of M and by $|M|$ the number of pairwise non-isomorphic indecomposable direct summands of M. Let Q_0 be the set of vertices of Q and Q_1 be the set of arrows of Q. Furthermore, let P_i (I_i, S_i, respectively) be an indecomposable projective (injective, simple, respectively) module in mod-kQ associated with vertex $i \in Q_0$ and τ be the Auslander–Reiten translation.

Tilting theory for kQ, or more generally for a finite dimensional basic k-algebra, was first appeared in [**4**] and have been central in the representation theory of finite-dimensional algebras since the early seventies. For the classical tilting modules and their mutation theory, there is a naturally associated quiver named tilting quiver which is defined in [**16**]. Happel and Unger defined a partial order on the set of basic tilting modules and showed that the tilting quiver coincides with the Hasse quiver of this poset [**5**]. A related partial order has been studied in the τ-tilting theory introduced in [**2**] and the analog result also holds, that is, the support τ-tilting quiver also coincide with the Hasse quiver of this related partial order.

Recently, the lattice structure of the poset of tilting modules and support τ-tilting modules have been studied in [**7**, **9**, **15**]. More precisely, Kase showed that for representation-infinite algebras kQ, the poset of its pre-projective tilting modules possess a distributive lattice structure if and only if the degree of all vertices in Q is greater than 1 [**9**]. Later Iyama, Reiten, Thomas and Todorov proved that for path algebras kQ, the poset of its support τ-tilting modules possess a lattice structure if and only if Q is a Dynkin quiver or has at most two vertices.

The aim of this paper is to study the following problem.

Problem 1.1. Let Q be a finite connected acyclic quiver.

(1) When does the poset of basic tilting kQ-modules possess a distributive lattice structure?
(2) When does the poset of basic support τ-tilting kQ-modules possess a distributive lattice structure?

Our main result is the following theorem.

Theorem 1.2. *Let Q be a Dynkin quiver. Then the following statements are equivalent.*

(1) *All tilting modules are slice modules.*
(2) *The full subquiver generated by any tilting module form a section of $\Gamma($mod $kQ)$.*
(3) *The tilting quiver $\vec{\mathcal{T}}(Q)$ is a distributive lattice.*
(4) *Any boundary orbit (see Definition 3.1) of $\Gamma($mod $kQ)$ contains at most two modules.*

For the representation-infinite case, see [**6**, **9**, **10**].
As a consequence, the answer to Problem 1.1(1) is given in the following theorem.

Theorem 1.3. *Let Q be a finite connected acyclic quiver.*

(1) [**9**, Theorem 3.1] *If Q is a non-Dynkin quiver, then the poset of basic pre-projective tilting kQ-modules is a distributive lattice if and only if the degree of all vertices in Q are greater than* 1.
(2) *If Q is a Dynkin quiver, then the poset of basic tilting kQ-modules is a distributive lattice if and only if Q is of types $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$ with a non-linear orientation.*

On the other hand, we also show the following result which answers Problem 1.1(2).

Theorem 1.4. *Let Q be a finite connected acyclic quiver. Then, the poset of basic support τ-tilting kQ-modules is a distributive lattice if and only if Q is of type $\mathbb{A}_1$.*

The paper is organized as follows. In Section 2, we recall some preliminary definitions and results of tilting theory, τ-tilting theory and lattice theory, especially about the tilting quiver, support τ-tilting quiver and distributive lattice. In Section 3.1, we first introduce the notions of boundary module and boundary orbit and then prove Theorem 1.2. In Section 3.2 we give the proof of Theorems 1.3 and 1.4.

2. Preliminaries.

2.1. Tilting theory and τ-tilting theory. We start with the following definitions of tilting modules and tilting quiver which was considered in [**9**], and was first introduced in [**5**, **16**].

Definition 2.1. A module $T \in$ mod-kQ is a tilting module if

(1) $\mathrm{Ext}^1_{kQ}(T, T) = 0$.
(2) $|T| = |Q_0|$.

We denote by $\mathcal{T}(Q)$ a complete set of representatives of the isomorphism classes of the basic tilting modules in mod-kQ.

DEFINITION 2.2. The tilting quiver $\vec{\mathcal{T}}(Q)$ is defined as follows:

(1) $\vec{\mathcal{T}}(Q)_0 := \mathcal{T}(Q)$.
(2) $T \to T'$ in $\vec{\mathcal{T}}(Q)$ if $T \cong M \oplus X$, $T' \cong M \oplus Y$ for some $X, Y \in$ ind-kQ, $M \in$ mod-kQ and there is a non-split exact sequence

$$0 \longrightarrow X \longrightarrow M' \longrightarrow Y \longrightarrow 0$$

with $M' \in \operatorname{add} M$.

Now we recall some basic definitions of τ-tilting theory, which was first introduced in [**2**], in order to "complete" the classical tilting theory from the viewpoint of mutation.

DEFINITION 2.3.

(1) We call $M \in$ mod-kQ τ-rigid if $\operatorname{Hom}_{kQ}(M, \tau M) = 0$.
(2) We call $M \in$ mod-kQ τ-tilting if M is τ-rigid and $|M| = |Q_0|$.
(3) We call $M \in$ mod-kQ support τ-tilting if there exists an idempotent e of kQ such that M is a τ-tilting $(kQ/\langle e\rangle)$-module.

Indeed for a path algebra, a support τ-tilting module is just a support tilting module introduced in [**8**].

We denote by $\mathcal{ST}(Q)$ a complete set of representatives of the isomorphism classes of the basic support τ-tilting modules in mod-kQ.

Recall that the Hasse-quiver $\vec{P}$ of a poset $(P, \leq)$ is defined as follows:

(1) $\vec{P}_0 := P$.
(2) $x \to y$ in $\vec{P}$ if $x > y$ and there is no $z \in P$ such that $x > z > y$.

The support τ-tilting quiver $\vec{\mathcal{ST}}(Q)$ is defined as follows.

PROPOSITION-DEFINITION 2.1. ([**2**, Theorem 2.7, Corollary 2.34])

(1) *Let $T, T' \in \mathcal{ST}(Q)$. Then the following relation $\leq$ defines a partial order on $\mathcal{ST}(Q)$,*

$$T \geq T' \overset{\text{def}}{\Leftrightarrow} \operatorname{Fac} T \supseteq \operatorname{Fac} T'.$$

(2) *The* support τ-tilting quiver $\vec{\mathcal{ST}}(Q)$ *is the Hasse quiver of the partially order set $(\mathcal{ST}(Q), \leq)$.*

We remark that there is the following similar result in the classical tilting theory.

THEOREM 2.4. ([**5**, Theorem 2.1])

(1) *Let $T, T' \in \mathcal{T}(Q)$. Then the following relation $\leq$ defines a partial order on $\mathcal{T}(Q)$,*

$$T \geq T' \overset{\text{def}}{\Leftrightarrow} \operatorname{Fac} T \supseteq \operatorname{Fac} T'.$$

(2) *The tilting quiver $\vec{\mathcal{T}}(Q)$ is the Hasse quiver of the partially order set $(\mathcal{T}(Q), \leq)$.*

https://doi.org/10.1017/S001708951600032X Published online by Cambridge University Press

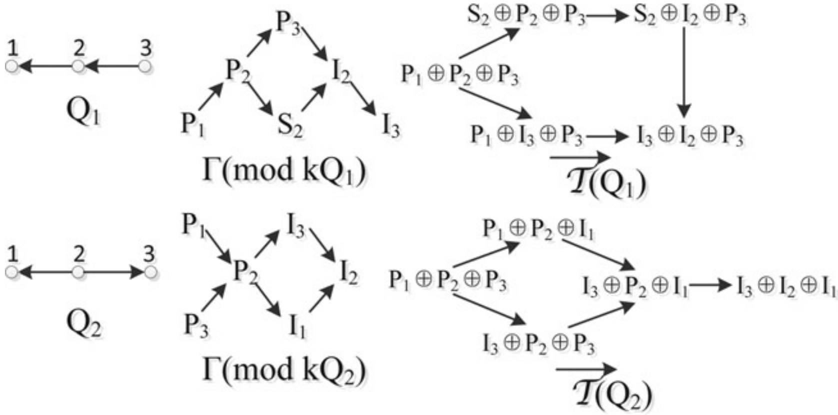

Figure 1. Tilting quiver.

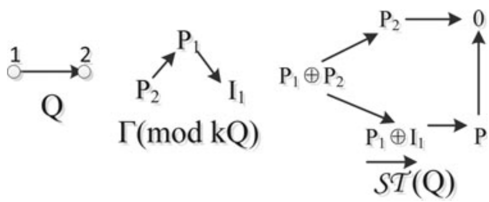

Figure 2. Support τ-tilting quiver.

We end this subsection with the following two examples.

EXAMPLE 2.1. Let Q_1, Q_2 be the following two different quivers, see Figure 1. Although they share the same underlying graph; however, the corresponding tilting quivers are different.

EXAMPLE 2.2. Let Q be of type $\mathbb{A}_2$. Then its support τ-tilting quiver $\vec{\mathcal{ST}}(Q)$ is shown in Figure 2.

2.2. Lattices and distributive lattices. In this subsection, we will recall definitions of lattices and distributive lattices.

DEFINITION 2.5. A poset $(L, \leq)$ is a lattice if for any $x, y \in L$ there is a minimum element of $\{z \in L | z \geq x, y\}$ and there is a maximum element of $\{z \in L | z \leq x, y\}$.

In this case, we denote by $x \vee y$ the minimum element of $\{z \in L | z \geq x, y\}$ and call it join of x and y. We also denote by $x \wedge y$ the maximum element of $\{z \in L | z \leq x, y\}$ and call it meet of x and y.

DEFINITION 2.6. A lattice L is a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ holds for any $x, y, z \in L$.

Immediately we have the following basic observation, which will be used frequently in this paper.

LEMMA 2.7. *For any $n \geq 2$, the following Hasse quiver in FIGURE 3 is not a distributive lattice.*

https://doi.org/10.1017/S001708951600032X Published online by Cambridge University Press

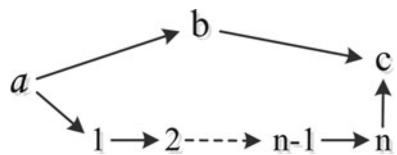

Figure 3. Hasse quiver not a distributive lattice.

Proof. Since $n \geq 2$, it is easy to see that

$$(b \vee 2) \wedge 1 = a \wedge 1 = 1 \neq 2 = c \vee 2 = (b \wedge 1) \vee (2 \wedge 1),$$

therefore it is not a distributive lattice. □

In the above Examples 2.1 and 2.2, it is easy to see that the lattice $(\mathcal{T}(Q_2), \leq)$ is a distributive lattice. On the other hand, it follows by Lemma 2.7 that both $(\mathcal{T}(Q_1), \leq)$ and $(\vec{\mathcal{ST}}(Q), \leq)$ are not distributive lattice.

3. Main results.

3.1. Boundary module and boundary orbit. From now on, we will not distinguish between an indecomposable kQ-module M and its corresponding vertex $[M]$ in the Auslander–Reiten quiver $\Gamma(\text{mod}\, kQ)$. By Theorem 2.4 and Proposition-Definition 2.1, it is easy to see that our problem reduces to the study of lattice structure of the tilting quiver $\vec{\mathcal{T}}(Q)$ and the support τ-tiling quiver $\vec{\mathcal{ST}}(Q)$.

Before proceeding further, let (Γ, τ) be a connected translation quiver, recall from [**1**] that a connected full subquiver Σ of Γ is called a *presection* (is also called a *cut* in [**12**]) in Γ if it satisfies the following two conditions:

(1) If $x \in \Sigma_0$ and $x \to y$ is an arrow, then either $y \in \Sigma_0$ or $\tau y \in \Sigma_0$.
(2) If $y \in \Sigma_0$ and $x \to y$ is an arrow, then either $x \in \Sigma_0$ or $\tau^{-1} x \in \Sigma_0$.

Moreover, in [**11**] a connected full subquiver Σ of Γ is a called *section* of Γ if the following conditions are satisfied:

(1) Σ contains no oriented cycle.
(2) Σ meets each τ-orbit in Γ exactly once.
(3) Σ is convex in Γ, that is, every path in Γ with end-points belonging to Σ lies entirely in Σ.

From [**14**] recall also that a module S is said to be a *slice module* if S is sincere and add S satisfies the following conditions:

(1) If there is a path $x_0 \to x_1 \to \cdots \to x_t$ with $x_0, x_t \in \text{add}\, S$, then $x_i \in \text{add}\, S$ $(i = 0, 1, \ldots, t)$.
(2) If M is indecomposable and not projective, then at most one of $M, \tau M$ belongs to add S.
(3) If there is an arrow $M \to X$ with $X \in \text{add}\, S$ in the Auslander–Reiten quiver, then either $M \in \text{add}\, S$ or M is not injective and $\tau^{-1} M \in \text{add}\, S$.

Now we introduce the notions of boundary module and boundary orbit.

DEFINITION 3.1.

(1) We call a module $M \in \Gamma(\text{mod } kQ)$ boundary module if M has at most one direct predecessor and at most one direct successor in Auslander–Reiten quiver $\Gamma(\text{mod } kQ)$.
(2) We call a τ-orbit Σ of $\Gamma(\text{mod } kQ)$ boundary orbit if Σ contains a boundary module.

The following two observations are useful.

LEMMA 3.2. *Let Q be a tree. If Q is not of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$ with a non-linear orientation, then the Auslander–Reiten quiver $\Gamma(\text{mod } kQ)$ has a boundary orbit containing at least three modules.*

Proof. Assume that Q is not of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$ with linear orientation. If Q is a linear quiver, then by our assumption we have $|Q_0| \geq 3$ and hence the result follows at once. Otherwise since Q is a tree, there exists $a \in Q_0$ such that the degree of a is 1. Without loss of generality we may assume that a is a source vertex, i.e., there is an arrow $a \to b \in Q_1$. Then $I_a = S_a$ is a simple non-projective module and by [**3**, Lemma IV, 3.9] the only direct predecessor of I_a in $\Gamma(\text{mod } kQ)$ is I_b.

If τI_a is not projective, then the result follows since I_a is a boundary module. Otherwise, there exists an almost split sequence $0 \to P_c \to I_b \to I_a \to 0$ in $\Gamma(\text{mod } kQ)$ for some $c \in Q_0$. In particular, $\text{Hom}_{kQ}(P_c, I_b) \neq 0$, i.e., Q contains the following subquiver: $c \to d \to \cdots \to b \leftarrow a$. Moreover since Q is not a linear quiver, there is no projective–injective module in $\Gamma(\text{mod } kQ)$ and thus P_c is also a boundary module, i.e., c has degree 1 in Q.

If τI_c is not projective, then the result follows since I_c is also a boundary module. Otherwise similarly we can show that Q contains the following subquiver: $e \to \cdots \to f \to d \leftarrow c$. If $f \neq a$, then $0 < (\mathbf{dim}\ I_b)_f = (\mathbf{dim}\ P_c)_f + (\mathbf{dim}\ I_a)_f = 0 + 0 = 0$, which is a contradiction. Here $(\mathbf{dim}\ M)_i$ denotes the ith component of the dimension vector of the module M. If $f = a$, then $b = d$ and now Q is of type $\mathbb{A}_3$ with a non-linear orientation, which is also a contradiction. It means that we can always construct an injective boundary module M such that τM is not projective. The proof of the lemma is completed. □

LEMMA 3.3. *Let Q be a Dynkin quiver. If one of its boundary orbits contains at least three modules, then the tilting quiver $\vec{\mathcal{T}}(Q)$ is not a distributive lattice.*

Proof. Since Q is a Dynkin quiver, $\Gamma(\text{mod } kQ)$ must be a full convex subquiver of $\mathbb{Z}Q$. Without loss of generality, by our assumption $\Gamma(\text{mod } kQ)$ will contain the following shaded area $\mathcal{T}$, see Figure 4.

Now we enlarge $\mathcal{T}$ for each type, for the type A, see the left-lower of Figure 4. For simplicity, we may continue with the type A, for the remaining two types, the argument is similar.

Let $|Q_0| = n$, it is easy to see that we can construct a section Σ of the lower $(n-2)$-rows starting with M_6 and denote the module corresponding to this section by M_Σ. Then we consider the following five modules

$$T_1 = M_\Sigma \oplus M_4 \oplus M_1,\ T_2 = M_\Sigma \oplus M_4 \oplus M_2,\ T_3 = M_\Sigma \oplus M_5 \oplus M_2,$$
$$T_4 = M_\Sigma \oplus M_5 \oplus M_3,\ T_5 = M_\Sigma \oplus M_1 \oplus M_3.$$

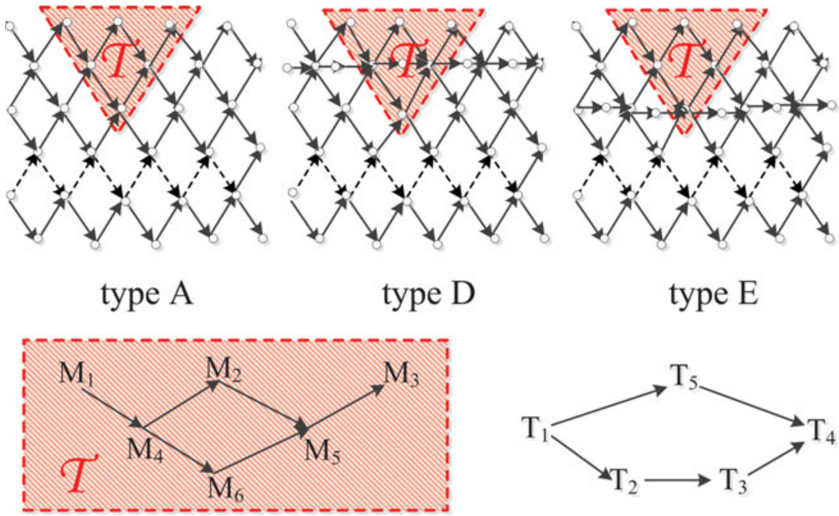

Figure 4. (Colour online) Shaded area.

Since $\Gamma(\text{mod}\, kQ)$ is a standard component, it is not hard to see that all of these five modules are tilting modules and they forms the right-lower of Figure 4, which is a full subquiver of the tilting quiver $\vec{\mathcal{T}}(Q)$, however, is not a distributive lattice by Lemma 2.7. Hence the tilting quiver $\vec{\mathcal{T}}(Q)$ is also not a distributive lattice, which completes the proof. □

Now we are ready to prove Theorem 1.2.

(1) $\Leftrightarrow$ (2): This is shown in [**13**] or [**17**].

(2) $\Rightarrow$ (3): Let $|Q_0| = n$, according to (2) it follows that any tilting module can be written as

$$T \cong \bigoplus_{i=1}^{n} \tau^{-r_i} P_i$$

for $r_i \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq n$ and if T, T' be two tilting modules, $T \to T'$ in $\vec{\mathcal{T}}(Q)$ if and only if there is an indecomposable direct summand X such that $T \cong M \oplus X$ and $T' \cong M \oplus \tau^{-1}X$. Thus, for any two tilting modules $T \cong \bigoplus_{i=1}^{n} \tau^{-r_i} P_i$, $T' \cong \bigoplus_{i=1}^{n} \tau^{-r'_i} P_i$, $T \geq T'$ if and only if $r_i \leq r'_i$, $1 \leq i \leq n$. From now on let Σ_T be the full subquiver of $\Gamma(\text{mod}\, kQ)$ generated by T. Since $\Sigma_T, \Sigma_{T'}$ form a section of $\Gamma(\text{mod}\, kQ)$, it is not hard to check that both $\Sigma_{\bigoplus_{i=1}^{n} \tau^{-\min\{r_i, r'_i\}} P_i}$ and $\Sigma_{\bigoplus_{i=1}^{n} \tau^{-\max\{r_i, r'_i\}} P_i}$ again form a section of $\Gamma(\text{mod}\, kQ)$, which implies that both $\bigoplus_{i=1}^{n} \tau^{-\min\{r_i, r'_i\}} P_i$ and $\bigoplus_{i=1}^{n} \tau^{-\max\{r_i, r'_i\}} P_i$ are tilting modules. Therefore the join and meet of T and T' are

$$T \vee T' \cong \bigoplus_{i=1}^{n} \tau^{-\min\{r_i, r'_i\}} P_i, \qquad T \wedge T' \cong \bigoplus_{i=1}^{n} \tau^{-\max\{r_i, r'_i\}} P_i$$

respectively, which makes the tilting quiver $\vec{\mathcal{T}}(Q)$ to be a distributive lattice. Indeed, it follows by the fact that $a \vee b = (\min(r_i, r'_i))_{1 \leq i \leq n}$ and

https://doi.org/10.1017/S001708951600032X Published online by Cambridge University Press

$a \wedge b = (\max(r_i, r'_i))_{1 \le i \le n}$ makes $(\mathbb{Z}^n, \le^{\mathrm{op}})$ to be a distributive lattice, where $a = (r_i)_{1 \le i \le n}$, $b = (r'_i)_{1 \le i \le n}$.

(3) $\Rightarrow$ (4): It follows from Lemma 3.3 at once.

(4) $\Rightarrow$ (2): Since Q is a Dynkin quiver and any boundary orbit of $\Gamma(\mathrm{mod}\ kQ)$ contains at most two modules, by Lemma 3.2 we have that Q is of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$ with a non-linear orientation. Then the result follows at once.

3.2. Proof of Theorem 1.3 and 1.4. First, we start with the proof of Theorem 1.3. For the non-Dynkin case, see [**9**, Theorem 3.1]. If Q is a Dynkin quiver, it follows from the equivalence between (2) and (4) in Theorem 1.2 and Lemma 3.2 at once. The proof of the Theorem 1.3 is completed.

Now we are going to prove Theorem 1.4.

Indeed, by [**7**, Theorem 0.3], it suffices to consider the following two cases.

Case 1: Q is of Dynkin type.

If $|Q_0| = 1$, then the support τ-tilting quiver is $\cdot \to \cdot$, it is clear.

If $|Q_0| = n \ge 2$, then Q contains $\mathbb{A}_2$ as its full subquiver. Without loss of generality we assume that $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents for kQ and there is an arrow α between the vertices 1 and 2. Let $e = e_3 + e_4 + \cdots + e_n$. Then $kQ/\langle e\rangle \cong k\mathbb{A}_2$.

By Example 2.2 the support τ-tilting quiver $\vec{\mathcal{ST}}(\mathbb{A}_2)$ is not a distributive lattice. On the other hand, according to [**2**, Proposition 2.27(a)] it can easily be seen that $\vec{\mathcal{ST}}(\mathbb{A}_2)$ is a full subquiver of $\vec{\mathcal{ST}}(Q)$, which implies that $\vec{\mathcal{ST}}(Q)$ is not a distributive lattice itself.

Case 2: Q has at most two vertices.

According to [**7**, Proposition 2.2], it follows that the support τ-tilting quiver $\vec{\mathcal{ST}}(Q)$ is isomorphic to the Figure 3 in Lemma 2.7, where n tends to $+\infty$. Now by Lemma 2.7 it is obvious that $\vec{\mathcal{ST}}(Q)$ is not a distributive lattice.

Finally, by combining the above two cases together, the proof of the Theorem 1.4 is completed.

ACKNOWLEDGEMENTS. The author is grateful to Professor Shiping Liu for his valuable discussions. He also warmly thank the referees for many helpful comments and suggestions in improving the quality and readability of this paper.

This work was carried out when the author is a post-doctoral fellow at Université de Sherbrooke, financed by Fonds Québécois de la Recherche sur la Nature et les Technologies (Québec, Canada) through the Merit Scholarship Program For Foreign Students.

REFERENCES

1. I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras and slices, *J. Algebra* **319**(8) (2008), 3464–3479.

2. T. Adachi, O. Iyama and I. Reiten, τ-tilting theory, *Compos. Math.* **150**(3) (2014), 415–452.

3. I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*, London Mathematical Society Student Texts, vol. 65 (Cambridge University Press, Cambridge, 2006).

4. S. Brenner and M. C. R. Butler, *Generalization of the Bernstein-Gelfand-Ponomarev reflection functors*, Lecture Notes in Math., vol. 839 (Springer-Verlag, Berlin, 1980), 103–169.

5. D. Happel and L. Unger, On a partial order of tilting modules, *Algebr. Represent. Theory* **8**(2) (2005), 147–156.

6. D. Happel and D. Vossieck, Minimal algebras of infinite representation type with preprojective component, *Manuscripta Math.* **42**(2–3) (1983), 221–243.

7. O. Iyama, I. Reiten, H. Thomas and G. Todorov, Lattice structure of torsion classes for path algebras, *Bull. Lond. Math. Soc.* **47**(4) (2015), 639–650.

8. C. Ingalls and H. Thomas, Noncrossing partitions and representations of quivers, *Compos. Math.* **145**(6) (2009), 1533–1562.

9. R. Kase, Distributive lattices and the poset of pre-projective tilting modules, *J. Algebra* **415**(1) (2014), 264–289.

10. O. Kerner and M. Takane, Mono orbits, epi orbits and elementary vertices of representation infinite quivers, *Comm. Algebra* **25**(1) (1997), 51–77.

11. S. Liu, Shapes of connected components of the Auslander-Reiten quivers of artin algebras, in *Representation theory of algebras and related topics* (Mexico City, 1994); Canad. Math. Soc. Conf. Proc., vol. 19 (1995), 109–137.

12. S. Liu, Another characterization of tilted algebras, *Arch. Math.* **104**(2) (2015), 111–123.

13. F. Li and Y. C. Yang, A note on section and slice for a hereditary algebra, *Int. J. Appl. Math. Stat.* **52**(9) (2014), 112–119.

14. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099 (Springer-Verlag, Berlin, 1984).

15. C. M. Ringel, Lattice structure of torsion classes for hereditary artin algebras, arXiv:1402.1260.

16. C. Riedtmann and A. Schofield, On a simplicial complex associated with tilting modules, *Comment. Math. Helv.* **66**(1) (1991), 70–78.

17. P. Zhang, Separating tilting modules, *Chinese Sci. Bull.* **37**(12) (1992), 975–978.