*J. Aust. Math. Soc.* **107** (2019), 381–391
doi:10.1017/S1446788718000447

# CHARACTERIZATIONS OF BMO AND LIPSCHITZ SPACES IN TERMS OF $A_{P,Q}$ WEIGHTS AND THEIR APPLICATIONS

DINGHUAI WANG✉, JIANG ZHOU and ZHIDONG TENG

(Received 7 March 2018; accepted 10 October 2018; first published online 30 January 2019)

Communicated by C. Meaney

**Abstract**

Let $0 < \alpha < n, 1 \le p < q < \infty$ with $1/p - 1/q = \alpha/n$, $\omega \in A_{p,q}$, $\nu \in A_\infty$ and let $f$ be a locally integrable function. In this paper, it is proved that $f$ is in bounded mean oscillation $BMO$ space if and only if

$$\sup_B \frac{|B|^{\alpha/n}}{\omega^p(B)^{1/p}}\bigg(\int_B |f(x) - f_{\nu,B}|^q \omega(x)^q\,dx\bigg)^{1/q} < \infty,$$

where $\omega^p(B) = \int_B \omega(x)^p\,dx$ and $f_{\nu,B} = (1/\nu(B))\int_B f(y)\nu(y)\,dy$. We also show that $f$ belongs to Lipschitz space $Lip_\alpha$ if and only if

$$\sup_B \frac{1}{\omega^p(B)^{1/p}}\bigg(\int_B |f(x) - f_{\nu,B}|^q \omega(x)^q\,dx\bigg)^{1/q} < \infty.$$

As applications, we characterize these spaces by the boundedness of commutators of some operators on weighted Lebesgue spaces.

2010 *Mathematics subject classification*: primary 47B47; secondary 42B35.

*Keywords and phrases*: $A_{p,q}$ weights, BMO space, characterize, commutators, Lipschitz spaces.

## 1. Introduction

The space of functions with bounded mean oscillation $BMO$ was introduced by John and Nirenberg in [11] and plays a crucial role in harmonic analysis and partial differential equations; see for example, [7, 15]. Recall that the space $BMO$ consists of all measurable functions $f$ satisfying

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty,$$

where $f_B = (1/|B|)\int_B f(x)\,dx$ and the supremum is taken over all balls $B$. Some characterizations of $BMO$ are given as follows.

A well-known immediate consequence of the John–Nirenberg inequality is the following result.

$$\|f\|_{BMO} \approx \sup_B \left( \frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right)^{1/p},$$

for all $1 < p < \infty$. Moreover, it can be proved that the above equivalence also holds for $0 < p < 1$ even though the right-hand side is not a norm in such a case (see [15]).

Another deep connection was made between Muckenhoupt weights and $BMO$ in the work of Muckenhoupt and Wheeden [13]. They proved that a function $f$ is in $BMO$ if and only if $f$ it is of $BMO$ with respect to $\omega$ for all $\omega \in A_\infty$. That is, if, for each $\omega \in A_\infty$, we define $BMO_\omega$ to be the collection of all $\omega$-locally integrable functions $f$ such that

$$\|f\|_{BMO_\omega} = \sup_B \frac{1}{\omega(B)} \int_B |f(x) - f_{\omega,B}| \omega(x) \, dx < \infty,$$

then $BMO = BMO_\omega$ and

$$\|f\|_{BMO} \approx \|f\|_{BMO_\omega}.$$

Here $\omega(B) = \int_B \omega(x) \, dx$ and

$$f_{\omega,B} = \frac{1}{\omega(B)} \int_B f(x) \omega(x) \, dx.$$

It was recently obtained by Hart and Torres [8] that, for $0 < p < \infty$ and $\omega, \nu \in A_\infty$,

$$\|f\|_{BMO} \approx \sup_B \left( \frac{1}{\omega(B)} \int_B |f(x) - f_{\nu,B}|^p \omega(x) \, dx \right)^{1/p}.$$

For $\nu \equiv 1$ and $1 \leq p < \infty$, the result above was obtained by Ho [9]. The aim of this paper is to show that $BMO$ space can be characterized by $A_{p,q}$ weights. To state our results, we first recall the definitions of $A_p$ and $A_{p,q}$ weights.

For $1 < p < \infty$ and a nonnegative locally integrable function $\omega$, we say that $\omega$ is in the Muckenhoupt $A_p$ class [12] if it satisfies the condition

$$[\omega]_{A_p} := \sup_B \left( \frac{1}{|B|} \int_B \omega(x) \, dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{-(1/p-1)} \, dx \right)^{p-1} < \infty.$$

A weight function $\omega$ belongs to the class $A_1$ if

$$[\omega]_{A_1} := \frac{1}{|B|} \int_B \omega(x) \, dx \left( \operatorname*{ess\,sup}_{x \in B} \omega(x)^{-1} \right) < \infty.$$

We write $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

Next, we recall the definition of $A_{p,q}$ weight introduced by Muckenhoupt and Wheeden [14]. For $1 < p, q < \infty$ and a nonnegative locally integrable function $\omega$, we say that $\omega$ is in the Muckenhoupt $A_{p,q}$ class if it satisfies the condition

$$\sup_B \left( \frac{1}{|B|} \int_B \omega(x)^q \, dx \right)^{1/q} \left( \frac{1}{|B|} \int_B \omega(x)^{-p'} \, dx \right)^{1/p'} < \infty.$$

https://doi.org/10.1017/S1446788718000447 Published online by Cambridge University Press

A weight function $\omega$ belongs to the class $A_{1,q}$ if there exists $C > 0$ such that, for every ball $B$,

$$\left(\frac{1}{|B|}\int_B \omega(x)^q\,dx\right)^{1/q} \le C \operatorname*{ess\,inf}_{x\in B} \omega(x).$$

Now we return to our first subject.

THEOREM 1.1. *Let* $0 < \alpha < n$, $1 \le p < q < \infty$ *with* $1/q = 1/p - \alpha/n$, $\omega \in A_{p,q}$ *and* $\nu \in A_\infty$. *The following statements are equivalent.*

(a1) $f \in BMO$.

(a2) *There exists a constant* $C > 0$ *such that*

$$\|f\|_{BMO^*} := \sup_B \frac{|B|^{\alpha/n}}{\omega^p(B)^{1/p}}\left(\int_B |f(x) - f_{\nu,B}|^q \omega(x)^q\,dx\right)^{1/q} \le C,$$

*where* $\omega^p(B) = \int_B \omega(x)^p\,dx$.

*Moreover, the norm* $\|\cdot\|_{BMO^*}$ *is mutually equivalent to* $\|\cdot\|_{BMO}$.

Another subject of this paper is to consider the characterizations of Lipschitz functions. For $0 < \beta < 1$, the Lipschitz space $Lip_\beta$ is the set of functions $f$ such that

$$\|f\|_{Lip_\beta} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

It is well known that

$$\|f\|_{Lip_\beta} \approx \sup_B \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q\,dx\right)^{1/q}.$$

The equivalence can be found in [5, pages 14 and 38] for $q = 1$ and in [10] for $1 < q < \infty$. Recently, we showed that the result holds for $0 < q < 1$ in [16].

In this paper, we characterize Lipschitz spaces by $A_{p,q}$ weights as follows.

THEOREM 1.2. *Let* $0 < \beta < 1$, $1 \le p < q < \infty$ *with* $1/q = 1/p - \beta/n$, $\omega \in A_{p,q}$ *and* $\nu \in A_\infty$. *The following statements are equivalent.*

(b1) $f \in Lip_\beta$.

(b2) *There exists a constant* $C > 0$ *such that*

$$\|f\|_{Lip^*_\beta} := \sup_B \frac{1}{\omega^p(B)^{1/p}}\left(\int_B |f(x) - f_{\nu,B}|^q \omega(x)^q\,dx\right)^{1/q} \le C.$$

*Moreover, the norm* $\|\cdot\|_{Lip^*_\beta}$ *is mutually equivalent to* $\|\cdot\|_{Lip_\beta}$.

THEOREM 1.3. *Let* $0 < \beta < 1$, $0 < \alpha < n$, $1 \le p < q < \infty$ *with* $1/q = 1/p - (\alpha+\beta)/n$, $\omega \in A_{p,q}$ *and* $\nu \in A_\infty$. *The following statements are equivalent.*

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(c1) $f \in Lip_\beta$.

(c2) *There exists a constant* $C > 0$ *such that*

$$\|f\|_{Lip_\beta^{**}} := \sup_B \frac{|B|^{\alpha/n}}{\omega^p(B)^{1/p}}\Big(\int_B |f(x) - f_{\nu,B}|^q \omega(x)^q \, dx\Big)^{1/q} \le C.$$

*Moreover, the norm* $\|\cdot\|_{Lip_\beta^{**}}$ *is mutually equivalent to* $\|\cdot\|_{Lip_\beta}$.

There are a number of classical results that demonstrate that *BMO* functions are the right collections for carrying out harmonic analysis on the boundedness of commutators. A well-known result of Coifman *et al.* [3] states that the commutator

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$$

is bounded on some $L^p$, $1 < p < \infty$, if and only if $b \in BMO$, where $T$ is the classical Calderón–Zygmund operator. Chanillo [2] proved that, if $b \in BMO$, the commutator

$$[b, I_\alpha](f)(x) = b(x)I_\alpha(f)(x) - I_\alpha(bf)(x)$$

is bounded from $L^p$ to $L^q$ with $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, where

$$I_\alpha(f)(x) = \int \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Moreover, if $n - \alpha$ is even, the reverse is also valid. Ding [6] showed that $b$ is in BMO if and only if the commutator $[b, T]$ of the Calderón–Zygmund operator $T$ is bounded on Morrey spaces. During the last thirty years, the theory has been extended and generalized in several directions. For instance, Bloom [1] investigated the characterization of *BMO* spaces in the weighted setting.

As an application of Theorems 1.1, 1.2 and 1.3 in this paper, we will study the characterization of *BMO* and Lipschitz spaces in terms of the boundedness of the commutator of some operator on weighted Lebesgue spaces.

THEOREM 1.4. *Let* $0 < \alpha < n$, $1 < p < q < \infty$ *with* $1/q = 1/p - \alpha/n$ *and* $\omega \in A_{p,q}$. *The following statements are equivalent.*

(d1) $b \in BMO$.

(d2) *There exists a constant* $C$ *such that*

$$\|[b, I_\alpha](f)\|_{L^q(\omega^q)} \le C\|f\|_{L^p(\omega^p)}.$$

THEOREM 1.5. *Let* $0 < \beta < 1$, $1 < p < q < \infty$ *with* $1/q = 1/p - \beta/n$ *and* $\omega \in A_{p,q}$. *The following statements are equivalent.*

(e1) $b \in Lip_\beta$.

(e2) *There exists a constant* $C$ *such that*

$$\|[b, T](f)\|_{L^q(\omega^q)} \le C\|f\|_{L^p(\omega^p)}.$$

THEOREM 1.6. *Let* $0 < \beta < 1, 0 < \alpha < n$, $1 < p < q < \infty$ *with* $1/q = 1/p - (\alpha + \beta)/n$ *and* $\omega \in A_{p,q}$. *The following statements are equivalent.*

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(f1) $b \in Lip_\beta$.
(f2) *There exists a constant* $C$ *such that*

$$\|[b, I_\alpha](f)\|_{L^q(\omega^q)} \le C\|f\|_{L^p(\omega^p)}.$$

Throughout this paper, all cubes are assumed to have their sides parallel to the coordinate axes. Given a Lebesgue measurable set $E$, $\chi_E$ will denote the characteristic function of $E$ and $|E|$ is the Lebesgue measure of $E$. The letter $C$ will be used for various constants, and may change from one occurrence to another.

## 2. Proof of Theorems 1.1, 1.2 and 1.3

Proof of Theorem 1.1. $(a1) \Rightarrow (a2)$. In [13], Muckenhoupt and Wheeden proved the John–Nirenberg inequality for $BMO_\nu$. That is, there are two constants $C_1, C_2 > 0$ such that, for any $\lambda > 0$,

$$\nu(\{x \in B : |f(x) - f_{\nu,B}| > \lambda\}) \le C_1 \exp\Big(-\frac{C_2\lambda}{\|f\|_{BMO_\nu}}\Big)\nu(B).$$

Since $\omega \in A_{p,q}$, we have $\mu := \omega^q \in A_q \subset A_\infty$. Then, for any ball $B$ and any measurable set $E$ contained in $B$, there are positive constants $C_0$ and $\epsilon$ such that

$$\frac{\mu(E)}{\mu(B)} \le C_0\Big(\frac{|E|}{|B|}\Big)^{\epsilon}.$$

Since $\nu \in A_\infty$, there exists a constant $N$ such that $\nu \in A_N$. Then

$$\Big(\frac{|E|}{|B|}\Big)^N \le C\frac{\nu(E)}{\nu(B)}.$$

This implies that

$$\frac{\mu(E)}{\mu(B)} \le C_0\Big(\frac{\nu(E)}{\nu(B)}\Big)^{\epsilon/N}$$

and

$$\mu(\{x \in B : |f(x) - f_{\nu,B}| > \lambda\}) \le C \exp\Big(-\frac{C_2\epsilon/N \cdot \lambda}{\|f\|_{BMO_\nu}}\Big)\mu(B).$$

For any ball $B$,

$$\begin{aligned}\|(f - f_{\nu,B})\chi_B\|^q_{L^q(\mu)} &= q\int_0^\infty \lambda^{q-1}\mu(\{x \in B : |f(x) - f_{\nu,B}| > \lambda\})\,d\lambda \\ &\le C\int_0^\infty \lambda^{q-1}\exp\Big(-\frac{C_2\epsilon/N \cdot \lambda}{\|f\|_{BMO_\nu}}\Big)\mu(B)\,d\lambda \\ &\le C\|f\|_{BMO_\nu}\mu(B).\end{aligned}$$

By the Hölder inequality,

$$|Q| \le \Big(\int_Q \omega(x)^p\,dx\Big)^{1/p}\Big(\int_Q \omega(x)^{-p'}\,dx\Big)^{1/p'}.$$

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Then it follows from $\omega \in A_{p,q}$ that

$$\begin{aligned}\frac{\mu(B)^{1/q}|B|^{\alpha/n}}{\omega^p(B)^{1/p}} &\le |B|^{1/p-1/q-1}\Big(\int_B \omega(x)^q\,dx\Big)^{1/q}\Big(\int_B \omega(x)^{-p'}\,dx\Big)^{1/p'}\\ &\le \Big(\frac{1}{|B|}\int_B \omega(x)^q\,dx\Big)^{1/q}\Big(\frac{1}{|B|}\int_B \omega(x)^{-p'}\,dx\Big)^{1/p'}\\ &\le C,\end{aligned}$$

and thus $f \in BMO_\nu$ implies that

$$\frac{|B|^{\alpha/n}}{\omega^p(B)^{1/p}}\Big(\int_B |f(x)-f_{\nu,B}|^q\omega(x)^q\,dx\Big)^{1/q} \le C\|f\|_{BMO_\nu}.$$

That $(a1) \Rightarrow (a2)$ follows from the equivalence of $BMO$ and $BMO_\nu$.

$(a2) \Rightarrow (a1)$. Now we prove that if there exists a constant $C$ such that, for any ball $B$,

$$\frac{1}{\omega^p(B)^{1/p}}\Big(\int_B |f(x)-f_{\nu,B}|^q\omega(x)^q\,dx\Big)^{1/q} \le C|B|^{-\alpha/n},$$

then $f \in BMO$.

When $p > 1$, the Hölder inequality gives us that

$$\begin{aligned}&\int_B |f(x)-f_{\nu,B}|\,dx\\ &\le \Big(\int_B |f(x)-f_{\nu,B}|^p\omega(x)^p\,dx\Big)^{1/p}\Big(\int_B \omega(x)^{-p'}\,dx\Big)^{1/p'}\\ &\le C|B|^{\alpha/n}\Big(\int_B |f(x)-f_{\nu,B}|^q\omega(x)^q\,dx\Big)^{1/q}\Big(\int_B \omega(x)^{-p'}\,dx\Big)^{1/p'}\\ &\le C\|f\|_{BMO^*}\Big(\int_B \omega(x)^{-p'}\,dx\Big)^{1/p'}\Big(\int_B \omega(x)^p\,dx\Big)^{1/p}\\ &\le C\|f\|_{BMO^*}|B|\Big(\frac{1}{|B|}\int_B \omega(x)^{-p'}\,dx\Big)^{1/p'}\Big(\frac{1}{|B|}\int_B \omega(x)^q\,dx\Big)^{1/q}\\ &\le C\|f\|_{BMO^*}|B|.\end{aligned}$$

When $p = 1$,

$$\begin{aligned}\int_B |f(x)-f_{\nu,B}|\,dx &\le \int_B |f(x)-f_{\nu,B}|\omega(x)\,dx \cdot \Big\|\frac{1}{\omega}\chi_B\Big\|_{L^\infty}\\ &\le C\Big(\int_B |f(x)-f_{\nu,B}|^q\omega(x)^q\,dx\Big)^{1/q} \cdot \Big\|\frac{1}{\omega}\chi_B\Big\|_{L^\infty}|B|^{\alpha/n}\\ &\le C\|f\|_{BMO^*}|B|.\end{aligned}$$

We can conclude that $f \in BMO$ from the result of Hart and Torres [8, Theorem 5.2] with $\upsilon \equiv 1$: that is, $f \in BMO$ if and only if

$$\sup_B\Big(\frac{1}{\upsilon(B)}\int_B |f(x)-f_{\nu,B}|^p\upsilon(x)\,dx\Big)^{1/p} < \infty$$

for $\upsilon, \nu \in A_\infty$ and $0 < p < \infty$. □

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Proof of Theorem 1.2. $(b2) \Rightarrow (b1)$. Let $x, y$ be two fixed points. Take $B = B(x, r)$ with $r \le |x-y|$ and $U = B(x, 2|x-y|)$, and define $B_k = B(x, 2^k r)$ for $0 \le k \le \tilde{k}$, where $\tilde{k}$ is the first integer such that $2^{\tilde{k}} r \ge |x-y|$.

Notice that, for any balls, $R_1 = B(x_1, r_1), R_2 = B(x_2, r_2)$ with $R_1 \subset R_2$ and $r_2 \le 2r_1$. When $p > 1$, then $\omega \in A_{p,q}$ and the Hölder inequality shows that

$$\begin{aligned}
&|f_{R_1} - f_{\nu,R_2}|\\
&\quad \le \frac{1}{|R_1|}\int_{R_1} |f(z) - f_{\nu,R_2}|\,dz\\
&\quad \le \frac{C}{|R_2|}\Big(\int_{R_2} |f(z) - f_{\nu,R_2}|^p \omega(z)^p\,dz\Big)^{1/p}\Big(\int_{R_2} \omega(z)^{-p'}\,dz\Big)^{1/p'}\\
&\quad \le \frac{C}{|R_2|^{1-\beta/n}}\Big(\int_{R_2} |f(z) - f_{\nu,R_2}|^q \omega(z)^q\,dz\Big)^{1/q}\Big(\int_{R_2} \omega(z)^{-p'}\,dz\Big)^{1/p'}\\
&\quad \le \frac{C\|f\|_{Lip^*_\beta}}{|R_2|^{1-\beta/n}}\Big(\int_{R_2} \omega(x)^p\,dx\Big)^{1/p}\Big(\int_{R_2} \omega(z)^{-p'}\,dz\Big)^{1/p'}\\
&\quad \le C\|f\|_{Lip^*_\beta} r_1^\beta.
\end{aligned}$$

When $p = 1$,

$$\begin{aligned}
|f_{R_1} - f_{\nu,R_2}| &\le \frac{1}{|R_1|}\int_{R_1} |f(z) - f_{\nu,R_2}|\,dz\\
&\le \frac{C}{|R_2|}\int_{R_2} |f(z) - f_{\nu,R_2}|\omega(z)\,dz \cdot \Big\|\frac{1}{\omega}\chi_{R_2}\Big\|_{L^\infty}\\
&\le \frac{C}{|R_2|^{1-\beta/n}}\Big(\int_{R_2} |f(z) - f_{\nu,R_2}|^q \omega(z)^q\,dz\Big)^{1/q} \cdot \Big\|\frac{1}{\omega}\chi_{R_2}\Big\|_{L^\infty}\\
&\le \frac{C\|f\|_{Lip^*_\beta}}{|R_2|^{1-\beta/n}}\int_{R_2} \omega(x)\,dx \cdot \Big\|\frac{1}{\omega}\chi_{R_2}\Big\|_{L^\infty}\\
&\le C\|f\|_{Lip^*_\beta}|R_2|^{\beta/n}\Big(\frac{1}{|R_2|}\int_{R_2} \omega(x)^q\,dx\Big)^{1/q} \cdot \Big\|\frac{1}{\omega}\chi_{R_2}\Big\|_{L^\infty}\\
&\le C\|f\|_{Lip^*_\beta} r_1^\beta.
\end{aligned}$$

By the same argument as for $|f_{R_1} - f_{\nu,R_2}|$, we also have

$$|f_{R_2} - f_{\nu,R_2}| \le C\|f\|_{Lip^*_\beta} r_1^\beta,$$

which implies that

$$|f_{R_1} - f_{R_2}| \le |f_{R_1} - f_{\nu,R_2}| + |f_{\nu,R_2} - f_{R_2}| \le C\|f\|_{Lip^*_\beta} r_1^\beta.$$

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This shows that

$$|f_B - f_U| \le \sum_{k=0}^{\tilde{k}-1} |f_{B_k} - f_{B_{k+1}}| + |f_{B_{\tilde{k}}} - f_U|$$
$$\le C\|f\|_{Lip^*_\beta} \sum_{k=0}^{\tilde{k}-1} (2^k r)^{\beta/n} \le C\|f\|_{Lip^*_\beta} |x-y|^\beta.$$

A similar argument can be made for the point $y$ with $B' = B(x, r')$ and $V = B(y, 3|x-y|)$. We conclude that

$$|f_B - f_{B'}| \le |f_B - f_U| + |f_U - f_V| + |f_V - f_{B'}| \le C\|f\|_{Lip^*_\beta} |x-y|^\beta.$$

Consider $x, y$ as in the Lebesgue difference theorem: that is,

$$\lim_{r_j \to 0} \frac{1}{|B(x, r_j)|} \int_{B(x,r_j)} f(z)\, dz = f(x).$$

Let $B_j = B(x, r_j), B'_j = B(y, r'_j)$ with $j \ge 1$ be two sequence balls with $r_j, r'_j \to 0 (j \to \infty)$. We obtain

$$|f(x) - f(y)| \le \lim_{j\to\infty} |f_{B_j} - f_{B'_j}| \le C\|f\|_{Lip^*_\beta} |x-y|^\beta.$$

$(b1) \Rightarrow (b2)$. For any ball $B = B(x_0, r)$,

$$\frac{1}{\omega^p(B)^{1/p}} \Big( \int_B |f(z) - f_{\nu,B}|^q \omega(z)^q\, dz \Big)^{1/q}$$
$$\le \frac{1}{\omega^p(B)^{1/p}} \Big( \int_B \Big( \frac{1}{\nu(B)} \int_B |f(z) - f(z')|\nu(z')\, dz' \Big)^q \omega(z)^q\, dz \Big)^{1/q}$$
$$\le C\|f\|_{Lip_\beta} |B|^{\beta/n-1} \omega^q(B)^{1/q} \omega^{-p'}(B)^{1/p'}$$
$$\le C\|f\|_{Lip_\beta}.$$

Which implies that

$$\frac{1}{\omega^p(B)^{1/p}} \Big( \int_B |f(z) - f_{\nu,B}|^q \omega(z)^q\, dz \Big)^{1/q} \le C\|f\|_{Lip_\beta}.$$

We complete the proof of Theorem 1.2. □

Proof of Theorem 1.3. $(c2) \Rightarrow (c1)$. We modify the proof of Theorem 1.2. We need only to check the estimates

$$|f_{R_1} - f_{\nu,R_2}|$$
$$\le \frac{1}{|R_1|} \int_{R_1} |f(z) - f_{\nu,R_2}|\, dz$$
$$\le \frac{C}{|R_2|^{1-(\alpha+\beta)/n}} \Big( \int_{R_2} |f(z) - f_{\nu,R_2}|^q \omega(z)^q\, dz \Big)^{1/q} \Big( \int_{R_2} \omega(z)^{-p'}\, dz \Big)^{1/p'}$$
$$\le \frac{C\|f\|_{Lip^{**}_\beta}}{|R_2|^{1-\beta/n}} \Big( \int_{R_2} \omega(x)^p\, dx \Big)^{1/p} \Big( \int_{R_2} \omega(z)^{-p'}\, dz \Big)^{1/p'}$$
$$\le C\|f\|_{Lip^{**}_\beta} r_1^\beta$$

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and

$$\begin{aligned}
&|f_{R_1} - f_{\nu,R_2}| \\
&\quad \le \frac{C}{|R_2|^{1-(\alpha+\beta)/n}}\Big(\int_{R_2}|f(z)-f_{\nu,R_2}|^q\omega(z)^q\,dz\Big)^{1/q}\cdot\Big\|\frac{1}{\omega}\chi_{R_2}\Big\|_{L^\infty} \\
&\quad \le \frac{C\|f\|_{Lip^{**}_\beta}}{|R_2|^{1-\beta/n}}\int_{R_2}\omega(x)\,dx\cdot\Big\|\frac{1}{\omega}\chi_{R_2}\Big\|_{L^\infty} \\
&\quad \le C\|f\|_{Lip^{**}_\beta} r_1^\beta.
\end{aligned}$$

$(c1) \Rightarrow (c2)$. For any ball $B = B(x_0, r)$,

$$\begin{aligned}
&\frac{1}{\omega^p(B)^{1/p}}\Big(\int_B|f(z)-f_{\nu,B}|^q\omega(z)^q\,dz\Big)^{1/q} \\
&\quad \le \frac{1}{\omega^p(B)^{1/p}}\Big(\int_B\Big(\frac{1}{\nu(B)}\int_B|f(z)-f(z')|\nu(z')\,dz'\Big)^q\omega(z)^q\,dz\Big)^{1/q} \\
&\quad \le C\|f\|_{Lip_\beta}\frac{|B|^{\beta/n}\omega^q(B)^{1/q}}{\omega^p(B)^{1/p}} \\
&\quad \le C\|f\|_{Lip_\beta}|B|^{-\alpha/n}.
\end{aligned}$$

This implies that

$$\frac{|B|^{\alpha/n}}{\omega^p(B)^{1/p}}\Big(\int_B|f(z)-f_{\nu,B}|^q\omega(z)^q\,dz\Big)^{1/q} \le C\|f\|_{Lip_\beta}.$$

The proof of Theorem 1.3 is complete. □

## 3. Proof of Theorems 1.4, 1.5 and 1.6

PROOF OF THEOREM 1.4. $(d2) \Rightarrow (d1)$. For any point $z_0 \neq 0$, let $\delta = (|z_0|/2\sqrt{n})$ and $Q_0(z_0, \delta)$ denote the open cube centered at $z_0$ with side length $2\delta$. Then, for $x \in Q_0(z_0, \delta)$, $|x|^{n-\alpha}$ has an absolutely convergent Fourier series

$$|x|^{n-\alpha} = \sum a_m e^{i\nu_m\cdot x}$$

with $\sum |a_m| < \infty$, where the exact form of the vectors $\nu_m$ is unrelated. Taking $z_1 = (z_0/\delta)$, we have the expansion

$$|x|^{n-\alpha} = \delta^{-n+\alpha}|\delta x|^{n-\alpha} = \delta^{-n+\alpha}\sum a_m e^{i\nu_m\cdot\delta x} \quad \text{for } |x - z_1| < \sqrt{n}.$$

Given cubes $Q = Q(x_0, r)$ and $Q' = Q(x_0 - rz_1, r)$, if $x \in Q$ and $y \in Q'$, then

$$\Big|\frac{x-y}{r} - \frac{z_0}{\delta}\Big| \le \Big|\frac{x-x_0}{r}\Big| + \Big|\frac{y-(x_0-(rz_0/\delta))}{r}\Big| < \sqrt{n}.$$

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This gives

$$\begin{aligned}
b(x) - b_{Q'} &= \frac{1}{|Q'|}\int_{Q'} (b(x) - b(y))\,dy \\
&= \frac{1}{|Q'|}\int_{Q'} \frac{r^{n-\alpha}(b(x) - b(y))}{|x-y|^{n-\alpha}}\left|\frac{x-y}{r}\right|^{n-\alpha} dy \\
&= |Q|^{-\alpha/n}\int_{Q'} \frac{(b(x) - b(y))}{|x-y|^{n-\alpha}} \sum a_m e^{iv_m\cdot\delta(x-y/r)}\,dy \\
&= |Q|^{-\alpha/n}\int_{Q'} \frac{(b(x) - b(y))}{|x-y|^{n-\alpha}} \sum a_m e^{iv_m\cdot\delta(x/r)} e^{-iv_m\cdot\delta(y/r)}\,dy.
\end{aligned}$$

Set

$$\begin{aligned}
f_m(y) &= e^{-iv_m\cdot(\delta/r)y}\chi_{Q'}(y), \\
g_m(x) &= |Q|^{-\alpha/n} e^{iv_m\cdot(\delta/r)x}\chi_Q(x).
\end{aligned}$$

Then, for any cube $Q$ and $x \in Q$,

$$\begin{aligned}
|(b(x) - b_Q)\chi_Q(x)| &= \left|\sum_m a_m [b, I_\alpha](f_m)(x) g_m(x)\right| \\
&\leq |Q|^{-\alpha/n}\sum_m |a_m||[b, I_\alpha](f_m)(x)|.
\end{aligned}$$

From the boundedness of $[b, I_\alpha]$ from $L^p(\omega^p)$ to $L^q(\omega^q)$ for $\omega \in A_{p,q}$, it follows that

$$\begin{aligned}
\frac{|Q|^{\alpha/n}}{\omega^p(Q)^{1/p}}\left(\int_Q |b(x) - b_Q|^q \omega(x)^q\,dx\right)^{1/q} &\leq C\sum_m |a_m| \frac{\|[b, I_\alpha](f_m)\|_{L^q(\omega^q)}}{\omega^p(Q)^{1/p}} \\
&\leq C\sum_m |a_m| \frac{\omega^p(Q')^{1/p}}{\omega^p(Q)^{1/p}}.
\end{aligned}$$

By the definitions of $Q$ and $Q'$, there exists a constant $c_0 = c_0(|z_0|, n)$ such that $Q' \subset c_0 Q$. Then

$$\left(\frac{\mu(Q')}{\mu(Q)}\right)^{1/p} \leq C\left(\frac{\mu(Q')}{\mu(c_0 Q)}\right)^{1/p} \leq C\left(\frac{|Q'|}{|c_0 Q|}\right)^{\epsilon/p} \leq C,$$

where $\mu := \omega^p \in A_\infty$. For any $Q$,

$$\frac{|Q|^{\alpha/n}}{\omega^p(Q)^{1/p}}\left(\int_Q |b(x) - b_Q|^q \omega(x)^q\,dx\right)^{1/q} \leq C.$$

We obtain that $b \in BMO$ by Theorem 1.1.

The proof of $(d1) \Rightarrow (d2)$ follows from [4]. Theorem 1.4 is proved. □

The proofs of Theorems 1.5 and 1.6 use very similar arguments, and hence we omit the details.

https://doi.org/10.1017/S1446788718000447 Published online by Cambridge University Press

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DINGHUAI WANG, School of Mathematical Sciences,
Nanjing Normal University, Nanjing 210023,
China
e-mail: Wangdh1990@126.com

JIANG ZHOU, College of Mathematics and System Sciences,
Xinjiang University, Urumqi 830046,
China
e-mail: zhoujiang@xju.edu.cn

ZHIDONG TENG, College of Mathematics and System Sciences,
Xinjiang University, Urumqi 830046,
China
e-mail: zhidong1960@163.com