Canad. Math. Bull. Vol. 34 (2), 1991 pp. 165–174

SOME APPLICATIONS OF A THEOREM OF MARCINKIEWICZ

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ABSTRACT. A classical theorem of Marcinkiewicz states that a function is Perron integrable iff it has one continuous major and one continuous minor function. Using an elaboration of this remarkable theorem three applications are made; to obtain a new proof of a recent characterization of the Perron integral, to proofs of some theorems on interchange of limits and integration and to extend classical existence theorems for ordinary differential equations.

1. **Introduction.** If f: $[a,b] \longrightarrow \mathbf{R}$ then M is called a major function, m a minor function of f on $[a,b]$, if both are real-valued functions on $[a,b]$, $M(a) = m(a) = 0$, and

$$\overline{D}m \leqq f \leqq \underline{D}M. \tag{1}$$

It is known, Saks [12, p. 201], that the theory of the P-integral, (the Perron or Denjoy-Perron integral), can be developed if we assume the existence of at least one M and m. Then $P\int_a^b f = \inf M(b) = \sup m(b)$, provided the last two real numbers are equal; f can then be proved measurable, being almost everywhere the derivative of its continuous primitive, and we write $f \in P(a,b)$. In practice it is convenient to use continuous M and m since then the inequalities (1) can be relaxed on certain exceptional sets; see Bullen [1] where the basic references are given.

The following theorem published in Saks [12, p. 253] is usually referred to as the Marcinkiewicz theorem, although it was also proved, independently, by Tolstov [16] and Denjoy [4].

THEOREM 1. *If f: $[a,b] \longrightarrow \mathbf{R}$ is measurable then $f \in P(a,b)$ iff it has at least one continuous major function, and one continuous minor function.*

The two hypotheses are critical for the proof: (a) the measurability of f is used to prove f summable on a perfect set where M and m are of bounded variation; (b) the continuity of M is used to deduce $f \in P(\alpha,\beta)$ from $f \in P(\phi,\eta)$ for all $\phi,\eta,\alpha < \phi < \eta < \beta$. Further Saks [12, p. 253] gives an example to show that if the continuity hypothesis is dropped the result is false.

However Sarkhel [13] pointed out that the Saks example was not convincing, and showed that Theorem 1 was still valid if M and m are regulated functions satisfying

$$\begin{aligned} \lim_{y\to x-} M(y) \leqq M(x) \leqq \lim_{y\to x+} M(y), \\ \lim_{y\to x-} m(y) \geqq m(x) \geqq \lim_{y\to x+} m(y). \end{aligned} \tag{2}$$

The work of the first author was done while visiting the University of Queensland.
Received by the editors February 7, 1989.

(As Sarkel points out (2) is a very mild requirement since in any case (1) implies that

$$\lim_{y \to x-} \sup M(y) \leqq M(x) \leqq \lim_{y \to x+} \inf M(y),$$

$$\lim_{y \to x-} \inf m(y) \geqq m(x) \geqq \lim_{y \to x+} \sup m(y).)$$

A more convincing counterexample can be obtained by modifying one due to Burkill [2]. Define Φ, f on $[0, 1/\pi]$ as follows:

$$\Phi(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0; \end{cases} \quad f(x) = \begin{cases} \Phi''(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is not Cauchy-Riemann integrable and so f is not in $P(0, 1/\pi)$. However, we can easily define a major function, and a minor function, for f as follows:

$$M(x) = \Phi'(x) + 2 \text{ for } x \neq 0,$$

$$m(x) = \Phi'(x) - 2 \text{ for } x \neq 0,$$

$$M(0) = m(0) = 0,$$

in fact M is lower semicontinuous, and m is upper semicontinuous. It follows from Theorem 1 that f cannot have any continuous major or minor functions.

In general once a function f has a major function we can define the upper P-integral, $P \int_a^{\bar{b}} f = \inf M(b)$, and similarly the lower P-integral $P \int_{\underline{a}}^b f = \sup m(b)$ is defined once f has a minor function. If both exist then it is easy to see that

$$-\infty < P \int_{\underline{a}}^b f \leqq P \int_a^{\bar{b}} f < \infty;$$

in the case of the above example the middle inequality is strict.

The concept of major and minor functions finds applications in the theory of differential equations. In section 4 we prove existence theorems concerning the Cauchy problem for the equation (and systems of equations) $y' = f(x, y)$. We employ the existence of functions m, M satisfying

$$\bar{D}m(t) \leqq f(t, g(t)) \leqq \underline{D}M(t)$$

for all suitable g (for details see Theorem 11). This condition can be viewed as a generalisation of the condition $f(t, x) \leqq m(t)$ with a summable m, used in the Caratheodory theory (see [3] p. 43). The theorems on interchange of limit and integration play an important role in Section 4 and we discuss them and their connection with the Marcinkiewicz theorem in Section 3.

2. **The Marcinkiewicz Theorem in the Henstock-Kurzweil Theory.** Suppose $\delta\colon [a,b] \longrightarrow]0,\infty[$ then π is a δ-fine partition of $[a,b], \pi \in \Pi(\delta) = \Pi(\delta; a,b)$ if $\pi = (a_0,\dots,a_n;\, y_1,\dots,y_n) = \{[a_{i-1}a_i], y_i, 1 \leqq i \leqq n\}$ for some $n \in \mathbf{N}, a_i, y_i \in [a,b], 1 \leqq i \leqq n$ satisfying

$$\text{(a) } a = a_0 < \cdots < a_n = b, \tag{3}$$
$$\text{(b) } a_{i-1} \leqq y_i \leqq a_i,\ 1 \leqq i \leqq n,$$
$$\text{(c) } a_i - a_{i-1} < \delta(y_i),\ 1 \leqq i \leqq n.$$

The $[a_{i-1}, a_i]$ are called the intervals, and the y_i the tags of π. It is known that the theory of the P-integral can be developed in the usual fashion using Riemann sums for δ-fine partitions of $[a,b]$; see Kurzweil [6], Henstock [5].

If now $f\colon [a,b] \longrightarrow \mathbf{R}, \pi \in \Pi(\delta; a,b)$ the associated Riemann sum is written

$$\sum_\pi f = \sum_{i=1}^n f(y_i)(a_i - a_{i-1}).$$

If we have $a \leqq c < d \leqq b$ we can consider δ to be restricted to $[c,d]$ and write $(c\sum_\pi d)f$, for a $\pi \in \Pi(\delta; c,d)$.

LEMMA 2. *If $f\colon [a,b] \longrightarrow \mathbf{R}$ then f has a major and minor function iff there is a $\delta\colon [a,b] \longrightarrow]0,\infty[$ and a $K > 0$ such that for all $\pi \in \Pi(\delta)$*

$$\left|\sum_\pi f\right| < K. \tag{4}$$

PROOF. Given a major function M, a minor function m and $\epsilon > 0$ define $\delta(x) > 0$ for all x by

$$M(v) - M(u) \geqq (v-u)\{f(x) - \epsilon\}$$
$$m(v) - m(u) \leqq (v-u)\{f(x) + \epsilon\}$$

when $x - \delta(x)/2 < u \leqq x \leqq v < x + \delta(x)/2$; then (4) is immediate.

Conversely suppose (4) holds then we easily prove that for some $K > 0$, not necessarily the same at each occurence,

(i) for all $\pi_1, \pi_2 \in \Pi(\delta)$,

$$\left|\sum_{\pi_1} f - \sum_{\pi_2} f\right| < K; \tag{5}$$

(ii) for all $c, d, a \leqq c < d \leqq b$, and $\pi \in \Pi(\delta; c,d)$

$$\left|\left(c\sum_\pi d\right)f\right| \leqq K;$$

(further if (i) or (ii) holds so does (4)).

Using (i) we define for $a \leqq x \leqq b$

$$(6)\qquad M_\delta(x) = \sup_{\pi\in\Pi(\delta)} \left(a\sum_\pi x\right) f; \qquad m_\delta(x) = \inf_{\pi\in\Pi(\delta)} \left(a\sum_\pi x\right) f.$$

It follows easily that if $x - \delta(x)/2 \leqq u \leqq x \leqq v < x + \delta(x)/2$ then $M_\delta(v) - M_\delta(u) \geqq f(x)(v-u)$ and M_δ is a major function of f on $[a,b]$. Similarly m_δ is a minor function. ∎

In general M_δ, m_δ given by (6) cannot be continuous since clearly the example in section 1 satisfies (4) of Lemma 2. If M_δ and m_δ exist then $P\bar{\int}_a^b = \inf_\delta \ M(b)$, and $P\underline{\int}_a^b f = \sup_\delta \ m(b)$; see Pfeffer [11].

In order that M_δ, m_δ of (6) be continuous some extra condition is needed; a condition that suggests itself is: for all $\epsilon > 0$ there is an $\xi = \xi(\epsilon, x) > 0$ such that if $x \in [c,d], d-c < \xi$, then for all $\pi \in \Pi(\delta)$

$$(7)\qquad \left|\left(c\sum_\pi d\right) f\right| < \epsilon.$$

However given both δ of (4) and ξ of (7) both could be replaced by min (ξ, δ); then given (7) a simple application of Cousin's lemma (see Mawhin p. 103) gives (4). In this form (7) implies the existence of M_δ and m_δ but we have not been able to prove they are continuous; on the other hand we have no example to show they need not be. We avoid this difficulty by generalising Theorem 1 as follows.

THEOREM 3. *If (i) $f\colon [a,b] \longrightarrow \mathbf{R}$ is measurable; (ii) U is a non-empty family of major functions of f, L is a non-empty family of minor functions of f; and for all $\epsilon > 0, x \in [a,b]$ there is a $\delta = \delta(x,\epsilon) > 0, M \in U, m \in L$ such that if $x - \delta < \alpha < \beta < x$ or $x < \alpha < \beta < x + \delta$ then $-\epsilon < m(\beta) - m(\alpha) \leqq M(\beta) - M(\alpha) < \epsilon$; then $f \in P(a,b)$.*

PROOF. As was remarked in (b) following Theorem 1 the only place where continuity is used is in proving that if $f \in P(\phi, \eta)$ for all ϕ, η such that $\alpha < \phi < \eta < \beta$ then $f \in P(\alpha, \beta)$; and to prove this it suffices to prove that $\lim_{\eta\to\beta-} \int_\phi^\eta f$ and $\lim_{\phi\to\alpha+} \int_\phi^\eta f$ exist. Consider the first limit, by the usual criterion it is sufficient to prove that given $\epsilon > 0$ there is a $\delta > 0$ such that if $\beta - \delta < x < y < \beta$ then $|\int_x^y f| < \epsilon$. If we pick $\delta = \delta(\beta, \epsilon)$ of the hypothesis (ii) then with the $M \in U, m \in L$ of the same hypothesis we get that $\epsilon > M(y) - M(x) \geqq \int_x^y f \geqq m(y) - m(x) > -\epsilon$ and we have the desired conclusion. ∎

Theorem 3 generalises Theorem 1 since if M is a continuous major function, m a continuous minor function then we can take $U = \{M\}, L = \{m\}$. In general, of course, M and m in the above proof depend on ϵ and on β. Now if (7) is satisfied it is easily checked that we can take $U = \{M_\delta;\ M_\delta$ defined by (6)$\}$ $L = \{m_\delta;\ m_\delta$ defined by (6)$\}$ and so using Theorem 3 we have

https://doi.org/10.4153/CMB-1991-027-x Published online by Cambridge University Press

THEOREM 4. *If $f\colon [a,b] \longrightarrow \mathbf{R}$ is measurable then* $f \in P(a,b)$ iff *for all* $\epsilon > 0$ *there is a* $\delta = \delta(\epsilon, x) > 0$ *such that if* $x \in [c,d]$, *and*

$$[c,\ d] \subset]x - \frac{\delta(x)}{2},\ x + \frac{\delta(x)}{2}[$$

then for all $\pi \in \Pi(\delta)$ *we have that*

$$\left|\left(c \sum_{\pi} d\right) f\right| < \epsilon.$$

This theorem is due to Schurle [14] who called the condition in Theorem 4 condition LSRS, for locally small Riemann sums. His proof is a longer and deeper; it can be regarded as a proof of the Marcinkiewicz theorem in the setting of the Kurzweil-Henstock theory. By analogy with the terminology of Schurle let us call the condition of Lemma 2 condition BRS, for bounded Riemann sums.

In the case of non-negative functions we have

THEOREM 5. *If $f\colon [a,b] \longrightarrow \mathbf{R}$ is measurable and non-negative then* $f \in L(a,b)$ iff *the condition BRS is satisfied.*

PROOF. In this case we can take $m = 0$ and M given by (6). Obviously M is monotonic and so satisfies (2). The result then follows from Sarkhel's modification of Theorem 1 and the well known result that $f \geqq 0$ and $f \in P(a,b)$ implies $f \in L(a,b)$, Saks [12. p. 203]. Alternatively, since M is monotonic $\underline{D}M$ is summable and the result follows from (1). Yet another proof can be obtained by considering Max (f,n), these functions have uniformly bounded integrals because of BRS. The summability of f follows now from the monotone convergence theorem. ∎

It is known that if (b) is omitted in (3) we get what could be called δ-fine absolute partitions, $\pi \in |\Pi|(\delta; a,b)$ say, and that using these the Riemann theory defines the L-integral; McShane [10]. Replacing $\Pi(\delta)$ by $|\Pi|(\delta)$ we can define condition $|$BRS$|$ and $|$LSRS$|$; also the obviously modified 5 (i) and (ii) are equal to $|$BRS$|$.

LEMMA 6. *If f satisfies $|BRS|$, then so does $|f|$.*

PROOF. We first prove $|$BRS$|$ holds iff there exists $\delta\colon [a,b] \longrightarrow]0,\infty[$ such that (5) holds for all $\pi_1, \pi_2 \in |\Pi|(\delta)$ having the same intervals. One way is trivial so let π_1, $\pi_2 \in |\Pi|(\delta)$ with $\pi_1 = (c_0, c_1, \ldots, c_m;\ u_1, u_2, \ldots, u_m)$, $\pi_2 = (d_0, d_1, \ldots, d_n;\ v_1, v_2, \ldots, v_n)$ and let $\{a_0, \ldots, a_1, a \equiv a_0 < \cdots < a_P = b\}$ be the set of distinct elements of $\{c_0, c_1, \ldots, c_m, d_0, d_1, \ldots, d_n\}$. Now define $\pi_3, \pi_4 \in |\Pi|(\delta)$ as follows $\pi_3 = (a_0, \ldots, a_p;\ y_1, \ldots, y_p\}$, $\pi_4 = (a_0, \ldots, a_p;\ z_1, \ldots, z_p)$ where if $[a_{i-1}, a_i] = [c_{j-1}, c_j] \cap [d_{k-1}, d_k]$, $y_i = u_j$, $z_i = v_k$. Since clearly $|\sum_{\pi_1} f - \sum_{\pi_2} f| = |\sum_{\pi_3} f - \sum_{\pi_4} f|$, we have (5) for $\pi_1 \in |\Pi|(\delta)$, $\pi_2 \in |\Pi|(\delta)$ and consequently $|$BRS$|$.

Suppose now f satisfies $|\mathrm{BRS}|$ and $\pi_3, \pi_4 \in |\Pi|(\delta)$ and have the same intervals, then

$$\left|\sum_{\pi_3}|f| - \sum_{\pi_4}|f|\right| \leqq \sum_{i=1}^{p} \big||f(y_i)| - |f(z_i)|\big|(a_i - a_{i-1})$$
$$\leqq \sum_{i=1}^{p} |f(y_i) - f(z_i)|(a_i - a_{i-1})$$
$$= \sum_{\pi_3^1} f - \sum_{\pi_4^1} f < K,$$

where $\pi_3^1, \pi_4^1 \in |\Pi|(\delta)$ are defined as $\pi_3^1 = (a_0, \dots, a_p;\ y_1^1, \dots, y_p^1)$, $\pi_4^1 = (a_0, \dots, a_p;\ z_1^1, \dots, z_p^1)$ where $y_i^1 = y_i, z^1 = z_i$ if $f(y_i) - f(z_i) \geqq 0$, and $y_i^1 = z_i, z_i^1 = y_i$ if $f(y_i) - f(z_i) < 0$. Hence, by the above result, $|f|$ satisfies $|\mathrm{BRS}|$. ∎

THEOREM 7. *If f: $[a,b] \to \mathbf{R}$ is measurable then $f \in L(a,b)$* iff $|\mathrm{BRS}|$ *holds.*

PROOF. If f satisfies $|\mathrm{BRS}|$ then by Lemma 5 the function $|f|$ also satisfies $|\mathrm{BRS}|$ and so also then f^+ and f^- satisfy $|\mathrm{BRS}|$, and in particular BRS. Hence f^+ and f^- are L-integrable by Theorem 4 and the theorem is proved. ∎

The above result with $|\mathrm{BRS}|$ replaced by $|\mathrm{LSRS}|$ was proved by Schurle [15] in a completely different way.

3. **The Marcinkiewicz Theorem and Some Convergence Theorems.**

THEOREM 8. *If (i) $f_n \in P(a,b), F_n(x) = P\int_a^x f_n, a \leqq x \leqq b, n \in \mathbf{N}$; (ii)* $\lim_{n\to\infty} f_n = f$ *a.e.; (iii)* $\lim_{n\to\infty} F_n = F$ *uniformly; (iv) f_n satisfies condition BRS uniformly in $n, n \in \mathbf{N}$; then $f \in P(a,b)$ and $F(x) = P\int_a^x f$ for $a \leqq x \leqq b$.*

PROOF. Since $f_n, n \in \mathbf{N}$, satisfies BRS uniformly in n, it follows that $\sup f_n$ and $\inf f_n$ both satisfy BRS and so by Lemma 2 $\sup f_n$ has a major function M, $\inf f_n$ has a minor function m. Clearly M is a major function of f_n, for all n, and m is a minor function of f_n, for all n. The result now follows from a convergence theorem of Lee [7, p. 20]. ∎

In fact Theorem 8 is equivalent to the theorem of Lee since obviously if $f_n, n \in \mathbf{N}$, have a common major and a common minor function, the argument of Lemma 2 shows that BRS holds for $f_n, n \in \mathbf{N}$, uniformly in n.

THEOREM 9. *If (i) $f_n \in P(a,b)$ for all $n \in \mathbf{N}$; (ii)* $\lim_{n\to\infty} f_n = f$ *a.e.; (iii)* LSRS *condition holds for $f_n, n \in \mathbf{N}$, uniformly in n: then $f \in P(a,b)$ and $P\int_a^b f =$* $\lim_{n\to\infty} P\int_a^b f_n$.

PROOF. As in the previous proof, it follows that $\sup f_n$ and $\inf f_n$ satisfy LSRS and so, by Theorem 3 are P-integrable. The result now follows from the dominated convergence theorem. ∎

THEOREM 10. *If (i) $f_n \in P(a,b)$ for every $n \in \mathbf{N}$; (ii)* $\lim_{n\to\infty} f_n = f$ *a.e.; (iii) for every $\epsilon > 0$ there exists a δ: $[a,b] \to]0,\infty[$ such that for all $\pi \in \Pi(\delta)$ and for all $n \in \mathbf{N}$*

$$\left|\sum_{\pi} f_n - P\int_a^b f_n\right| < \epsilon \tag{8}$$

then $f \in P(a,b)$ *and*

$$P\int_a^b f = \lim_{n\to\infty} P\int_a^b f_n.$$

PROOF. It is immediate that (8) implies 5(i) and so BRS for $f_n, n \in \mathbf{N}$, uniformly in n.

In addition (8) implies the same result for any interval $[a,x]$, from which $F_n(x) = P\int_a^x f_n$ converges uniformly and so the theorem follows from Theorem 7. ■

This last theorem is due to Kurzweil [6, p. 41] who gives a different proof.

4. **The Marcinkiewicz Theorem and Differential Equations.** We first give a generalization of a classical result due to Caratheodory, see Coddington [3], for which we introduce the following notation.

If $x,y \in \mathbf{R}^n, x = (x_1,\dots,x_n), y = (y_1,\dots,y_n)$ then we write $x \leqq y$ iff $x_i \leqq y_i$ for $i = 1,2,\dots,n$. For $\xi \in \mathbf{R}^n, b \in \mathbf{R}^n$ let $[\xi - b, \xi + b]$ be the Cartesian product of the intervals $[\xi_i - b_i, \xi_i + b_i]$ for $i = 1,2,\dots,n$ and for $g\colon [a,b] \longrightarrow \mathbf{R}^n$ let $\bar{D}g(x) := [\bar{D}g_1(x),\dots,\bar{D}g_n(x)]$; similarly $\underline{D}g(x) := [\underline{D}g_1(x),\dots,\underline{D}g_n(x)]$. Further let $I = [\tau - a, \tau + a] \subset \mathbf{R}$, and $J =]\xi - b, \xi + b[\subset \mathbf{R}^n$.

Given $f\colon I \times J \longrightarrow \mathbf{R}, K$ a compact interval, $K \subset I$ and $g\colon K \longrightarrow J$ we define f_g by $f_g(t) = f(t,g(t)), t \in K$. Finally, we shall say that $g\colon K \longrightarrow \mathbf{R}^n$ in ACG$_*$ (on K) if each component of g is ACG$_*$ (on K).

THEOREM 11. *If* $f\colon I \times J \longrightarrow \mathbf{R}$ *is such that (i)* $f(t,.)$ *is continuous on* J *for almost all* $t \in J$; *(ii) there exists* $\beta > 0$ *and two continuous functions* $m, M\colon [\tau - \beta, \tau + \beta] \longrightarrow J$ *with* $m(\tau) = M(\tau) = 0$ *such that if* g *is a continuous* ACG$_*$ *function,* $g\colon [\tau - \beta, \tau + \beta] \longrightarrow J$, *with* $g(\tau) = \xi$ *then* f_g *is measurable and* $\bar{D}m \leqq f_g \leqq \underline{D}M$; *then there is a continuous* ACG$_*$ *function* ϕ *satisfying* $\phi(t) = \xi + \int_\tau^t f(s,\phi(s))ds$ *on* $[\tau - \beta, \tau + \beta]$.

REMARK. ϕ obviously satisfies

$$y' = f(x,y) \tag{9}$$

almost everywhere on $[\tau - \beta, \tau + \beta]$ and $\phi(\tau) = \xi$.

PROOF. As usual we assume $t \geqq \tau$, as the case $t \leqq \tau$ can be treated in a similar manner. On the interval $[\tau, \tau + \beta]$ we defined the approximations $\phi_j (j = 1,2,\dots)$ as follows:

$$\begin{aligned} &\phi_j(t) = \xi \text{ if } \tau \leqq t \leqq \tau + \frac{\beta}{j} \\ &\phi_j(t) = \xi + \int_\tau^{t-\beta/j} f_j, \text{ if } \tau + \frac{\beta}{j} < t \leqq \tau + \beta, \end{aligned} \tag{10}$$

where we write f_j for f_{ϕ_j}. The integral in (10) is a Perron integral whose existence follows from Hypothesis (ii) and Theorem 1. We prove this. First we define ϕ_j^1 by $\phi_j^1(t) = \xi$ on $[\tau, \tau + \beta]$. Then ϕ_j^1 is continuous ACG$_*$ and it follows from hypothesis (ii) that

$$P\int_\tau^{t-\beta/j} f_{\phi_j^1}$$

exists. Now we define ϕ_j^2 by $\phi_j^2(t) = \phi_j^1(t)$ on $[\tau, \tau + \beta/j]$,

$$\phi_j^2(t) = \xi + \int_\tau^{t-\beta/j} f_{\phi_j^1} \text{ for } t \in \left]\tau + \frac{\beta}{j}, \tau + \frac{2\beta}{j}\right] \text{ and}$$

$$\phi_j^2(t) = \phi_j^2\left(\tau + \frac{2\beta}{j}\right) \text{ for } t \in \left]\tau + \frac{2\beta}{j}, \tau + \beta\right].$$

Continuing this process finally gives $\phi_j^j\colon [\tau, \tau + \beta] \longrightarrow J$ and $\phi_j = \phi_j^j$ satisfies (10). It follows that ϕ_j is a continuous ACG$_*$ function on $[\tau, \tau + \beta]$.

Since M is a major function of f_j on $[\tau, \tau + \beta]$ and m a minor function we have for all $u, v, \tau \leqq u \leqq v \leqq \tau + \beta$

$$(11) \qquad m(v) - m(u) \leqq \phi_j(v) - \phi_j(u) \leqq M(v) - M(u)$$

In particular, taking $u = \tau$, we see from that $(\phi_j : j = 1, 2, \ldots)$ is uniformly bounded and equicontinuous. It follows from Ascoli's theorem that we can assume $\lim_{j'-\infty} \phi_j = \phi$, uniformly on $[\tau, \tau + \beta]$: and ϕ is continuous.

Further, by hypothesis (i), $\lim_{j-\infty} f_j = f_\phi$ a.e. Clearly from this, and (ii), $(f_j, j = 1, 2, \ldots)$ satisfies the conditions of the convergence theorem of Lee ([7] p. 20) quoted in the proof of Theorem 8. Hence

$$\lim_{j-\infty} \int_\tau^t f_j = \int_\tau^t f_\phi$$

from which the result is immediate. ∎

It has been shown, Vyborny [17], that this method of Tonelli can be modified to obtain the maximum solution of (9) in case $n = 1$. We now state and prove a generalization of that result. A function ϕ is said to be a maximum solution of equation (9) on $[\tau - \beta, \tau + \beta]$ satisfying $\phi(\tau) = \xi$ if (i) ϕ is a solution of equation (9) on $[\tau - \beta, \tau - \beta]$, (ii) $\phi(\tau) = \xi$, (iii) any solution ψ of (9) defined on some interval $[\tau - \gamma, \tau + \gamma]$ and satisfying $\psi(\tau) = \xi$ has the property that $\psi(t) \leqq \phi(t)$ for all t with $|t - \tau| < \operatorname{Min}(\beta, \gamma)$.

THEOREM 12. *If $f\colon I \times J \longrightarrow \mathbf{R}$ is such that (i) $\{f(t, .); t \in I\}$ is equicontinuous; (ii) for all continuous ACG$_*$ $g\colon I \longrightarrow \mathbf{R}$ the function f_g is a derivative; (iii) as (ii) in Theorem 11; Then there is a maximum solution ψ of (9) on $[\tau - \beta, \tau + \beta]$ satisfying $\phi(\tau) = \xi$.*

PROOF. As in the proof of Theorem 11 we restrict our discussion to $[\tau, \tau + \beta]$. Following the idea of [17] we define approximations as follows:

$$\phi_n(t) = \xi, \; t \leqq \tau$$

$$\phi_n(t) = \xi + \int_\tau^t f(t, \; \phi_n(t - h_n))dt + (2/4^n)(t - \tau)$$

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where we choose h_n later, but in any case $h_n > 0$, and the above definition easily is seen to define ϕ_n on $[\tau, \tau + \beta]$. As in Theorem 11, the integrals are Perron integrals and ϕ_j is continuous and ACG$_*$. Also we have, with the notation of (11)

$$m(v) - m(u) + (2/4^n(v-u) \leqq \phi_n(v) - \phi_n(u) \leqq M(v) - M(u) + (2/4^n)(v-u)$$

Hence

$$(12) \qquad |\phi_n(v) - \phi_n(u)| \leqq (2/4^n)|v-u| + \max\{|M(v) - M(u)|, |m(v) - m(u)|\}$$

Now given $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for all $t \in I$ $|f(t,x) - f(t,x')| < \epsilon$ if $|x - x'| < \delta(\epsilon)$. Using (12) choose h_n so that

$$|\phi_n(t - h_n) - \phi_n(t)| < \delta(1/4^n)$$

for then

$$|f(t, \phi_n(t-h_n)) - f(t, \phi_n(t))| < (1/4^n).$$

Hence we easily see that

$$\phi_n'(t) > f(t, \phi_n(t)) + (1/4^n),$$

$$\phi_{n+1}'(t) < f(t, \phi_{n+1}(t)) + (1/4^n),$$

The derivatives existing everywhere by Hypothesis (ii). Hence by a well-known lemma, Vyborny [17], $\phi_n > \phi_{n+1}$. Since $\{\phi_n, n = 1, 2, \ldots\}$ is, as in Theorem 11, uniformly bounded, and so in particular bounded below, $\lim_{n\to\infty} \phi_n = \phi$, uniformly. The proof that ϕ satisfies (9) now proceeds as in Theorem 11. If ψ is another solution of (9) then by the just quoted lemma $\phi_n \geqq \psi$ and consequently $\phi \geqq \psi$. ϕ is the maximum solution. ∎

One can of course define a minimum solution and prove and analogue of Theorem 12.

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