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BILINEAR INTEGRATION WITH POSITIVE VECTOR MEASURES

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Abstract

The integration of vector (and operator) valued functions with respect to vector (and operator) valued measures can be simplified by assuming that the measures involved take values in the positive elements of a Banach lattice.

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1. Introduction

Given a countably additive vector measure $m : \mathscr{S} \to L^p(\mu)$ and a strongly m-measurable function $f : \Omega \to X$ with values in a Banach space X, the question of the existence of the integral

$$(1.1) \qquad \int_A f \otimes dm \in L^p(\mu; X), \quad A \in \mathscr{S},$$

arises in numerous contexts. For the case where m is either positive, or dominated by a positive measure, we provide a simple condition involving the scalar function $\|f\| : \Omega \to \mathbb{R}$ which guarantees the existence of the integral (1.1) as well as establishing an upper bound for the norm of the function $\gamma \mapsto \left(\int_A f \otimes dm\right)(\gamma)$ for μ-almost all $\gamma \in \Gamma$ (Theorem 2.2).

We also consider the case where m is replaced by either a positive or dominated operator valued measure $M : \mathscr{S} \to \mathscr{L}(L^p(\mu))$ and f is replaced by an operator valued function $F : \Omega \to \mathscr{L}(X)$. In this setting we provide a simple condition involving the scalar function $\|F\|_{\mathscr{L}(X)} : \Omega \to \mathbb{R}$ which establishes the existence of

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the integral $\int_A F \otimes dM \in \mathscr{L}(L^p(\mu; X))$ as well as providing an upper bound for its value (Theorem 3.8).

Bilinear integration, under various guises, has been investigated by several authors. See for example, [1, 3, 4, 5, 10, 12]. Also, see Remark 3.3 for a short comparison of the bilinear integrals of Jefferies and Okada [9] and Bartle [1] in the operator valued setting.

We establish some basic notation. Throughout $(\Omega, \mathscr{S})$ will denote a measurable space, X a Banach space with norm $\|\cdot\|_X$, and $\mathscr{L}(X)$ will denote the space of bounded linear operators acting on X (equipped with the strong operator topology). For the sake of brevity we drop the term linear and just refer throughout to bounded operators.

We shall always assume that $(\Gamma, \mathscr{E}, \mu)$ is a nonzero σ-finite measure space and that $1 \leq p < \infty$. Let $\mathbb{F}$ denote either the real or complex numbers. The set of all equivalence classes of $\mathbb{F}$-valued functions f on Γ for which $f\,|f\,|^{p-1}$ is μ-integrable is denoted by $L^p(\mathbb{F})$ and is equipped with the usual L^p-norm $\|\cdot\|_{L^p}$. By $L^p(X)$ we denote the vector space of μ-equivalence classes of strongly μ-measurable functions $y : \Gamma \to X$ such that $\|y\|_{L^p(X)} := \left(\int_\Gamma \|y\|_X^p \, d\mu\right)^{1/p}$ is finite. It is a Banach space under the norm $\|\cdot\|_{L^p(X)}$.

Let $y \in L^p(\mathbb{F})$. We write $y \geq 0$ and say that y is *positive* if $y(\gamma) \geq 0$ for μ-almost every $\gamma \in \Gamma$. A bounded operator $S \in \mathscr{L}(L^p(\mathbb{F}))$ is *positive* if $Sy \geq 0$ holds for all $y \geq 0$.

The algebraic tensor product $L^p(\mathbb{F}) \otimes X$ is the set of all finite sums $\sum_{j=1}^k y_j x_j$, where $x_j \in X, y_j \in L^p(\mathbb{F})$ and $k \in \mathbb{N}$. We shall assume that the vector space $L^p(\mathbb{F}) \otimes X$ is endowed with the relative norm of $L^p(X)$. In this case the completion of $L^p(\mathbb{F}) \otimes X$ is given by $L^p(X)$. The bilinear mapping of relevance here is the canonical map $(y, x) \mapsto yx$ for $y \in L^p(\mathbb{F})$ and $x \in X$, defining the algebraic tensor product $L^p(\mathbb{F}) \otimes X$.

A σ-additive set function $m : \mathscr{S} \to L^p(\mathbb{F})$ will be referred to as a *vector measure* while a set function $M : \mathscr{S} \to \mathscr{L}(L^p(\mathbb{F}))$ which is countably additive with respect to the strong operator topology of $\mathscr{L}(L^p(\mathbb{F}))$ will be called an *operator valued measure*. Our definition of a vector measure differs from that of [2] where only finite additivity is assumed. We shall use the notation My to denote the vector measure $My : A \mapsto M(A)y$ for all $A \in \mathscr{S}$. Similarly, for the case of an operator valued function $F : \Omega \to \mathscr{L}(X)$, we use the notation Fx to denote the vector valued function $Fx : \omega \mapsto F(\omega)x$ for all $\omega \in \Omega$.

2. Dominated measures: the vector case

In this section we establish Bochner (that is, scalar) type conditions guaranteeing a vector valued function's integrability with respect to a dominated vector measure

(Theorem 2.2) and a useful estimate (Theorem 2.4).

Throughout this section m and n denote vector measures and $f : \Omega \to X$ is a vector valued function. We use $\|f\|_X$ to denote the scalar function $\|f\|_X : \Omega \to \mathbb{R}$.

We write $m \geq 0$ and say that m is *positive* if $m(A) \geq 0$ for all $A \in \mathcal{S}$. We say that m is *order bounded* if there exists a positive vector measure, u, for which $u(A) \geq |m(A)|$ holds for all $A \in \mathcal{S}$. In the present context, this is equivalent to saying that m has order bounded range in the Banach lattice $L^p(\mathbb{F})$ (see [10, Lemma 4.4.4]; note that the condition that the Banach lattice should have an order continuous norm is missing from the final part of [10, Lemma 4.4.4]) The smallest positive measure satisfying this requirement is denoted by $|m|$. We write $|m| \geq |n|$ and say that m *dominates* n if $|m|(A) \geq |n|(A)$ for all $A \in \mathcal{S}$.

A scalar function $h : \Omega \to \mathbb{F}$ is said to be *strongly m-integrable* if it is integrable with respect to the scalar measure $\langle m, y' \rangle : A \mapsto \langle m(A), y' \rangle$ for every $y' \in L^q(\mathbb{F})$. Here q satisfies $1/p + 1/q = 1$ with the usual convention that $q = \infty$ if $p = 1$. The *integral* of h over A with respect to m, is an element of $L^p(\mathbb{F})$ which we denote by $\int_A h\,dm$ and which satisfies $\left\langle \int_A h\,dm, y' \right\rangle = \int_A h\,d\langle m, y' \rangle$ for all $y' \in L^q(\mathbb{F})$ and $A \in \mathcal{S}$. Every bounded, $\mathcal{S}$-measurable function is strongly m-integrable, [11, Lemma II.3.1]. This notion of integration is developed more generally for locally convex space valued measures by [11].

An *X-valued $\mathcal{S}$-simple function* is a function $g : \Omega \to X$ for which there exist $k \in \mathbb{N}$, sets $E_j \in \mathcal{S}$ and vectors $x_j \in X$, $j = 1, \dots, k$, such that $g = \sum_{j=1}^k x_j \chi_{E_j}$. For the vector measure m define $\int_A g \otimes dm = \sum_{j=1}^k x_j \left[m(A \cap E_j) \right] \in L^p(X)$ for all $A \in \mathcal{S}$. A vector valued function f is *strongly m-measurable* if it is the limit m-almost everywhere of X-valued $\mathcal{S}$-simple functions.

DEFINITION 2.1 ([9, Definition 1.5]). A function f is said to be *m-integrable* in $L^p(X)$ if there exist X-valued $\mathcal{S}$-simple functions f_j, $j \in \mathbb{N}$, such that $f_j \to f$ pointwise m-almost everywhere as $j \to \infty$ and $\left\{ \int_A f_j \otimes dm \right\}_{j=1}^{\infty}$ converges in $L^p(X)$ for each $A \in \mathcal{S}$. Let $\int_A f \otimes dm$ denote this limit.

The above limit is well defined and independent of the approximating sequence [10, Lemma 4.1.4]. The set function $A \to \int_A f \otimes dm$, $A \in \mathcal{S}$, is σ-additive in $L^p(X)$ by the Vitali-Hahn-Saks theorem [2, Theorem I.5.6]. Clearly, the map $(f, m) \mapsto \int f \otimes dm$ is bilinear in the obvious sense. Also, for the case $X = \mathbb{F}$, f is m-integrable in $L^p(\mathbb{F})$ if and only if it is strongly m-integrable [10, Remark after Definition 4.1.5].

THEOREM 2.2. *Suppose that m is order bounded and that n is dominated by m. If the function $f : \Omega \to X$ is strongly m-measurable and $\|f\|_X$ is strongly $|m|$-integrable,*

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then f is n-integrable in $L^p(X)$ and the inequality

$$\left\|\left(\int_A f\otimes dn\right)(\gamma)\right\|_X \le \left(\int_A \|f\|_X d|m|\right)(\gamma)$$

holds for all $A\in\mathcal{S}$ and for μ-almost every $\gamma\in\Gamma$.

PROOF. We begin by considering the special case $n = m \geq 0$. First we establish the estimate for simple functions. Let $l \in \mathbb{N}$. Suppose that $h = \sum_{j=1}^{l} x_j \chi_{E_j}$ is an X-valued $\mathcal{S}$-simple function with $x_j \in X$ and $E_j \in \mathcal{S}$ pairwise disjoint for $j = 1, \dots, l$. Let $A \in \mathcal{S}$. Then for μ-almost every $\gamma \in \Gamma$,

$$\begin{aligned}(2.1)\qquad \left\|\left(\int_A h\otimes dm\right)(\gamma)\right\|_X &= \left\|\sum_{j=1}^{l} x_j(m(E_j\cap A))(\gamma)\right\|_X\\ &\le \sum_{j=1}^{l}\|x_j\|_X(m(E_j\cap A))(\gamma)\\ &= \left(\int_A \|h\|_X\,dm\right)(\gamma).\end{aligned}$$

The positivity of the vector measure is crucial here.

Next we prove that f is m-integrable in $L^p(X)$. By assumption f is strongly m-measurable and so there exists a sequence $\{\psi_j\}_{j=1}^{\infty}$ of X-valued $\mathcal{S}$-simple functions such that $\psi_j \to f$ m-almost everywhere as $j \to \infty$. Now let

$$f_j(\omega) = \begin{cases}\psi_j(\omega) & \text{if } \|\psi_j(\omega)\|_X \le 2\|f(\omega)\|_X\\ 0 & \text{if } \|\psi_j(\omega)\|_X > 2\|f(\omega)\|_X.\end{cases}$$

Then each f_j is an X-valued $\mathcal{S}$-simple function such that $f_j \to f$ m-almost everywhere and further $\|f_j(\omega)\|_X \le 2\|f(\omega)\|_X$ for all $\omega \in \Omega$. Thus to ensure integrability it suffices to show that the sequence $\{\int_A f_j \otimes dm\}_{j=1}^{\infty}$ converges in $L^p(X)$ for each $A \in \mathcal{S}$.

Let $j, k \in \mathbb{N}$. By construction, $\|f_j(\cdot) - f(\cdot)\|_X \le 3\|f(\cdot)\|_X$ and by assumption $\|f\|_X$ is strongly m-integrable so, making use of inequality (2.1) and dominated convergence for vector measures [11, II.4] we have

$$\begin{aligned}\left\|\int_A (f_j - f_k)\otimes dm\right\|_{L^p(X)} &\le \int_A \|f_j - f_k\|_X\,dm\\ &\le \int_A \|f_j - f\|_X\,dm + \int_A \|f - f_k\|_X\,dm \to 0\end{aligned}$$

as $j, k \to \infty$. Thus f is m-integrable in $L^p(X)$.

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Finally, we establish that inequality (2.1) holds for the function f. We know that $\lim_{j\to\infty}\int_A f_j \otimes dm = \int_A f \otimes dm$ in $L^p(X)$ for each $A \in \mathscr{S}$. By taking an appropriate subsequence, if necessary, we may assume that $\left\|\left(\int_A f_j \otimes dm\right)(\gamma)\right\|_X \to \left\|\left(\int_A f \otimes dm\right)(\gamma)\right\|_X$ for μ-almost every $\gamma \in \Gamma$ also. Since $\|f\|_X$ is assumed to be strongly m-integrable, dominated convergence for vector measures again ensures that $\int_A \|f_j\|_X\, dm \to \int_A \|f\|_X\, dm$ in $L^p(\mathbb{F})$ as $j \to \infty$ for all $A \in \mathscr{S}$. By taking a further subsequence, if necessary, we may assume that $\left(\int_A \|f_j\|_X\, dm\right)(\gamma) \to \left(\int_A \|f\|_X\, dm\right)(\gamma)$ for μ-almost every $\gamma \in \Gamma$ as well. See [6, Corollary 2.32]. This guarantees that inequality (2.1) holds for the function f and establishes the result for the case $n = m \geq 0$.

The general case is obtained by reduction to the above special case via the inequality

$$\begin{aligned}\left\|\left(\int_A h \otimes dn\right)(\gamma)\right\|_X &= \left\|\sum_{j=1}^{l} x_j (n(E_j \cap A))(\gamma)\right\|_X \\ &\leq \sum_{j=1}^{l} \|x_j\|_X (|n|\,(E_j \cap A))(\gamma) \\ &\leq \sum_{j=1}^{l} \|x_j\|_X (|m|\,(E_j \cap A))(\gamma) \\ &= \left(\int_A \|h\|_X d|m|\right)(\gamma)\end{aligned}$$

and a repetition of the arguments used previously. □

Next we introduce an adaptation of the notion of semivariation to the bilinear setting. It was originally introduced in [7] and used extensively by Bartle, [1]. The following definition is taken from [9, Section 2].

DEFINITION 2.3. The X-semivariation, $\beta_X(m) : \mathscr{S} \to [0, \infty]$ of m is defined by

$$\beta_X(m)(A) = \sup\left\{\left\|\sum_{j=1}^{k} x_j m(E_j \cap A)\right\|_{L^p(X)}\right\},$$

where the supremum is taken over all pairwise disjoint sets $E_1, \ldots, E_k$ from $\mathscr{S}$ and vectors $x_1, \ldots, x_k$ from X, such that $\|x_j\|_X \leq 1$ for all $j = 1, \ldots, k$ and $k \in \mathbb{N}$. For the special case where $X = \mathbb{F}$, X-semivariation reduces to the usual notion of semivariation and we write $\|m\|(A)$ in place of $\beta_X(m)(A)$ [2, Proposition I.1.11 (a)].

If in the above definition, $\beta_X(m)(\Omega) < \infty$, then we say that the vector measure m has *finite* X-semivariation. We say that the vector measure m has *continuous* X-semivariation if, for all sets $A_k \in \mathscr{S}$ decreasing to the empty set, $\beta_X(m)(A_k) \to 0$ as

$k \to \infty$. The implications of continuous X-semivariation were explored by Dobrakov in [3, 4, 5]. Bartle in [1] refers to continuous X-semivariation as the *-property. According to [3, *-Theorem], if $1 \le p < \infty$ and X contains no subspace isomorphic to c_0, then the X semivariation of m is continuous once it is finite. We note here that if m has continuous X-semivariation then the class of m-integrable functions coincides with the class of functions integrable in the sense of Bartle [1] and also coincides with the class of functions integrable in the sense of Dobrakov [3]. See [10, Remark end of Section 4.2] for a discussion on these points. A result that will be of use to us later is that if m has continuous X-semivariation then every strongly m-measurable, bounded function f is m-integrable in $L^p(X)$ [1, Theorem 7 and Lemma 3].

Theorem 2.2 can be used to show that dominated vector measures have continuous X-semivariation. This result is presented next as:

THEOREM 2.4. *Suppose that m is order bounded and that n is dominated by m. Then $\beta_X(n)$ is continuous and $\beta_X(n)(A) \le \||m|\|(A) = \||m|(A)\|_{L^p(\mathbb{F})}$ holds true for all $A \in \mathscr{S}$.*

PROOF. Let $A \in \mathscr{S}$. For $k \in \mathbb{N}$, let $x_j \in X$ satisfy $\|x_j\|_X \le 1$ for all $j = 1, \dots, k$. Let $E_j, j = 1, \dots, k$, be pairwise disjoint sets belonging to $\mathscr{S}$. Then by Theorem 2.2 and the definition of semivariation we have

$$\left\| \sum_{j=1}^{k} x_j n(E_j \cap A) \right\|_{L^p(X)} \le \left\| \sum_{j=1}^{k} \|x_j\| |m|(E_j \cap A) \right\|_{L^p(\mathbb{F})} \le \||m|\|(A).$$

Thus $\beta_X(n)(A) \le \||m|\|(A)$ for all $A \in \mathscr{S}$.

We next make the observation that if $A, B \in \mathscr{S}$ with $A \subseteq B$ then

$$\||m|(A)\|_{L^p(\mathbb{F})} \le \||m|(B)\|_{L^p(\mathbb{F})}.$$

This is easily seen by noting that $|m|(B) - |m|(A) = |m|(B \backslash A) \ge 0$. This observation, in combination with [2, Proposition I.11], gives us that $\||m|(A)\|_{L^p(\mathbb{F})} \le \||m|\|(A)$. To prove the reverse inequality we again make use of Theorem 2.2 (with the Banach space $X = \mathbb{F}$), and the above observation. Let $k \in \mathbb{N}$ and let $\alpha_j \in \mathbb{F}$ with $|\alpha_j| \le 1$ for $j = 1, \dots, k$. Also let $E_j, j = 1, \dots, k$ be pairwise disjoint subsets of $\mathscr{S}$. Then

$$\begin{aligned} \left\| \sum_{j=1}^{k} \alpha_j |m|(E_j \cap A) \right\|_{L^p(\mathbb{F})} &\le \left\| \sum_{j=1}^{k} |\alpha_j| |m|(E_j \cap A) \right\|_{L^p(\mathbb{F})} \le \left\| \sum_{j=1}^{k} |m|(E_j \cap A) \right\|_{L^p(\mathbb{F})} \\ &= \left\| |m| \left(\bigcup_{j=1}^{k} E_j \cap A \right) \right\|_{L^p(\mathbb{F})} \le \||m|(A)\|_{L^p(\mathbb{F})}. \end{aligned}$$

From the definition of semivariation this implies $\||m|\|(A) \le \||m|(A)\|_{L^p(\mathbb{F})}$ and we have established the equality. The continuity of $\beta_X(n)$ follows immediately. □

3. Dominated measures: the operator case

In this section we consider positive or dominated operator valued measures. We present Bochner type conditions guaranteeing the integrability of an operator valued function with respect to a positive or dominated operator valued measure.

Throughout this section M and N are operator valued measures and $F : \Omega \to \mathscr{L}(X)$ is an operator valued function. We use $\|F\|_{\mathscr{L}(X)}$ to denote the scalar function $\|F\|_{\mathscr{L}(X)} : \Omega \to \mathbb{R}$.

If $M(A)$ is a positive operator on $L^p(\mathbb{F})$ for each $A \in \mathscr{S}$ then M is said to be a *positive operator valued measure*. We write $M \geq 0$ if this is the case. We write $M \geq |N|$, and say that N is *dominated* by M if $M(A)y \geq |N(A)y|$ for all $y \geq 0 \in L^p(\mathbb{F})$ and $A \in \mathscr{S}$.

The question of when an operator valued measure is guaranteed to be dominated by a positive operator valued measure does not, in general, seem to have a straightforward answer. Partial results, involving Jordan decompositions of operator valued measures, can be found in the monograph by Schmidt, [16]. Nevertheless, a large class of operator valued measures that arise in applications to the Feynman-Kac formula in L^p-spaces is so dominated [8].

We first introduce a notion of integration for scalar valued functions with respect to operator valued measures. It is analogous to the scheme introduced in Section 2 for integrating scalar functions with respect to vector measures.

A scalar function $h : \Omega \to \mathbb{F}$ is said to be *M-integrable* in $\mathscr{L}(L^p(\mathbb{F}))$ if for each $y \in L^p(\mathbb{F})$ and $y' \in L^q(\mathbb{F})$, it is integrable with respect to the scalar measure $\langle My, y'\rangle : A \mapsto \langle M(A)y, y'\rangle$, $A \in \mathscr{S}$. The *integral* of h over A with respect to M, is an element of $\mathscr{L}(L^p(\mathbb{F}))$ which we denote by $\int_A h\,dM$ and which satisfies $\left\langle \left(\int_A h\,dM\right) y, y'\right\rangle = \int_A h\,d\langle My, y'\rangle$ for all $y \in L^p(\mathbb{F})$, $y' \in L^q(\mathbb{F})$ and $A \in \mathscr{S}$. Every bounded, $\mathscr{S}$-measurable function is M-integrable, [11, Lemma II.3.1]. Again see Kluvánek and Knowles, [11], for more details.

Now we define a notion of integration for operator valued functions with respect to operator valued measures.

DEFINITION 3.1 ([10, Definition 4.3.2]). A function F is said to be *M-integrable in $\mathscr{L}(L^p(X))$*, if for each $A \in \mathscr{S}$, there exists an operator $\int_A F \otimes dM \in \mathscr{L}(L^p(X))$ such that for every $x \in X$ and $y \in L^p(\mathbb{F})$, the X-valued function Fx, is My-integrable in $L^p(X)$ and the equality

$$\left(\int_A F \otimes dM\right)(yx) = \int_A [Fx] \otimes d[My]$$

holds for every $A \in \mathscr{S}$.

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We observe that the set function $A \mapsto \int_A F \otimes dM$, $A \in \mathscr{S}$, is σ-additive in the strong operator topology of $\mathscr{L}(L^p(X))$ [10, Lemma 4.3.3].

Next we introduce an operator valued analogue of the notion of X-semivariation.

DEFINITION 3.2 ([10, Definition 4.3.6]). We say that M has *finite $\mathscr{L}(X)$-semivariation* if

(i) $T \otimes (M(A)) \in \mathscr{L}(L^p(X))$ for each $T \in \mathscr{L}(X)$ and $A \in \mathscr{S}$, and

(ii) there exists $C > 0$ such that $\left\|\sum_{j=1}^n T_j \otimes (M(A_j))\right\|_{\mathscr{L}(L^p(X))} \leq C$, for all $T_j \in \mathscr{L}(X)$ with $\|T_j\|_{\mathscr{L}(X)} \leq 1$ and pairwise disjoint sets $A_j \in \mathscr{S}$, $j = 1, \dots, n$, and $n \in \mathbb{N}$.

Let $\beta_{\mathscr{L}(X)}(M)(A)$ be the smallest such number C as the sets A_j above range over all subsets of $A \in \mathscr{S}$. Then the set function $\beta_{\mathscr{L}(X)}(M)$ is called the *$\mathscr{L}(X)$-semivariation* of M.

REMARK 3.3. The well established Bartle integral [1] is perhaps the obvious candidate for a bilinear integral here. However when applied to the current situation, it requires stronger assumptions than does the integral presented above. We demonstrate this next.

Let $T : X \to X$ and $S : L^p(\mathbb{F}) \to L^p(\mathbb{F})$ be bounded operators and suppose that S is also positive. Then the bilinear map $(S, T) \mapsto S \otimes T$ from $\mathscr{L}_+(L^p(\mathbb{F})) \times \mathscr{L}(X) \to \mathscr{L}(L^p(X))$ exists [15, Exercise IV.22 (b) for the case where X is a Banach lattice. The extension to a general Banach space presents no difficulty.]. To apply Bartle's integral here we would need some control over the $\mathscr{L}(X)$-semivariation of M. Typically, this would involve assuming that M has continuous $\mathscr{L}(X)$-semivariation with the resulting implication that M is σ-additive in the operator norm on $L^p(\mathbb{F})$. Further, the resulting indefinite bilinear integral would be σ-additive in the uniform operator topology of $\mathscr{L}(L^p(X))$. For most applications, σ-additivity in the strong operator topologies of $\mathscr{L}(L^p(\mathbb{F}))$ and $\mathscr{L}(L^p(X))$ is more appropriate. See [10, Notes 4.7] for a more detailed discussion on these points.

PROPOSITION 3.4. *Suppose that $\beta_{\mathscr{L}(X)}(M)$ is finite. Then for each $y \in L^p(\mathbb{F})$, the vector measure My has finite X-semivariation.*

PROOF. Let $n \in \mathbb{N}$, and suppose that $x_j \in X$ satisfy $\|x_j\|_X \leq 1$ for all $j = 1, \dots, n$. Let E_j, $j = 1, \dots, n$, be pairwise disjoint sets belonging to $\mathscr{S}$. We claim that there exists a family $\{T_j\}_{j=1}^n$ of bounded linear operators on X with $\|T_j\| \leq 1$ for $j = 1, \dots, n$, associated with a vector $x_0 \in X$, so that $T_j x_0 = x_j$. To see this, fix $x_0 \in X$ with $\|x_0\|_X = 1$. By the Hahn-Banach theorem [6, Theorem 5.2.5], there exists $x_0' \in X'$ such that $\langle x_0, x_0' \rangle = 1$ and $\|x_0'\|_{X'} = 1$. For $j = 1, \dots, n$ we define the operator T_j by $T_j x = \langle x, x_0' \rangle x_j$, for all $x \in X$. It is easy to check that these operators satisfy our needs.

Now let $y \in L^p(\mathbb{F})$. Then

$$\begin{aligned}
\left\| \sum_{j=1}^{n} x_j [My](E_j) \right\|_{L^p(X)} &= \left\| \sum_{j=1}^{n} T_j x_0 [My](E_j) \right\|_{L^p(X)} \\
&= \left\| \left(\sum_{j=1}^{n} T_j \otimes M(E_j) \right) (y x_0) \right\|_{L^p(X)} \\
&\le \left\| \left(\sum_{j=1}^{n} T_j \otimes M(E_j) \right) \right\|_{\mathscr{L}(L^p(X))} \| y x_0 \|_{L^p(X)} \\
&\le \beta_{\mathscr{L}(X)}(M)(\Omega) \| y \|_{L^p(\mathbb{F})}.
\end{aligned}$$

Since we have assumed finite $\mathscr{L}(X)$-semivariation, by Definition 2.3 the required result is achieved. □

Next we provide a convergence theorem for operator valued functions. It is an operator valued version of [9, Theorem 2.6] and will be used in the proof of this section's main result.

THEOREM 3.5. *Suppose $\beta_{\mathscr{L}(X)}(M)$ is finite and that $F_j : \Omega \to \mathscr{L}(X)$, $j \in \mathbb{N}$, are M-integrable functions such that for each $x \in X$ and $y \in L^p(\mathbb{F})$ the following conditions hold:*

(a) *$F_j x$ converges to Fx pointwise My-almost everywhere;*
(b) *$\left\{ \int_A F_j x \otimes d[My] \right\}_{j=1}^{\infty}$ converges in $L^p(X)$ for each $A \in \mathscr{S}$, and*
(c) *For each $A \in \mathscr{S}$ the family of operators $\left\{ \int_A F_j \otimes dM \right\}_{j=1}^{\infty}$ is equicontinuous in $L^p(X)$.*

Then the function F is M-integrable in $\mathscr{L}(L^p(X))$.

PROOF. Set $G_j(A) = \int_A F_j \otimes dM$, for all $A \in \mathscr{S}$ and $j \in \mathbb{N}$. Then by assumption $G_j(A)\phi$ will converge in $L^p(X)$ for each $\phi \in L^p \otimes X$ and $A \in \mathscr{S}$. Next, the assumed equicontinuity of the family $\left\{ \int_A F_j \otimes dM \right\}_{j=1}^{\infty}$ and a standard $\epsilon/3$ argument shows that $G_j(A)\phi$ will converge for all $\phi \in L^p(X)$. See [13, Section I.5], for example. An application of the uniform boundedness principle [6, Theorem 5.12] establishes the existence of a limit operator $G(A) \in \mathscr{L}(L^p(X))$ such that $G(A)\phi = \lim_{j\to\infty} G_j(A)\phi$ for each $A \in \mathscr{S}$ and $\phi \in L^p(X)$. This operator will be the candidate for our integral.

To show now that F is M-integrable we need confirm two things:

(1) For each $x \in X$ and $y \in L^p(\mathbb{F})$, Fx is My-integrable in $L^p(X)$, and
(2) $G(A)(yx) = \int_A Fx \otimes d[My]$ holds for all $A \in \mathscr{S}$.

By Proposition 3.4, for each $y \in L^p(\mathbb{F})$, My has finite X-semivariation. Also, by assumption $F_j x$ converges to Fx, My-almost everywhere and $\{\int_A F_j x \otimes d[My]\}_{j=1}^\infty$ converges in $L^p(X)$. By [9, Theorem 2.6] this implies that Fx is My-integrable for each $x \in X$ and $y \in Y$.

The equality (2) is easily verified since $\int_A F_j x \otimes d[My]$ converges to $G(A)(yx)$ by the first part of the proof, and also to $\int_A Fx \otimes d[My]$ by [9, Theorem 2.6]. □

By [10, Theorem 4.3.7] this next result guarantees that bounded measurable functions are integrable with respect to dominated measures.

PROPOSITION 3.6. *Suppose M is positive and that N is dominated by M. Then $\beta_{\mathcal{L}(X)}(N)$ is finite, and for each $y \in L^p(\mathbb{F})$ the vector measure Ny has continuous X-semivariation.*

PROOF. We first establish the result for the special case $N = M \geq 0$. For $l \in \mathbb{N}$ and $1 \leq i \leq l$ let $A_i \in \mathcal{L}(X)$ with $\|A_i\|_{\mathcal{L}(X)} \leq 1$. For the same finite set of i's, let $\{B_i\}_{i=1}^l$ be pairwise disjoint subsets of $B \in \mathcal{S}$. For $k \in \mathbb{N}$, let $g = \sum_{j=1}^k x_j \chi_{G_j}$ be an X-valued $\mathcal{E}$-simple function with $x_j \in X$ and $G_j \in \mathcal{E}$ pairwise disjoint for $j = 1, \dots, k$. Also assume that $\|g\|_{L^p(X)} \leq 1$.

Then

$$\begin{aligned}
\left\|\sum_i (A_i \otimes [M(B_i)])g\right\|_{\mathcal{L}(L^p(X))}^p &= \int_\Gamma \left\|\sum_{i,j} A_i x_j [M(B_i)\chi_{G_j}](\gamma)\right\|_X^p d\mu(\gamma) \\
&\leq \int_\Gamma \left(\sum_{i,j} \|A_i x_j\|_X [M(B_i)\chi_{G_j}](\gamma)\right)^p d\mu(\gamma) \\
&\leq \int_\Gamma \left(\sum_{i,j} \|x_j\|_X [M(B_i)\chi_{G_j}](\gamma)\right)^p d\mu(\gamma) \\
&= \left\|\sum_i M(B_i)\left(\sum_j \|x_j\|_X \chi_{G_j}\right)\right\|_{L^p(\mathbb{F})}^p \\
&\leq \left\|\sum_i M(B_i)\right\|_{\mathcal{L}(L^p(\mathbb{F}))}^p \left\|\sum_j \|x_j\|_X \chi_{G_j}\right\|_{L^p(\mathbb{F})}^p \\
&= \left\|\sum_i M(B_i)\right\|_{\mathcal{L}(L^p(\mathbb{F}))}^p \|g\|_{L^p(X)}^p \leq \|M(B)\|_{\mathcal{L}(L^p(\mathbb{F}))}^p.
\end{aligned}$$

The last equality follows from the observation that if $A, B \in \mathcal{S}$ with $A \subseteq B$ then $\|M(A)\|_{\mathcal{L}(L^p(X))} \leq \|M(B)\|_{\mathcal{L}(L^p(X))}$. This is easily seen by noting that $M(B)f -$

$M(A)f = M(B\backslash A)f \geq 0$ holds for all $f \geq 0$. Since X-valued $\mathscr{E}$-simple functions are dense in $L^p(X)$, by Definition 3.2 this implies $\beta_{\mathscr{L}(X)}(M)(B) \leq \|M(B)\|_{\mathscr{L}(L^p(X))}$.

Next we show that for an arbitrary $y \in L^p(\mathbb{F})$, the $L^p(\mathbb{F})$-valued measure My has continuous X-semivariation. This follows immediately from Theorem 2.4 if we can show that My is dominated by a positive vector measure. However, decomposing y first into its real and imaginary parts and then into its positive and negative parts gives us that $|My| \leq 2M|y|$. This is easily seen since, for all $A \in \mathscr{S}$,

$$\begin{aligned}|M(A)y| &= |M(A)y_1 + iM(A)y_2| \\ &= \left|M(A)y_1^+ - M(A)y_1^- + iM(A)y_2^+ - iM(A)y_2^-\right| \\ &\leq M(A)(y_1^+ + y_1^-) + M(A)(y_2^+ + y_2^-) = M(A)(|y_1| + |y_2|) \leq 2M(A)|y|.\end{aligned}$$

Since M is a positive operator valued measure it follows that $M|y| \geq 0$. This completes the proof for the positive case.

To prove that the dominated measure N has finite $\mathscr{L}(X)$-semivariation it suffices to note that $|N(A)\chi_G| \leq M(A)\chi_G$ for all $A \in \mathscr{S}$ and $G \in \mathscr{E}$ and repeat the argument for the positive case. To prove Ny has continuous X-semivariation for each $y \in L^p(\mathbb{F})$ it suffices to note that $|Ny| \leq 2M|y|$, that is, that the vector measure Ny is dominated by a positive measure. The result then follows from Theorem 2.4. □

The principle result of this section is an operator valued version of Theorem 2.2. We give a preliminary result below in Proposition 3.7. The assumption of integrability on the operator valued function will be removed in Theorem 3.8.

PROPOSITION 3.7. *Suppose that M is positive and that N is dominated by M. Further suppose F is N-integrable in $\mathscr{L}(L^p(X))$ and that $\|F\|_{\mathscr{L}(X)}$ is M-integrable in $\mathscr{L}(L^p(\mathbb{F}))$. Then the estimate*

$$\left\|\int_A F \otimes dN\right\|_{\mathscr{L}(L^p(X))} \leq \left\|\int_A \|F\|_{\mathscr{L}(X)}\, dM\right\|_{\mathscr{L}(L^p(\mathbb{F}))}$$

holds for all $A \in \mathscr{S}$.

PROOF. We establish the result for the special case $N = M \geq 0$ first. For $n \in \mathbb{N}$, let $g = \sum_{j=1}^n x_j \chi_{G_j}$ be an X-valued $\mathscr{E}$-simple function with $x_j \in X$ and G_j pairwise disjoint for $j = 1, \dots, n$, such that $\|g\|_{L^p(X)} \leq 1$. Then, making use of Proposition 2.2, we have

$$\begin{aligned}&\left\|\left(\int_A F(\omega) \otimes dM(\omega)\right) g\right\|_{L^p(X)}^p \\ &\quad = \left\|\sum_j \int_A [Fx_j](\omega) \otimes d[M\chi_{G_j}](\omega)\right\|_{L^p(X)}^p\end{aligned}$$

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$$
\begin{aligned}
&= \int_\Gamma \left\| \left(\sum_j \int_A [Fx_j](\omega) \otimes d[M\chi_{G_j}](\omega) \right)(\gamma) \right\|_X^p d\mu(\gamma) \\
&\le \int_\Gamma \left(\sum_j \left\| \left(\int_A [Fx_j](\omega) \otimes d[M\chi_{G_j}](\omega) \right)(\gamma) \right\|_X \right)^p d\mu(\gamma) \\
&\le \int_\Gamma \left(\sum_j \left(\int_A \|F(\omega)x_j\|_X \, d[M\chi_{G_j}](\omega) \right)(\gamma) \right)^p d\mu(\gamma) \\
&= \left\| \sum_j \int_A \|F(\omega)x_j\|_X \, d[M\chi_{G_j}](\omega) \right\|_{L^p(\mathbb{F})}^p \\
&\le \left\| \sum_j \int_A \|F(\omega)\|_{\mathscr{L}(X)} \|x_j\|_X \, d[M\chi_{G_j}](\omega) \right\|_{L^p(\mathbb{F})}^p \\
&= \left\| \left(\int_A \|F(\omega)\|_{\mathscr{L}(X)} \, dM(\omega) \right) \left(\sum_j \|x_j\|_X \chi_{G_j} \right) \right\|_{L^p(\mathbb{F})}^p \\
&\le \left\| \int_A \|F(\omega)\|_{\mathscr{L}(X)} \, dM(\omega) \right\|_{\mathscr{L}(L^p(\mathbb{F}))}^p \left\| \sum_j \|x_j\|_X \chi_{G_j} \right\|_{L^p(\mathbb{F})}^p \\
&= \left\| \int_A \|F(\omega)\|_{\mathscr{L}(X)} \, dM(\omega) \right\|_{\mathscr{L}(L^p(\mathbb{F}))}^p \|g\|_{L^p(X)}^p
\end{aligned}
$$

holding for all $A \in \mathscr{S}$. Since X-valued $\mathscr{E}$-simple functions are dense in $L^p(X)$ this establishes the required inequality and completes the proof for the positive case.

The inequality for the case where N is dominated by M follows analogously to the positive case taking note that, when $x \in X$ and $E \in \mathscr{E}$,

$$
\left\| \left(\int_A [Fx](\omega) \otimes d[N\chi_E](\omega) \right)(\gamma) \right\|_X \le \left(\int_A \|F(\omega)x\|_X \, d[M\chi_E](\omega) \right)(\gamma)
$$

holds true for all $A \in \mathscr{S}$ and μ-almost every $\gamma \in \Gamma$ (Proposition 2.2). □

Finally, we provide a Bochner type condition guaranteeing the integrability of an operator valued function with respect to a positive or dominated operator valued measure. It is an operator valued version of Theorem 2.2 and substantially strengthens the results of Proposition 3.7.

THEOREM 3.8. *Suppose that M is positive and that N is dominated by M. Let F be such that for each $x \in X$ and $y \in L^p(\mathbb{F})$, Fx is strongly My-measurable and $\|F\|_{\mathscr{L}(X)}$ is M-integrable in $\mathscr{L}(L^p(\mathbb{F}))$. Then the function F is N-integrable in $\mathscr{L}(L^p(X))$ and* *the estimate*

$$\left\| \int_A F \otimes dN \right\|_{\mathscr{L}(L^p(X))} \le \left\| \int_A \|F\|_{\mathscr{L}(X)}\, dM \right\|_{\mathscr{L}(L^p(\mathbb{F}))}$$

holds for all $A \in \mathscr{S}$.

PROOF. As usual we consider the special case $N = M \ge 0$ initially. For $j \in \mathbb{N}$, let $A_j = \{\omega \in \Omega : \|F(\omega)\|_{\mathscr{L}(X)} \le j\}$. Next define a family of bounded functions $F_j : \Omega \to \mathscr{L}(X)$ by setting $F_j = F.\chi_{A_j}$. From Proposition 3.6 we have finite $\mathscr{L}(X)$-semivariation and continuous pointwise X-semivariation in this setting. By [10, Theorem 4.3.7] this implies that bounded $\mathscr{S}$-measurable functions are M-integrable. Thus we have a sequence $\{F_j\}_{j=1}^{\infty}$ of M-integrable functions converging pointwise in the operator norm to F. Verifying the conditions of Theorem 3.5 will show F to be M-integrable in $\mathscr{L}(L^p(X))$.

Condition (a) of Theorem 3.5 is obviously satisfied since, for each fixed $\omega \in \Omega$, $F_j(\omega) = F(\omega)$ if j is taken large enough.

Let $x \in X$ and $y \in L^p(\mathbb{F})$. We show that the sequence $\left\{\int_A F_j x \otimes d[My]\right\}_{j=1}^{\infty}$ converges in $L^p(X)$ for each $A \in \mathscr{S}$. Recall from the proof of Proposition 3.6 that the vector measure My is dominated by the positive vector measure $2M|y|$. Thus we can apply Theorem 2.2 to obtain (assuming $j > k$)

$$\begin{aligned}
&\left\| \int_A (F_j - F_k)(\omega) x \otimes d[My](\omega) \right\|_{L^p(X)} \\
&\quad = \left\| \int_A F(\omega) x\, (\chi_{A_j} - \chi_{A_k})(\omega) \otimes d[My](\omega) \right\|_{L^p(X)} \\
&\quad \le \left\| \int_A \|F(\omega)x\|_X (\chi_{A_j} - \chi_{A_k})(\omega)\, d[2M|y|](\omega) \right\|_{L^p(\mathbb{F})} \\
&\quad = \left\| \int_{\bigcup_{i=k+1}^{j} B_i} \|F(\omega)x\|_X\, d[2M|y|](\omega) \right\|_{L^p(\mathbb{F})}.
\end{aligned}$$

Here $B_i = \{\omega \in A : i - 1 < \|F(\omega)\|_{\mathscr{L}(X)} \le i\}$. By assumption $\|Fx\|_X$ is $M|y|$-integrable so the set function $\int_{(\cdot)} \|Fx\|_X\, d[2M|y|] : \mathscr{S} \to L^p(\mathbb{F})$ is σ-additive. Since the B_i's are pairwise disjoint, it follows from the unconditional summability of the resulting sequence that if j and k are made large enough then the difference between the corresponding integrals will be arbitrarily small. Thus our sequence of integrals is Cauchy and condition (b) is satisfied.

Finally, the equicontinuity of the family $\{\int_A F_j \otimes dM\}_{j=1}^{\infty}$ is established by making use of Proposition 3.7. We have

$$\left\| \int_A F_j \otimes dM \right\|_{\mathscr{L}(L^p(X))} \le \left\| \int_A \|F_j\|_{\mathscr{L}(X)}\, dM \right\|_{\mathscr{L}(L^p(\mathbb{F}))} \le \left\| \int_A \|F\|_{\mathscr{L}(X)}\, dM \right\|_{\mathscr{L}(L^p(\mathbb{F}))}.$$

This uniform bound gives the required equicontinuity and condition (c). Thus F is M-integrable in $\mathscr{L}(L^p(X))$. The estimate now follows from an application of Proposition 3.7 and completes the proof for the positive case.

To prove the result for dominated N it suffices to repeat the above proof, noting that $|Ny| \leq 2M|y|$ for all $y \in L^p$ and that

$$\left\| \int_A F_j \otimes dN \right\|_{\mathscr{L}(L^p(X))} \leq \left\| \int_A \|F_j\|_{\mathscr{L}(X)} \, d[2M] \right\|_{\mathscr{L}(L^p(\mathbb{F}))}$$

holds true for $A \in \mathscr{S}$ and $j \in \mathbb{N}$. □

4. Example: a Feynman-Kac formula

Finally, we give an example where a bilinear integral provides a representation for the solution of an initial value problem.

We define Ω to be the collection of all functions $\omega : [0, \infty) \to \mathbb{R}$ such that in each time interval $[0, T]$, there exist finitely many times $0 = t_0 < t_1 < t_2 < \cdots < t_k < T$, such that $\omega(t) = \omega(t_{j-1})$ for each time t such that $t_{j-1} \leq t < t_j$ and $\omega(t) = \omega(t_k)$ for all $t_k \leq t < T$. Thus Ω is the collection of all piecewise constant functions having a finite number of discontinuities in each finite time interval. For all $t > 0$ let $\mathscr{S}_t$ denote the σ-algebra $\sigma(\omega(s) : 0 \leq s \leq t)$ generated by the family $\{\omega(s) : 0 \leq s \leq t\}$.

Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Recall that a bounded operator $S \in \mathscr{L}(L^p(\mathbb{F}))$ is a *regular* operator if it can be written as a linear combination of positive operators. Let $k : \mathbb{R} \times \mathbb{R} \to \mathbb{F}$ be a measurable function such that $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |k(x, y)|^q \, dy \right)^{p/q} dx < \infty$. Define a *Hilbert-Schmidt* operator acting on $L^p(\mathbb{R})$ with kernel k, to be the operator, A, given by $(Af)(x) = \int_{\mathbb{R}} k(x, y) f(y) \, dy$ for each $f \in L^p(\mathbb{R})$. It is easily seen that A is bounded and regular.

THEOREM 4.1 ([14, Section 7]). *Let $1 \leq p \leq s < \infty$ and $p < r \leq \infty$ satisfy $1/p = 1/s + 1/r$. Let A be a Hilbert-Schmidt operator acting on $L^p(\mathbb{R}) \cap L^s(\mathbb{R})$ with kernel k. Suppose that B is the infinitesimal generator of a C_0-semigroup acting on $L^p(\mathbb{R})$ such that the Banach space $K = D(B)$ with norm $\|y\|_K = \|y\|_{L^p(\mathbb{R})} + \|By\|_{L^p(\mathbb{R})}$ is reflexive and that $a : \mathbb{R} \to (0, \infty)$ is a function which is an element of $L^r(\mathbb{R}) + L^\infty(\mathbb{R})$.*

Let $\phi \in L^p(\mathbb{R}^2) \cap L^s(\mathbb{R}^2)$ be such that $A_x\phi \in L^p(\mathbb{R}^2)$ and $B_y\phi \in L^p(\mathbb{R}^2) \cap L^s(\mathbb{R}^2)$. Then for each $t \geq 0$ there exists an operator valued measure $M_t : \mathscr{S}_t \to \mathscr{L}(L^p(\mathbb{R}))$ which is dominated by a positive operator valued measure, and also there exists an operator valued function $F_t : \Omega \to \mathscr{L}(L^p(\mathbb{R}))$ (in fact, a Multiplicative Operator Functional) *which is M_t-integrable in $\mathscr{L}(L^p(\mathbb{R}; L^p(\mathbb{R})))$. Moreover the function*

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$u(t, x, y) = \left(\left[\int_\Omega F_t \otimes dM_t\right]\phi\right)(x, y)$, $t \geq 0$ *solves the initial value problem*

$$\frac{\partial u}{\partial t}(t, x, y) = a(x)B_y u(t, x, y) + \int_{\mathbb{R}} k(x, z)u(t, z, y)\,dz,$$
$$u(0, x, y) = \phi(x, y), \quad \textit{for almost all } x, y \in \mathbb{R}.$$

Here A_x denotes the operator A acting on the x variable of the function $u : (t, x, y) \mapsto (t, x, y)$ and B_y denotes the operator B acting on the y variable of the function u. The derivative with respect to t is in the sense of convergence in $L^p(\mathbb{R}^2)$.

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