ENDOMORPHISM RINGS OF QUASI-INJECTIVE MODULES

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Y. Utumi (**14** and **15**) obtained some interesting results on self-injective rings. He showed that, if R is right self-injective, then so is R/J, where J is the Jacobson radical of R. Also, if R is right self-injective and regular, then R is left self-injective $\Leftrightarrow$ for any set of orthogonal idempotents $\{e_i\}$, $\Pi_i\, e_i R$ is an essential extension of $\sum_i \oplus e_i R$. This note extends these results to endomorphism rings of quasi-injective modules.

Let R be a ring with identity 1, M_R a unital right R-module. M_R is called injective if it is a direct summand of every R-module containing it. The following are well-known properties of injective modules (see **1**, Chapter 1).

(I.1) A direct summand of an injective module is injective.

(I.2) A finite direct sum of injective modules is injective.

(I.3) If $A_R \subseteq B_R$, and $f\colon A \to M$, where M is injective, then f extends to a map $f'\colon B \to M$.

(I.4) If for any right ideal I of R and map $f\colon I \to M$, there is an $m \in M$ such that $f(x) = mx$ for all $x \in I$, then M is injective.

M_R is called an essential extension of N_R, written $M \,'\!\supseteq N$ or $N \subseteq' M$, if $M \supseteq N$ and for all $K_R \subseteq M$, $K \cap N = 0 \Leftrightarrow K = 0$. Every M_R has an injective essential extension called the injective hull of M. Every injective module containing M contains an isomorphic copy of this injective hull. (See Eckmann and Schopf **2**.)

N_R is called closed in M_R if N_R has no essential extension in M_R. It is called complemented in M_R if there exists $B_R \subseteq M$ such that N is a maximal element in the set of all submodules of M which have zero intersection with B.

M_R is called quasi-injective if, for all $N_R \subseteq M_R$ and all $f\colon N \to M$, f extends to a map from M to M. This condition is weaker than injectivity. For example, any simple module is quasi-injective but not necessarily injective.

LEMMA 1. *Let $N_R \subseteq M_R$. Then N is closed in $M \Leftrightarrow N$ is complemented in M.*

Proof. This is an immediate consequence of Proposition 1.7 of Miyashita (**9**).

LEMMA 2. *Any closed submodule of a quasi-injective module M is quasi-injective and a direct summand of M.*

Proof. This is Proposition 4.3 of (**9**).

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COROLLARY 3. *Let N_R be a closed submodule of the quasi-injective module M_R. Let $P \subseteq M$, f: $P \to N$. Then f extends to a map from M to N.*

Proof. Extend f to a map from M to M and follow this by a projection onto the direct summand N.

LEMMA 4. *Let M be quasi-injective, $N \subseteq M$. Then there is a direct summand P of M such that $P\,'\supseteq N$.*

Proof. By Zorn's lemma, one can find a maximal essential extension P of N in M. Since $K\,'\supseteq P\,'\supseteq N$ implies $K\,'\supseteq N$, P is closed. By Lemma 2, P is a direct summand of M.

If M, N, and P are as in Lemma 4, P will be called a quasi-injective closure of N in M. Unlike the injective hull of N, quasi-injective closures of N in distinct quasi-injective modules need not be isomorphic. However, two quasi-injective closures P and P' of N in M must be isomorphic. For the identity of N extends to a map f from P to P' whose kernel has zero intersection with N. Since $P\,'\supseteq N$, f is a monomorphism. By (**9**, Proposition 4.4), $P \approx P'$.

Property I.2 for injective modules does not hold for quasi-injective modules. However, the following weaker version is sufficient for our purpose.

LEMMA 5. *Let M be quasi-injective, $M = A \oplus B = C \oplus D$ where $A \cap C = 0$. Then the projection of C on B is a direct summand of B, and $A \oplus C$ is a direct summand of M.*

Proof. Let π project M on B with kernel A. Then π restricted to C is a monomorphism. Let σ: $\pi\ (C) \to C$ be its inverse. σ extends to a map: $B \to C$. Then $\sigma\pi$ is the identity of C, so $B = \pi(C) \oplus$ kernel σ. We show that $M = A \oplus C \oplus$ kernel σ.

Let $x \in M$. Then $x = a + \pi(c) + d$ for some $a \in A$, $c \in C$, $d \in$ kernel σ. Moreover, $\pi(c) = c - a'$ for some $a' \in A$. Then

$$x = (a - a') + c + d \in A + C + \text{kernel } \sigma,$$

and therefore $A + C +$ kernel $\sigma = M$.

Let $a + c + d = 0$, $a \in A$, $c \in C$, $d \in$ kernel σ. Then $c = -a - d$, thus $\pi(c) = -d \in \pi(C) \cap$ kernel $\sigma = 0$. Thus $a + c = 0$, and therefore a and c are in $A \cap C = 0$. We conclude that the sum is direct.

A ring R is called regular if every finitely generated right ideal is generated by an idempotent. Every finitely generated left ideal of a regular ring is also generated by an idempotent.

We note that, for any module M_R, $N_R \subseteq M$ is a direct summand of M if and only if there is an idempotent $e = e^2 \in \text{Hom}_R(M, M)$ such that $N = eM$.

In the following, M will denote a quasi-injective R-module; $\Lambda = \text{Hom}_R(M_R, M_R)$; e, f, and g will denote idempotents in Λ; $J =$ the Jacobson radical of Λ; $\bar{\Lambda} = \Lambda/J$; and for $x \in \Lambda$, $\bar{x}$ will denote its image in $\bar{\Lambda}$. If $N_R \subseteq P_R$, where P is a direct summand of M, then $E(N)$ will denote any quasi-injective closure of

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N in P. By the above, N is a closed quasi-injective submodule of $M \Leftrightarrow N = eM$ for some $e = e^2 \in \Lambda$.

References to (**14**) indicate that Utumi's proof goes through with very minor changes.

LEMMA 6. *Let* $\bar{x} = \bar{x}^2 \in \bar{\Lambda}$. *Then there exists an* $e = e^2 \in \Lambda$ *such that* $\bar{x} = \bar{e}$.

Proof. See (**7**), or (**14**, Corollary 3.2).

We will henceforth write idempotents in $\bar{\Lambda}$ as $\bar{e}$, $\bar{f}$, or $\bar{g}$. Of course, the lifting idempotent need not be a unique idempotent of Λ.

LEMMA 7. $J = \{x \in \Lambda | \text{ kernel } x \subseteq' M\}$, *and* $\bar{\Lambda}$ *is regular.*

Proof. See Utumi (**11**, or **14**, Lemma 4.1).

1. Right self-injectivity of $\bar{\Lambda}$. It is well known that Λ need not be a right self-injective ring. For example, the p-adic integers equal $\mathrm{Hom}_Z(Z_{p^\infty}, Z_{p^\infty})$ are not self-injective. However, if $J = 0$, then Λ_Λ must be injective. (See Johnson and Wong (**6**).) Moreover, if $M = R$, $\Lambda = R$, then M is injective by I.4 and $\bar{\Lambda}_{\bar{\Lambda}}$ is injective by Utumi's theorem. In this section we show that Utumi's result is true in general; that is, the quasi-injectivity of M implies injectivity of $\bar{\Lambda}_{\bar{\Lambda}}$.

LEMMA 8. *Let* $\bar{e}\bar{\Lambda} \cap \bar{f}\bar{\Lambda} = 0$. *Then there exists a* $g = g^2 \in \Lambda$ *such that* $f\Lambda = g\Lambda$ *and* $\bar{g}\bar{e} = 0$.

Proof. Since $\bar{\Lambda}$ is regular by Lemma 7, $\bar{\Lambda} = \bar{e}\bar{\Lambda} \oplus \bar{f}\bar{\Lambda} \oplus \bar{I}$. Let $\bar{f}\bar{\Lambda} = \bar{h}\bar{\Lambda}$, $\bar{e}\bar{\Lambda} \oplus \bar{I} = (\bar{1} - \bar{h})\bar{\Lambda}$ for $h = h^2 \in \Lambda$. Then $\bar{h}\bar{f} = \bar{f}$, $\bar{f}\bar{h} = \bar{h}$, and

$$\bar{h} = \bar{h}\bar{f} + \bar{h}(\bar{1} - \bar{f}) = \bar{f} + \bar{f}\bar{h}(\bar{1} - \bar{f}).$$

Set $g = f + fh(1 - f)$. Then $g\Lambda = f\Lambda$, $g^2 = g$, and $\bar{g} = \bar{h}$, thus

$$\bar{e} = (\bar{1} - \bar{h})\bar{e} + \bar{g}\bar{e} \Rightarrow \bar{g}\bar{e} = 0.$$

LEMMA 9. $N \subseteq' M \Rightarrow eN \subseteq' eM$ *for all* $e = e^2 \in \Lambda$.

Proof. This is (**14**, Lemma 2.2) with right ideals replaced by submodules of M. Let $eM \supseteq K$, $K \cap eN = 0$. Let $y \in K \cap N$. Then $y = ey$ since $K \subseteq eM$, and therefore $y \in eN \cap K = 0$. Hence $K = 0$ since $N \subseteq' M$.

LEMMA 10. *Let* $\bar{e}, \bar{f} \in \bar{\Lambda}$, $\bar{e}\bar{\Lambda} \cap \bar{f}\bar{\Lambda} = 0$. *Then* $eM \cap fM = 0$, $e\Lambda \cap f\Lambda = 0$, *and* $e\Lambda + f\Lambda = g\Lambda$ *for some* $g = g^2 \in \Lambda$.

Proof. By Lemma 8, we may assume that $f\bar{e} = 0$. Then $fe \in J$. By Lemma 7, $N = \text{kernel} fe \subseteq' M$. Then $eN \cap fM = 0$, so by Lemma 9, $eM \cap fM = 0$. (See **14**, Lemma 3.4.) Since $x \in e\Lambda \cap f\Lambda$ takes M to $eM \cap fM$, $e\Lambda \cap f\Lambda = 0$. Moreover, $eM \oplus fM$ is a direct summand of M by Lemma 5. Hence there is a $g = g^2 \in \Lambda$ such that $eM \oplus fM = gM$. $ge = e$ and $gf = f$, so $g\Lambda \supseteq e\Lambda + f\Lambda$. Let π_1 project M on eM with kernel $fM + (1 - g)M$, π_2 project M on fM with kernel $eM + (1 - g)M$. Then $g = e\pi_1 + f\pi_2 \in e\Lambda + f\Lambda$. Thus

$$e\Lambda + f\Lambda = g\Lambda.$$

LEMMA 11. *Let* $\{e_i | i \in \mathscr{I}\}$ *be idempotents in* Λ *such that* $\sum \bar{e}_i\bar{\Lambda}$ *is direct. Then* $\sum e_i\Lambda$ *is direct,* $\sum e_iM$ *is direct, and if* $\mathscr{I}$ *is finite,* $\sum e_i\Lambda = e\Lambda$ *for some* $e = e^2 \in \Lambda$, *where* $eM = \sum \oplus e_iM$.

Proof. Since independence of modules depends only on finite sums, it suffices to consider only the case when $\mathscr{I}$ is finite. If $\mathscr{I}$ has only one element, the lemma is true. Now assume the lemma for all sets of $n-1$ idempotents e_i. Then if $\sum_{i=1}^{n} \bar{e}_i\bar{\Lambda}$ is direct,

$$\sum_{i=1}^{n-1} e_i\Lambda = e'\Lambda, \qquad e'M = \sum_{i=1}^{n-1} e_iM,$$

and the sums are direct. Since

$$\sum_{i=1}^{n} \bar{e}_i\bar{\Lambda} = \bar{e}'\bar{\Lambda} \oplus \bar{e}_n\bar{\Lambda},$$

by Lemma 10, $e'\Lambda \cap e_n\Lambda = 0$, and

$$\sum_{i=1}^{n} \oplus e_i\Lambda = e'\Lambda \oplus e_n\Lambda = e\Lambda$$

for some $e = e^2 \in \Lambda$, where

$$eM = e'M \oplus e_nM = \sum_{i=1}^{n} \oplus e_iM.$$

THEOREM 12. $\bar{\Lambda}_{\bar{\Lambda}}$ *is injective.*

Proof. Let I be a right ideal of $\bar{\Lambda}$, $f: I \to \bar{\Lambda}$. Let $\{\bar{e}_i | i \in \mathscr{I}\}$ be a maximal set of idempotents in I such that $\sum \bar{e}_i\bar{\Lambda}$ is direct. Then $I' \supseteq \sum_{i \in \mathscr{I}} \bar{e}_i\bar{\Lambda}$, since $\bar{x}\bar{\Lambda} \cap \sum_{i\in\mathscr{I}} \bar{e}_i\bar{\Lambda} = 0$ implies $\bar{x}\bar{\Lambda} + \sum_{i\in\mathscr{I}}\bar{e}_i\bar{\Lambda}$ is a direct sum. Since $\bar{\Lambda}$ is regular by Lemma 7, $\bar{x}\bar{\Lambda} = \bar{f}\bar{\Lambda}$ for some $\bar{f} = \bar{f}^2 \in \bar{\Lambda}$. Then $\bar{f} = 0$ by the maximality of $\{\bar{e}_i | i \in \mathscr{I}\}$.

By Lemma 11, $\sum_{i\in\mathscr{I}} e_i\Lambda$ is direct. Hence we may define a Λ-homomorphism $\phi: \sum_{i\in\mathscr{I}} e_i\Lambda \to \Lambda$ by $\phi(e_i) = x_ie_i$ for all $i \in \mathscr{I}$, where x_i is any element in Λ such that $f(\bar{e}_i) = \bar{x}_i\bar{e}_i$.

Define $\psi: \sum_{i\in\mathscr{I}} e_iM \to M$ by

$$\psi\left(\sum_{j=1}^{m} e_jm_j\right) = \sum_{j=1}^{m} \phi(e_j)m_j.$$

ψ is an R-homomorphism if it is well-defined. Let $\sum_{j=1}^{m} e_jm_j = 0$. By Lemma 11, $\sum_{j=1}^{m} e_j\Lambda = e\Lambda$ for some $e = e^2 \in \Lambda$. Then $e_j = ee_j$, and

$$\psi\left(\sum_{j=1}^{m} e_jm_j\right) = \sum_{j=1}^{m} \phi(e_j)m_j = \sum_{j=1}^{m} \phi(e)e_jm_j$$

$$= \phi(e) \sum_{j=1}^{m} e_jm_j = 0,$$

thus ψ is well-defined. Since M is quasi-injective, ψ extends to an element $\lambda \in \Lambda$. Then for all $u \in M$,

$$\lambda e_i(u) = \lambda(e_i(u)) = \psi(e_i(u)) = \phi(e_i)(u) = x_i e_i(u).$$

Then $\bar{\lambda}\bar{e}_i = \bar{x}_i\bar{e}_i = f(\bar{e}_i)$ for all $i \in \mathscr{I}$, and $(\bar{\lambda} - f)\sum_{i\in\mathscr{I}}\bar{e}_i\bar{\Lambda} = 0$. Let $\bar{x} \in I$, and set $K = \{\bar{z} \in \bar{\Lambda} | \bar{x}\bar{z} \in \sum_{i\in\mathscr{I}}\bar{e}_i\bar{\Lambda}\}$. Then $(\bar{\lambda}\bar{x} - f(\bar{x}))K = 0$. Since

$$I' \supseteq \sum_{i\in\mathscr{I}}\bar{e}_i\bar{\Lambda}, \qquad \bar{\Lambda}' \supseteq K.$$

Let $\bar{\Lambda}(\bar{\lambda}\bar{x} - f(\bar{x})) = \bar{\Lambda}\bar{e}$, where $\bar{e} = \bar{e}^2$. Then $\bar{e}\bar{\Lambda} \cap K = 0$, so $\bar{e}\bar{\Lambda} = 0$ and $f(\bar{x}) = \bar{\lambda}\bar{x}$ for all $\bar{x} \in I$. Then $\bar{\Lambda}_{\bar{\Lambda}}$ is injective by I.4.

The argument used in proving Theorem 12 is basically that of Johnson and Wong; see (**6**).

We have as immediate corollaries the theorems whose proofs were modified to prove Theorem 1.

COROLLARY 13 (Utumi). *Let R be a right self-injective ring. Then $R/J(R)$ is a right self-injective ring.*

COROLLARY 14 (Johnson-Wong). *Let M_R be quasi-injective, and $\Lambda = \mathrm{Hom}_R(M_R, M_R)$ regular. Then Λ_Λ is injective.*

As in Utumi's work, we can show that an arbitrary set of orthogonal idempotents in $\bar{\Lambda}$ lifts orthogonally to Λ. Indeed, the following lemma and theorem are a minor modification of (**14**, Theorem 4.9).

LEMMA 15. *Let $\sum\bar{e}_i\bar{\Lambda}$ be direct, and $eM' \supseteq \sum e_iM$. Then $\bar{e}\bar{\Lambda}' \supseteq \sum\bar{e}_i\bar{\Lambda}$.*

Proof. Since $\bar{\Lambda}$ is regular, we need only show any non-zero right ideal generated by an idempotent intersects $\sum\bar{e}_i\bar{\Lambda}$ non-trivially. Let $\bar{f} \in \bar{e}\bar{\Lambda}$, $\bar{f}\bar{\Lambda} \cap \sum\bar{e}_i\bar{\Lambda} = 0$. By Lemma 11, $fM \oplus \sum e_iM$ is a direct sum. Since $ef - f \in J$, by Lemma 7 there is an $N \subseteq' M$ such that $(ef - f)N = 0$. Hence $fN \subseteq eM$. Since $eM' \supseteq \sum e_iM$, $fN' \supseteq fN \cap \sum e_iM = 0$. Thus $fN = 0$. By Lemma 9, $fM' \supseteq fN = 0$, so $fM = 0$. Then $\bar{f} = 0$ and $\bar{e}\bar{\Lambda}' \supseteq \sum\bar{e}_i\bar{\Lambda}$.

THEOREM 16. *Let $\{\bar{e}_i | i \in \mathscr{I}\}$ be a set of orthogonal idempotents in $\bar{\Lambda}$. Then there exists $\{f_i | i \in \mathscr{I}\} \subseteq \Lambda$ of orthogonal idempotents such that $\bar{f}_i = \bar{e}_i$.*

Proof. By Lemma 11, $\sum e_iM$ is direct. Let $eM = E(\sum e_iM)$. For $i \in \mathscr{I}$, let $g_iM = E(\sum_{j\neq i}e_jM)$ be a quasi-injective closure of $\sum_{j\neq i}e_jM$ in eM. Then $e_iM \cap \sum_{j\neq i} e_jM = 0 \Rightarrow e_iM \cap g_iM = 0$ so

$$M = e_iM \oplus g_iM \oplus (1 - e)M = e_iM \oplus (1 - e_i)M.$$

By Lemma 5,

$$M = e_iM \oplus g_iM \oplus (1 - e_i)(1 - e)M \oplus K,$$

where $K \subseteq (1 - e_i)M$. Since $(1 - e) = e_i(1 - e) + (1 - e_i)(1 - e)$, $K=0$. Now let f_i project M on e_iM with kernel $g_iM \oplus (1 - e_i)(1 - e)M$. Then

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$f_i^2 = f_i$, $e_if_i = f_i$, $f_ie_i = e_i$, and if $i \neq j$, $f_ie_j = 0$ so $f_if_j = f_ie_jf_j = 0$. Thus $\{f_i | i \in \mathscr{I}\}$ are orthogonal idempotents, and $e_i\Lambda = f_i\Lambda$, $\bar{e}_i\bar{\Lambda} = \bar{f}_i\bar{\Lambda}$ for all i. By Lemma 15, $\bar{g}_i\bar{\Lambda}\ ' \supseteq \sum \bar{e}_j\bar{\Lambda}$. Since $\bar{f}_i(\sum_{j\neq i} \bar{e}_j\bar{\Lambda}) = (\sum_{j\neq i} \bar{f}_i\bar{e}_j\bar{\Lambda}) = 0$, we have, as in the proof of Theorem 12, that $\bar{f}_i(\bar{g}_i\bar{\Lambda}) = \bar{e}_i(\bar{g}_i\bar{\Lambda}) = 0$. Moreover, $\bar{f}_i(\bar{1} - \bar{e}_i)(\bar{1} - \bar{e}) = 0$. Hence $(\bar{1} - \bar{f}_i)\bar{\Lambda} \supseteq \bar{g}_i\bar{\Lambda} \oplus (\bar{1} - \bar{e}_i)(\bar{1} - \bar{e})\bar{\Lambda}$. Since

$$\bar{\Lambda} = \bar{e}_i\bar{\Lambda} \oplus \bar{g}_i\bar{\Lambda} \oplus (\bar{1} - \bar{e}_i)(\bar{1} - \bar{e})\bar{\Lambda},$$

$$(\bar{1} - \bar{f}_i)\bar{\Lambda} = (\bar{1} - \bar{e}_i)\bar{\Lambda} = \bar{g}_i\bar{\Lambda} \oplus (\bar{1} - \bar{e}_i)(\bar{1} - \bar{e})\bar{\Lambda}.$$

Then

$$\bar{f}_i = \bar{f}_i\bar{e}_i + \bar{f}_i(\bar{1} - \bar{e}_i) = \bar{f}_i\bar{e}_i = \overline{(f_ie_i)} = \bar{e}_i.$$

2. Left self-injectivity of $\bar{\Lambda}$. Let $\{e_i | i \in \mathscr{I}\}$ be a family of orthogonal idempotents of Λ. Then there exists a map ϕ: $M \to \Pi_{i\in\mathscr{I}}\, e_iM$ given by $\phi(m) = \langle e_im\rangle$. ϕ is an isomorphism between $\sum_{i\in\mathscr{I}} e_iM \subseteq M$ and

$$\textstyle\sum_{i\in\mathscr{I}} \oplus\, e_iM \subseteq \Pi_{i\in\mathscr{I}}\, e_iM.$$

ϕ is then a monomorphism on any essential extension of $\sum e_iM$ in M, in particular on $E(\sum e_iM)$. As in Utumi's work (**12**), we investigate when ϕ is onto.

LEMMA 17. (i) *ϕ is onto $\Leftrightarrow$ for all f: $\sum \Lambda e_i \to {}_\Lambda M$, there exists an $m \in M$ such that $f(\lambda) = \lambda m$ for all $\lambda \in \sum\Lambda e_i$.*

(ii) *ϕ onto $\Rightarrow$ for all f: $\sum\Lambda e_i \to {}_\Lambda\Lambda$, there exists a $\gamma \in \Lambda$ such that $f(\lambda) = \lambda\gamma$ for all $\lambda \in \sum\Lambda e_i$.*

Proof. (i) Let ϕ be onto, and let f: $\sum\Lambda e_i \to M$, $f(e_i) = e_ix_i$. Let $\langle e_ix_i\rangle = \phi(m)$. Then $e_im = e_ix_i = f(e_i)$ for all $i \in \mathscr{I}$, so $\lambda m = f(\lambda)$ for all $\lambda \in \sum\Lambda e_i$.

Conversely, let every f: $\sum\Lambda e_i \to M$ be given by right multiplication by some $m \in M$. Let $\langle e_ix_i\rangle \in \Pi e_iM$. Define f: $\sum\Lambda e_i \to M$ by $f(e_i) = e_ix_i$. Then there exists $m \in M$ with $e_im = e_ix_i$, so $\phi(m) = \langle e_ix_i\rangle$.

(ii) Let f: $\sum\Lambda e_i \to \Lambda$, $f(e_i) = e_i\lambda_i$. Define $\gamma \in \Lambda$ by $\gamma(x) = \phi^{-1}\langle e_i\lambda_ix\rangle$ for all $x \in M$. Then $e_i\gamma(x) = e_i\lambda_i(x)$ for all $x \in M$, $i \in \mathscr{I}$, so $e_i\gamma = e_i\lambda_i = f(e_i)$ for all $i \in \mathscr{I}$.

LEMMA 18. *Let ϕ be onto for every set $\{e_i | i \in \mathscr{I}\}$ of orthogonal idempotents. Let $\{N_j | j \in \mathscr{J}\}$ be a family of independent submodules of M. Then $\Pi N_j\ ' \supseteq \sum N_j$. Moreover, if M in injective, then $\Pi N_j\ ' \subseteq \sum N_j$ for every set of independent submodules implies ϕ is onto for every set of orthogonal idempotents.*

Proof. Let $\{N_j | j \in \mathscr{J}\}$ be a family of independent submodules of M, and let $eM = E(\sum_{j\in\mathscr{J}} E(N_j))$. Let N_j^* be a quasi-injective closure of $\sum_{i\neq j} N_i$ in eM. Let e_j project M on $E(N_j)$ with kernel $N_j^* + (1 - e)M$. Then $\{e_j | j \in \mathscr{J}\}$ is a set of orthogonal idempotents, and $e_je = e_j$ for all j. Let $\langle e_jx_j\rangle \in \Pi e_jM$. Since ϕ is onto, there exists $m \in M$ such that $\phi(m) = \langle e_jx_j\rangle$. Then $\phi(em) = \langle e_jem\rangle = \langle e_jm\rangle$, so ϕ restricts to an isomorphism from eM to $\Pi_{j\in\mathscr{J}}\, e_jM$. Then $\Pi e_j M\ ' \supseteq \sum e_jM$. Since $e_jM\ ' \supseteq N_j$ for all j, $\sum e_jM\ ' \supseteq \sum N_j$. Hence $\sum N_j \subseteq' \Pi N_j \subseteq \Pi e_jM$.

Conversely, assume M is injective. Let $\{e_i | i \in \mathscr{I}\}$ be a set of orthogonal idempotents of Λ. Then $E(\sum e_i M)$ is injective, so $\phi(E(\sum e_i M))$ is a direct summand of $\Pi e_i M$. Thus $\Pi e_i M \,' \supseteq \sum e_i M$ implies ϕ is onto.

LEMMA 19. *Let S be a right self-injective regular ring. Then S is left self-injective $\Leftrightarrow$ for every set $\{e_i | i \in \mathscr{I}\}$ of orthogonal idempotents and $f\colon \sum Se_i \to S$, there is an $m \in S$ such that $f(u) = um$ for all $u \in \sum Se_i$.*

Proof. See Utumi (**15**, § 3).

THEOREM 20. *Let ϕ be onto for every orthogonal set of idempotents $\{e_i | i \in \mathscr{I}\} \subseteq \Lambda$. Then ${}_{\bar\Lambda}\bar\Lambda$ is injective.*

Proof. By Theorem 12, $\bar\Lambda_{\bar\Lambda}$ is injective. Hence we need only show that for any set $\{\bar e_i | i \in \mathscr{I}\}$ of orthogonal idempotents of $\bar\Lambda$, and $f\colon \sum \bar\Lambda \bar e_i \to \bar\Lambda$, there exists $\bar\gamma \in \Lambda$ with $f(\bar e_i) = \bar e_i \bar\gamma$ for all $i \in \mathscr{I}$.

By Theorem 16, we may assume that $\{e_i | i \in \mathscr{I}\}$ are orthogonal idempotents of Λ. Let $f(\bar e_i) = \bar e_i \bar\lambda_i$. By Lemma 17 (ii), there exists $\gamma \in \Lambda$ such that $e_i\gamma = e_i\lambda_i$ for all $i \in \mathscr{I}$. Then $\bar e_i \bar\gamma = \bar e_i \bar\lambda_i = f(\bar e_i)$ for all $i \in \mathscr{I}$. Lemma 19 then shows that ${}_{\bar\Lambda}\bar\Lambda$ is injective.

COROLLARY 21. *If M is injective, then $\Pi N_i \,' \supseteq \sum N_i$ for any independent family of submodules of M implies ${}_{\bar\Lambda}\bar\Lambda$ is injective.*

Proof. Apply Lemma 18 and Theorem 20.

We note that ϕ is not always onto, even if ${}_{\bar\Lambda}\bar\Lambda$ is injective. Let $M_Z = \sum \oplus Z_{p^\infty}$, where the sum is over all primes p. Then M is injective, $\Lambda = \Pi$ p-adic integers is commutative, so ${}_{\bar\Lambda}\bar\Lambda$ is injective, but ΠZ_{p^∞} is not an essential extension of M. This same example shows that the converse to Lemma 17 (ii) is false.

The hypothesis M injective cannot be removed from the second part of Lemma 18. For let R be a direct product of fields ΠF_i, and let $M = \sum F_i$. Then M is quasi-injective, $\Pi F_i \,' \supseteq \sum F_i$, but ϕ is clearly not onto.

3. Singular submodule equal to 0.

For each $x \in M$, $N \subseteq M$, let

$$(N : x)_R = \{r \in R | xr \in N\}, \qquad (N : x)_\Lambda = \{\lambda \in \Lambda | \lambda x \in N\}.$$

$$Z(M_R) = \{x \in M | (0 : x)_R \subseteq' R_R\}, \qquad Z({}_\Lambda M) = \{x \in M | (0 : x)_\Lambda \subseteq' {}_\Lambda\Lambda\}.$$

Then $Z(M_R)$ $(Z({}_\Lambda M))$ is a submodule of M_R $({}_\Lambda M)$.

In this section we will assume that $Z(M_R) = 0$. We can then get nicer forms of the theorems of § 2.

LEMMA 22. *$Z(M_R) = 0$ implies $J = 0$. Hence $\Lambda = \bar\Lambda$.*

Proof. Let $\lambda \in J$. Then kernel $\lambda \subseteq' M$ by Lemma 2. Let $x \in M$. Then $(0 : \lambda x)_R =$ (kernel $\lambda : x)_R \subseteq' R$ so $\lambda x \in Z(M_R) = 0$. Hence $M =$ kernel λ so $\lambda = 0$.

LEMMA 23. *Let* $x \in M$, $E(xR) = eM$. *Then* $\lambda x = 0 \Leftrightarrow \lambda e = 0$ *for all* $\lambda \in \Lambda$.

Proof. If $\lambda e = 0$, then $\lambda(ex) = \lambda x = 0$.

Let $\lambda x = 0$. For $y \in M$, $(xR : ey)_R \subseteq' R_R$ and $\lambda ey(xR : ey)_R = 0$. Hence $\lambda ey \in Z(M_R) = 0$ so $\lambda e = 0$.

COROLLARY 24. $Z({}_\Lambda M) = 0$.

Proof. Let $0 \neq x \in M$, $E(xR) = eM$. Then $(0 : x)_\Lambda \cap \Lambda e = 0$ by Lemma 23.

THEOREM 25. *If* $Z(M_R) = 0$, ${}_\Lambda M$ *is injective* $\Leftrightarrow \phi$ *is onto for all* $\{e_i | i \in \mathscr{I}\}$ *orthogonal idempotents.*

Proof. By Lemma 17, ϕ is onto $\Leftrightarrow$ for $f: \sum \Lambda e_i \to {}_\Lambda M$ there exists $m \in M$ such that $f(\lambda) = \lambda m$ for all $\lambda \in \sum \Lambda e_i$. It is then clear that ${}_\Lambda M$ injective $\Rightarrow \phi$ is onto.

Let ϕ be onto for all $\{e_i | i \in \mathscr{I}\}$, ${}_\Lambda I \subseteq \Lambda$, $f: I \to M$. Let $\{\Lambda x_i\}$ be a maximal set of independent principal subideals of I. By Theorem 20, ${}_\Lambda\Lambda$ is injective. Let Λe be an injective hull of I in Λ. Then $\sum \Lambda x_i + \Lambda(1 - e)$ is a direct sum, and since $\Lambda = \bar{\Lambda}$ is regular, $I' \supseteq \sum \Lambda x_i$ so $\Lambda' \supseteq \sum \Lambda x_i + \Lambda(1-e)$. Let e_i project Λ on Λx_i with kernel the injective hull of $\sum_{j \neq i} \Lambda x_j \oplus \Lambda(1 - e)$. Then $\{e_i\} \cup \{1 - e\}$ is a set of orthogonal idempotents. If we set $f(1 - e) = 0$, f maps $\sum \Lambda e_i + \Lambda(1 - e) \to M$. By Lemma 17, there exists $m \in M$ such that $f(u) - um = 0$ for all $u \in \sum \Lambda e_i + \Lambda(1 - e)$. Let $y \in \Lambda$, and set $\bar{f}(y) = ym$. $\bar{f}: \Lambda \to M$. Let $z \in I$. Then $\Lambda' \supseteq (\sum \Lambda e_i : z)$, and

$$(\sum \Lambda e_i : z)[\bar{f}(z) - f(z)] = 0$$

since f and $\bar{f}$ agree on $\sum \Lambda e_i$. Thus $\bar{f}(z) = f(z)$ by Corollary 24, and ${}_\Lambda M$ is injective by I.4.

Even in the case where $Z(M_R) = 0$, we cannot conclude that ${}_\Lambda\Lambda$ injective implies ${}_\Lambda M$ injective, $M = \sum F_i$, $R = \Pi F_i$ is a counterexample to this. M here is only quasi-injective, not injective. It is unknown whether one can have $Z(M_R) = 0$, M_R injective, and ${}_\Lambda\Lambda$ injective, but not ${}_\Lambda M$ injective. We also note that in the above example, ${}_\Lambda M$ *is* quasi-injective. Whether this is always true is also unknown.

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