On the Asymptotic Growth of Bloch–Kato–Shafarevich–Tate Groups of Modular Forms Over Cyclotomic Extensions

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Abstract. We study the asymptotic behaviour of the Bloch–Kato–Shafarevich–Tate group of a modular form f over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ under the assumption that f is non-ordinary at p. In particular, we give upper bounds of these groups in terms of Iwasawa invariants of Selmer groups defined using p-adic Hodge Theory. These bounds have the same form as the formulae of Kobayashi, Kurihara, and Sprung for supersingular elliptic curves.

1 Introduction

Let p be an odd prime, let f be a normalised new cuspidal modular eigenform of weight $k \geq 2$, and let p an odd prime that does not divide the level of f. For notational simplicity, we assume in this introduction that all the Fourier coefficients of f lie in $\mathbb{Z}$. We let V_f be the *cohomological p-adic Galois representation* attached to f (so the determinant of V_f is χ^{1-k} times a finite-order character). Then V_f has Hodge–Tate weights $\{0, 1 - k\}$, where our convention[1] is that the Hodge–Tate weight of the cyclotomic character is 1. Let T_f be the canonical $G_{\mathbb{Q}}$-stable $\mathbb{Z}_p$-lattice in V_f defined by Kato [Kat04, 8.3].

Let K_∞ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and write K_n for the unique sub-extension of degree p^n. Our aim is to study the asymptotic behaviour of the Bloch–Kato–Shafarevich–Tate groups $\mathrm{III}(K_n, T_f(j))$ (with $j \in [1, k-1]$), whose definition we recall below.

When $k = 2$, the form f corresponds to an isogeny class of elliptic curves, and we can choose a curve $\mathcal{E}$ in this isogeny class such that $T_f(1) = T_p(\mathcal{E})$, where the latter is the p-adic Tate module of $\mathcal{E}$. In this case it can be shown that the group $\mathrm{III}(K_n, T_f(1))$ is the quotient of the classical p-primary Shafarevich–Tate group $\mathrm{III}_p(K_n, \mathcal{E})$ by its

[1]This is usual in p-adic Hodge theory, but the opposite convention appears to be common in papers on modularity lifting.

maximal divisible subgroup; hence if the latter group is finite (which is a well-known conjecture), the two groups are equal.

The Ordinary Case

The behaviour of the Selmer and Shafarevich–Tate groups over the cyclotomic extension depends sharply on whether $\mathcal{E}$ has ordinary or supersingular reduction at p. If $\mathcal{E}$ is ordinary, then the p-Selmer group

$$\operatorname{Sel}_p(K_\infty, \mathcal{E}) = \varinjlim_n \operatorname{Sel}_p(K_n, \mathcal{E})$$

of A over K_∞ is cotorsion over the Iwasawa algebra $\mathbb{Z}_p[\![\operatorname{Gal}(K_\infty/\mathbb{Q})]\!]$, by a theorem of Kato [Kat04, Theorem 17.4]. By Mazur's control theorem [Maz72], this implies that **if** the groups $\mathrm{III}_p(K_n, \mathcal{E})$ are finite for all n, then we must have

$$\operatorname{len}_{\mathbb{Z}_p} \mathrm{III}_p(K_n, \mathcal{E}) = \mu p^n + \lambda n + O(1)$$

for some Iwasawa invariants μ and λ associated with $\operatorname{Sel}_p(\mathcal{E}/K_\infty)$.

The Supersingular Case

The case of supersingular elliptic curves with $a_p(\mathcal{E}) = 0$ has been studied by Kurihara [Kur02] and Kobayashi [Kob03]. Suppose that $\mathrm{III}_p(K_n, \mathcal{E})$ is finite for all n and write $s_n(\mathcal{E}) = \operatorname{len}_{\mathbb{Z}_p} \mathrm{III}_p(K_n, \mathcal{E})$. They showed that for n sufficiently large,

$$s_n(\mathcal{E}) - s_{n-1}(\mathcal{E}) = q_n + \lambda_\pm + \mu_\pm(p^n - p^{n-1}) - r_\infty(\mathcal{E}),$$

where q_n is an explicit sum of powers of p, $r_\infty(\mathcal{E})$ is the rank of $\mathcal{E}$ over K_∞, $\lambda_\pm$, and $\mu_\pm$ are the Iwasawa invariants of some cotorsion signed Selmer groups, and the sign $\pm$ depends on the parity of n.

For supersingular elliptic curves with $a_p(\mathcal{E}) \neq 0$ (which can only occur when $p = 2$ or 3), Sprung [Spr13] proved a similar formula:

$$s_n(\mathcal{E}) - s_{n-1}(\mathcal{E}) = q_n^\star + \lambda_\star + \mu_\star(p^n - p^{n-1}) - r_\infty(\mathcal{E}),$$

for $n \gg 0$, where $q_n^\star$ is again an explicit sum of powers of p, $\star \in \{\#, \flat\}$, $\lambda_\star$ and $\mu_\star$ are Iwasawa invariants of some cotorsion Selmer groups defined in [Spr12] and the choice of $\star$ depends on the "modesty algorithm". An analytic version of this formula has been generalised to arbitrary weight 2 modular forms in [Spr15].

Higher Weights

The main result of this article is that a similar formula for modular forms of higher weight would give us an upper bound on the growth of the Bloch–Kato–Shafarevich–Tate groups. Suppose that $\operatorname{ord}_p(a_p(f)) > \frac{k-1}{2p}$ and $3 \leq k \leq p$, where $a_p(f)$ is the p-th Fourier coefficient of the modular form f. We will see below that the Selmer coranks

$$r_n(f) = \operatorname{corank}_{\mathbb{Z}_p} \operatorname{Sel}(K_n, T_f(j))$$

stabilise for $n \gg 0$, and we define $r_\infty(f)$ to be the limiting value (see Proposition 5.4). We define

$$s_n(f) = \operatorname{len}_{\mathbb{Z}_p} \mathrm{III}_p(K_n, T_f(j))$$

(which is finite by definition). We prove the inequality (see Theorem 5.5 for the precise statement)

$$s_n(f) - s_{n-1}(f) \le q_n^\star + \lambda_\star + \mu_\star(p^n - p^{n-1}) + \kappa - r_\infty(f),$$

for $n \gg 0$, where $q_n^\star$ is once again a sum of powers of p that depends on k and the parity of n, $\lambda_\star$ and $\mu_\star$ are the Iwasawa invariants of the Selmer groups defined in [LLZ10] for some choice of basis of the Wach module of T_f, κ is some integer that depends on the image of some Coleman maps that we shall review in §3 of this article and the choice of $\star$ is given by an explicit algorithm (similar to the "modesty algorithm" of Sprung).

The fact that we have an inequality is a result of the growth of the logarithmic matrix contributed from the twists of $T_f(i)$ for $i \ne j$. In the appendix to this paper, we relate the defect of this inequality to the Tamagamwa numbers of $T_f(j)$ using the method developed by Perrin–Riou in [PR03].

2 Background from p-adic Hodge Theory

We recall the necessary notation and definitions from p-adic analysis and p-adic Hodge theory. For more details, see [LLZ11, §1.3]. We fix (for the duration of this article) a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, which will be the coefficient field for all the representations we will consider.

2.1 Iwasawa Algebras and Distribution Algebras

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. This group is isomorphic to a direct product $\Delta \times \Gamma_1$, where Δ is a finite group of order $p-1$ and $\Gamma_1 = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. We choose a topological generator γ of Γ_1, which determines an isomorphism $\Gamma_1 \cong \mathbb{Z}_p$. We also fix a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ that will be our field of coefficients (*i.e.*, we will consider representations of Galois groups on E-vector spaces).

We write $\Lambda = \mathcal{O}[[\Gamma]]$, the Iwasawa algebra of Γ. The subalgebra $\mathcal{O}[[\Gamma_1]]$ can be identified with the formal power series ring $\mathcal{O}[[X]]$, via the isomorphism sending γ_1 to $1 + X$; this extends to an isomorphism

$$(2.1) \qquad\qquad\qquad \Lambda = \mathcal{O}[\Delta][[X]].$$

For a character η of Δ and a Λ-module M, M^η denotes its η-isotypic component, which is regarded as an $\mathcal{O}[[X]]$-module. For $n \ge 1$, we write Γ_n for the subgroup $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_{p^n}))$ and $\Lambda_n = \mathcal{O}[\Gamma/\Gamma_n]$. Note that

$$\Lambda_n = \mathcal{O}[\Delta][[X]]/(\omega_{n-1}(X)),$$

where $\omega_{n-1}(X)$ denotes the polynomial $(1 + X)^{p^{n-1}} - 1$.

We can consider Λ as a subring of the ring $\mathcal{H}$ of locally analytic E-valued distributions on Γ. The isomorphism (2.1) extends to an identification between $\mathcal{H}$ and the subring of power series $F \in E[\Delta][[X]]$, which converge on the open unit disc $|X| < 1$.

2.2 Power Series Rings

Let $\mathbb{A}^+_{\mathbb{Q}_p} = \mathcal{O}[\![\pi]\!]$, where π is a formal variable. We equip this ring with a $\mathcal{O}$-linear *Frobenius endomorphism* φ, defined by $\pi \mapsto (1+\pi)^p - 1$, and with an $\mathcal{O}$-linear action of Γ defined by $\pi \mapsto (1+\pi)^{\chi(\sigma)} - 1$ for $\sigma \in \Gamma$, where χ denotes the p-adic cyclotomic character.

The Frobenius φ has a left inverse ψ, satisfying

$$(\varphi \circ \psi)(f)(\pi) = \tfrac{1}{p} \sum_{\zeta : \zeta^p = 1} f\big(\zeta(1+\pi) - 1\big).$$

The map ψ is not a morphism of rings, but it is $\mathcal{O}$-linear, and commutes with the action of Γ.

We write $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{A}^+_{\mathbb{Q}_p}[1/p] \subset E[\![\pi]\!]$, and

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} = \big\{ F(\pi) \in E[\![\pi]\!] : F \text{ converges on the open unit disc} \big\},$$

so there are natural inclusions

$$\mathbb{A}^+_{\mathbb{Q}_p} \hookrightarrow \mathbb{B}^+_{\mathbb{Q}_p} \hookrightarrow \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}.$$

The actions of φ, ψ, and Γ extend to these larger rings (via the same formulae as before). We shall write $q = \varphi(\pi)/\pi \in \mathbb{A}^+_{\mathbb{Q}_p}$ and $t = \log(1+\pi) \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$.

2.3 The Mellin Transform

The action of Γ on $1 + \pi \in \big(\mathbb{A}^+_{\mathbb{Q}_p}\big)^{\psi=0}$ extends to an isomorphism of Λ-modules

$$\mathfrak{M} \colon \Lambda \xrightarrow{\cong} \big(\mathbb{A}^+_{\mathbb{Q}_p}\big)^{\psi=0}$$
$$1 \longmapsto 1 + \pi,$$

called the *Mellin transform*. This can be further extended to an isomorphism of $\mathcal{H}$-modules

$$\mathcal{H} \xrightarrow{\cong} \big(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\big)^{\psi=0},$$

which we denote by the same symbol.

Theorem 2.1 *For every $n \geq 1$, the Mellin transform induces an isomorphism of Λ-modules*

$$\Lambda_n \cong \big(\mathbb{A}^+_{\mathbb{Q}_p}\big)^{\psi=0} / \varphi^n(\pi)\big(\mathbb{A}^+_{\mathbb{Q}_p}\big)^{\psi=0}.$$

Proof If $\mu \in \omega_{n-1}(X)\Lambda$, then $\mathfrak{M}(\mu) \in \varphi^n(\pi)\big(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\big)^{\psi=0}$, by [LLZ10, Theorem 5.4]. However, $\varphi^n(\pi)$ is a monic polynomial in π, so if an element of $\mathbb{A}^+_{\mathbb{Q}_p}$ is divisible by $\varphi^n(\pi)$ in $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, it is divisible by the same element in $\mathbb{A}^+_{\mathbb{Q}_p}$. Hence, the Mellin transform induces a map $\Lambda_n \to \big(\mathbb{A}^+_{\mathbb{Q}_p}\big)^{\psi=0} / \varphi^n(\pi)\big(\mathbb{A}^+_{\mathbb{Q}_p}\big)^{\psi=0}$, and this map is surjective, because the Mellin transform itself is surjective. Since both sides are free $\mathcal{O}$-modules of the same rank, namely $(p-1)p^n$, it follows that the map must in fact be an isomorphism. ∎

We write ∂ for the differential operator $(1 + \pi)\frac{\mathrm{d}}{\mathrm{d}\pi}$ on $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and Tw for the ring automorphism of $\mathcal{H}$ defined by $\sigma \mapsto \chi(\sigma)\sigma$ for $\sigma \in \Gamma$. Then one has the compatibility relation $\mathfrak{M} \circ \mathrm{Tw} = \partial \circ \mathfrak{M}$.

Let $u = \chi(\gamma)$ be the image of our topological generator γ under the cyclotomic character, so that Tw maps X to $u(1 + X) - 1$. If $m \geq 0$ is an integer, we define $\omega_{n,m}(X) = \omega_n(u^{-m}(1 + X) - 1)$ and $\widetilde{\omega}_{n,m} = \prod_{i=0}^{m} \omega_{n,i}$. By exactly the same argument as Theorem 2.1, this gives the following isomorphism of Λ-modules:

$$(2.2) \qquad \Lambda_{n,m} \coloneqq \Lambda / \widetilde{\omega}_{n-1,m}\Lambda \cong (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0} / \varphi^n(\pi^{m+1})(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}.$$

We will need below the following technical result, regarding the interaction between Mellin transforms and the Iwasawa invariants of power series. We recall the *Weierstrass preparation theorem*, which states that any $F \in \mathcal{O}[[X]]$ can be factorized uniquely as

$$F(X) = \varpi^{\mu(F)} \cdot \left(X^{\lambda(F)} + \varpi G(X) \right) \cdot u(X),$$

where ϖ is a uniformizer of $\mathcal{O}$, $\lambda(F)$ and $\mu(F)$ are non-negative integers, $G \in \mathcal{O}[X]$ is a polynomial of degree $< \lambda(F)$, and $u \in \mathcal{O}[[X]]^{\times}$. The quantities $\lambda(F)$ and $\mu(F)$ are called the *Iwasawa invariants* of F.

It is clear that, for $x \in \mathcal{O}_{\mathbf{C}_p}$ with $\mathrm{ord}_p(x) > 0$, we have the lower bound

$$(2.3) \qquad \mathrm{ord}_p F(x) \geq \min\left(\tfrac{\mu+1}{e}, \tfrac{\mu}{e} + \lambda\,\mathrm{ord}_p(x) \right),$$

where $e = 1/\mathrm{ord}_p(\varpi)$ is the absolute ramification degree of F. Moreover, if $\mathrm{ord}_p(x)$ is sufficiently small (depending on F), this lower bound is an equality (it suffices to take $\mathrm{ord}_p(x) < 1/(e\lambda)$).

Proposition 2.2   *Let $f \in \mathbb{A}^+_{\mathbb{Q}_p}$, and let g be the unique element of $\Lambda(\Gamma_1)$ such that $\mathfrak{M}(g) = (1 + \pi)\varphi(f)$. Then the λ- and μ-invariants of f (as an element of $\mathcal{O}[[\pi]]$) coincide with those of g (as an element of $\mathcal{O}[[X]]$).*

Proof   This is a consequence of [LZ13, Proposition 7.2], which shows that for any $f \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and $g \in \mathcal{H}$ such that $\mathfrak{M}(g) = (1 + \pi)\varphi(f)$, and any real s with $0 < s < 1$, we have $v_s(f) = v_s(g)$, where

$$v_s(f) \coloneqq \inf\left\{ \mathrm{ord}_p f(x) : \mathrm{ord}_p(x) \geq s \right\}.$$

When $f \in \mathcal{O}[[X]]$ and s is sufficiently small, $v_s(f)$ is determined by the Iwasawa invariants of f; from the inequality (2.3) and the discussion following, we have $v_s(f) = \frac{1}{e}\mu(f) + \lambda(f)s$ for any $s < \frac{1}{e\lambda(f)}$. So the cited proposition implies the equalities $\lambda(f) = \lambda(g)$ and $\mu(f) = \mu(g)$. ∎

2.4   Crystalline Representations and Wach Modules

Fontaine has defined a certain topological $\mathbb{Q}_p$-algebra $\mathbb{B}_{\mathrm{cris}}$, equipped with an action of $G_{\mathbb{Q}_p}$, a filtration $\mathrm{Fil}^{\bullet}$, and a Frobenius endomorphism φ.

For any p-adic representation V of $G_{\mathbb{Q}_p}$, we define the *crystalline Dieudonné module* of V by $\mathbb{D}_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}})^{G_{\mathbb{Q}_p}}$. The space $\mathbb{D}_{\mathrm{cris}}(V)$ inherits a filtration and a Frobenius endomorphism from those of $\mathbb{B}_{\mathrm{cris}}$. It is known that $\dim_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V) \leq$

$\dim_{\mathbb{Q}_p} V$, and we say V is *crystalline* if equality holds. If in fact V is an E-linear representation, then $\mathbb{D}_{\mathrm{cris}}(V)$ is naturally an E-vector space (and its filtration and Frobenius are E-linear).

Definition 2.3 Let $a \leq b$ be integers. A *Wach module* over $\mathbb{B}^+_{\mathbb{Q}_p}$ with weights in $[a, b]$ is a finite free $\mathbb{B}^+_{\mathbb{Q}_p}$-module N, equipped with an action of Γ and a Frobenius

$$\varphi \colon N[1/\pi] \to N[1/\varphi(\pi)],$$

compatible with those of $\mathbb{B}^+_{\mathbb{Q}_p}$, satisfying the following conditions:

- Γ acts trivially on $N/\pi N$;
- $\varphi(\pi^b N) \subseteq \pi^b N$;
- if $\varphi^*(\pi^b N)$ is the $\mathbb{B}^+_{\mathbb{Q}_p}$-submodule of $\pi^b N$ generated by $\varphi(\pi^b N)$, then the quotient $\pi^b N / \varphi^*(\pi^b N)$ is killed by q^{b-a}.

In [Ber04, Definition III.4.1] it is shown how to attach to every crystalline E-linear representation V of $G_{\mathbb{Q}_p}$ a Wach module $\mathbb{N}(V)$ over $\mathbb{B}^+_{\mathbb{Q}_p}$, in such a way that there is a canonical isomorphism

$$\mathbb{N}(V) \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t] \cong \mathbb{D}_{\mathrm{cris}}(V) \otimes_E \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t].$$

Moreover, the definition of Wach modules also makes sense integrally, *i.e.*, over $\mathbb{A}^+_{\mathbb{Q}_p}$, and we can associate with each $\mathcal{O}$-lattice T in V that is stable under $G_{\mathbb{Q}_p}$ an integral Wach module $\mathbb{N}(T) \subset \mathbb{N}(V)$ (Lemme II.1.3 of *op.cit.*).

Definition 2.4 We say V satisfies the *Fontaine–Laffaille condition* if it is crystalline and has Hodge–Tate weights in $[a, a + (p-1)]$ for some $a \in \mathbb{Z}$.

If V satisfies the Fontaine–Laffaille condition, and V is irreducible of dimension ≥ 2, then one has a particularly convenient parametrisation of $G_{\mathbb{Q}_p}$-stable lattices in V. We say a $\mathcal{O}$-lattice $M \subset \mathbb{D}_{\mathrm{cris}}(V)$ is a *strongly divisible lattice* if the equality

$$\varphi\big(M \cap \mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(V)\big) \subset p^i M$$

holds for all $i \in \mathbb{Z}$. Then there is a bijection $T \mapsto \mathbb{D}_{\mathrm{cris}}(T)$ between $G_{\mathbb{Q}_p}$-stable lattices in V, and strongly divisible lattices in $\mathbb{D}_{\mathrm{cris}}(V)$, given by defining $\mathbb{D}_{\mathrm{cris}}(T)$ to be the image of $\mathbb{N}(T)$ in $\mathbb{N}(V)/\pi\mathbb{N}(V) \cong \mathbb{D}_{\mathrm{cris}}(V)$; *cf.* [Ber04, Propositions V.2.1 & V.2.3].

We shall need the following technical result.

Theorem 2.5 *Let T be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice in a crystalline E-linear representation V. Then $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is a free Λ-module of rank $d = \dim_E V$. Moreover, if $\{n_1, \ldots, n_d\}$ is an $\mathbb{A}^+_{\mathbb{Q}_p}$-basis of $\mathbb{N}(T)$ that satisfies the condition*

$$(\gamma - 1)n_i \in \pi^2 \mathbb{N}(T)$$

for all i, then $\{(1 + \pi)\varphi(n_i) : i = 1, \ldots, d\}$ is a Λ-module basis of $(\varphi^*\mathbb{N}(T))^{\psi=0}$.

Proof This is shown in the course of the proof of [LLZ10, Theorem 3.5]. The condition on the basis modulo π^2 is the conclusion of Lemma 3.9 in *op. cit.* ∎

2.5 Iwasawa Cohomology and the Fontaine Isomorphism

If V is an E-linear p-adic representation of $G_{\mathbb{Q}_p}$, and $T \subset V$ is a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice, then we define *Iwasawa cohomology* groups by

$$H^i_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) = \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T)$$

(where the inverse limit is with respect to the corestriction maps). These groups are finitely-generated Λ-modules, zero unless $i \in \{1, 2\}$. If $H^0(\mathbb{Q}_p(\mu_{p^\infty}), T/pT) = 0$, which is the case in our applications below, then H^2_{Iw} is zero, and H^1_{Iw} is a free Λ-module of rank equal to the $\mathcal{O}$-rank of T.

The following theorem is the starting-point for our study of Iwasawa cohomology.

Theorem 2.6 (Fontaine–Berger) *If V is crystalline with all Hodge–Tate weights ≥ 0, and V has no non-zero quotient on which $G_{\mathbb{Q}_p}$ acts trivially, then there is a canonical Λ-module isomorphism*

$$h^1_T \colon \mathbb{N}(T)^{\psi=1} \longrightarrow H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}), T\big).$$

See [CC99, §II.1], where it is shown that (for any T) there is an isomorphism $\mathbb{D}(T)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)$, where $\mathbb{D}(T)$ is the (φ, Γ)-module of T; and [Ber03, §A], where it is shown that $\mathbb{N}(T)^{\psi=1} = \mathbb{D}(T)^{\psi=1}$ under the above hypotheses.

3 Wach Modules and Coleman Maps

3.1 Review on the Definition of Coleman Maps

Let $f = \sum a_n q^n$ be a normalised new cuspidal modular eigenform of weight $k \geq 3$ (note that the case $k = 2$ can be dealt with using the method of Sprung in [Spr13]), nebentypus ε and level N with $(p, N) = 1$. We take E to be the completion of the smallest number field containing all the coefficients of f at some fixed prime above p. We assume that f is non-ordinary at p and that $k \leq p$. We write T_f for the $\mathcal{O}$-linear representation of $G_{\mathbb{Q}}$ associated with f as defined by Kato [Kat04, 8.3]. It is crystalline, with Hodge–Tate weights 0 and $1 - k$. We fix an integer $j \in [1, k-1]$ and write $T = T_f(j)$ and $\mathcal{T} = T_f(k-1)$. Note that $T = \mathcal{T}(j - k + 1)$.

The representation $T/\varpi T$ (where ϖ is a uniformiser of $\mathcal{O}$) is irreducible as a representation of $G_{\mathbb{Q}_p}$, so, in particular, we have

$$H^0(\mathbb{Q}_p(\mu_{p^\infty}), T/\varpi T) = 0.$$

Both T_f and $\mathcal{T}$ are $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattices in crystalline representations of $G_{\mathbb{Q}_p}$, so we can consider their Wach modules and Dieudonné modules. By [Ber04, Proposition III.2.1], there are inclusions of $\mathbb{B}^+_{\mathrm{rig}, \mathbb{Q}_p}$-modules

$$\mathbb{B}^+_{\mathrm{rig}, \mathbb{Q}_p} \otimes_{\mathbb{A}^+_{\mathbb{Q}_p}} \mathbb{N}(\mathcal{T}) \subset \mathbb{B}^+_{\mathrm{rig}, \mathbb{Q}_p} \otimes_{\mathcal{O}} \mathbb{D}_{\mathrm{cris}}(\mathcal{T}),$$

$$\mathbb{B}^+_{\mathrm{rig}, \mathbb{Q}_p} \otimes_{\mathcal{O}} \mathbb{D}_{\mathrm{cris}}(T_f) \subset \mathbb{B}^+_{\mathrm{rig}, \mathbb{Q}_p} \otimes_{\mathbb{A}^+_{\mathbb{Q}_p}} \mathbb{N}(T_f),$$

where the elementary divisors of the inclusions are given by 1 and $(t/\pi)^{k-1}$ in both cases.

Lemma 3.1 *There exists an $\mathcal{O}$-basis $\mathfrak{v}_1, \mathfrak{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ such that $\mathfrak{v}_1 \in \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ and $\mathfrak{v}_2 = \varphi(\mathfrak{v}_1)$, where φ is the Frobenius action on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$.*

Proof The Fontaine–Laffaille condition of [FL82] implies that for all integers i

(a) $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$;
(b) $\varphi(\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})) \subset p^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$;
(c) $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \sum_i p^{-i} \varphi(\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T}))$.

The Hodge–Tate weights of $\mathcal{T}$ are 0 and $k-1$, so $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ is of rank 1, say $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \mathfrak{v}_1$ and (b) says that $\mathfrak{v}_2 := \varphi(\mathfrak{v}_1) \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$. Furthermore, (a) tells us that there exists some $\mathfrak{v}' \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ such that

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \mathfrak{v}_1 \oplus \mathcal{O} \cdot \mathfrak{v}'.$$

By (c), we have

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \varphi(\mathfrak{v}_1) + p^{k-1} \varphi(\mathbb{D}_{\mathrm{cris}}(\mathcal{T})).$$

Combing the last two equations gives

$$(3.1) \qquad \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \varphi(\mathfrak{v}_1) \oplus \mathcal{O} \cdot p^{k-1} \varphi(\mathfrak{v}').$$

Let D be the $\mathcal{O}$-lattice generated by $\widehat{A} - \mathfrak{v}_1$ and $\mathfrak{v}_2$. Note that (3.1) implies that

$$(3.2) \qquad \mathfrak{v}' \in D + \mathcal{O} \cdot p^{k-1} \varphi(\mathfrak{v}').$$

As $\mathfrak{v}_2 = \varphi(\mathfrak{v}_1)$ and

$$\varphi^2 - \frac{a_p}{p^{k-1}} \varphi + \frac{\varepsilon(p)}{p^{k-1}} = 0$$

on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, we have $p^{k-1} \varphi(\mathfrak{v}_2) = a_p \mathfrak{v}_2 - \varepsilon(p) \mathfrak{v}_1$. In particular, this implies that $p^{k-1} \varphi(D) \subset D$. Hence, we may iterate the inclusion (3.2) to deduce that

$$\mathfrak{v}' \in D + \mathcal{O} \cdot (p^{k-1} \varphi)^n (\mathfrak{v}')$$

for all $n \geq 0$. However, as f is non-ordinary at p, $p^{k-1} \varphi$ is an $\mathcal{O}$-operator on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ with strictly positive slope. This implies that $(p^{k-1} \varphi)^n \to 0$ as $n \to \infty$, which forces that $\mathfrak{v}' \in D$. Hence, $D = \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, as required. ∎

We fix an $\mathcal{O}$-basis $\mathfrak{v}_1, \mathfrak{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, as given by Lemma 3.1. Since $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathbb{N}(\mathcal{T})/\pi \mathbb{N}(\mathcal{T})$, this basis can be lifted to a basis $\mathfrak{n}_1, \mathfrak{n}_2$ of $\mathbb{N}(\mathcal{T})$ as an $\mathbb{A}_{\mathbb{Q}_p}^+$-module. There is a change of basis matrix $M \in M_{2 \times 2}(\mathbb{B}_{\mathrm{rig}, \mathbb{Q}_p}^+)$ such that

$$(3.3) \qquad \begin{pmatrix} \mathfrak{n}_1 & \mathfrak{n}_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} M$$

and $M \equiv I_2 \mod \pi$, where I_2 is the 2×2 identity matrix. We write

$$v_i = \mathfrak{v}_i \cdot t^{k-j-1} e_{-k+j+1}, \qquad n_i = \mathfrak{n}_i \cdot \pi^{k-j-1} e_{-k+j+1},$$
$$v_{f,i} = \mathfrak{v}_i \cdot t^{k-1} e_{1-k}, \qquad n_{f,i} = \mathfrak{n}_i \cdot \pi^{k-1} e_{1-k}$$

for the corresponding bases of $\mathbb{D}_{\mathrm{cris}}(T)$, $\mathbb{N}(T)$, $\mathbb{D}_{\mathrm{cris}}(T_f)$, and $\mathbb{N}(T_f)$ respectively. Here e_r denotes a basis of the Tate motive $\mathcal{O}(\chi^r)$ for $r \in \mathbb{Z}$. By [Ber04, proof of Proposition V.2.3] and [Lei15, Proposition 4.2], we can choose our bases so that

$$(3.4) \qquad M \equiv I_2 \mod \pi^{k-1}$$

and that the matrices of φ with respect to $v_{1,f}, v_{2,f}$ and $n_{1,f}, n_{2,f}$ are given by

$$\begin{pmatrix} 0 & -\varepsilon(p) \\ p^{k-1} & a_p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\varepsilon(p) \\ (\delta q)^{k-1} & a_p \end{pmatrix},$$

respectively, where $\delta = p/(q - \pi^{p-1}) \in (\mathbb{A}_{\mathbb{Q}_p}^+)^\times$. Then the matrices of φ with respect to $\mathfrak{v}_1, \mathfrak{v}_2$ and $\mathfrak{n}_1, \mathfrak{n}_2$ are given by

$$A = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{p^{k-1}} \\ 1 & \frac{a_p}{p^{k-1}} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{q^{k-1}} \\ \delta^{k-1} & \frac{a_p}{q^{k-1}} \end{pmatrix}.$$

Definition 3.2   We define the *logarithmic matrix* $M_{\log}$ (with respect to the chosen bases) to be $\mathfrak{M}^{-1}((1+\pi)A\varphi(M))$.

Theorem 3.3   *Let $\mathfrak{n}_1, \mathfrak{n}_2$ be the basis of $\mathbb{N}(\mathcal{T})$ chosen above. Then $(1+\pi)\varphi(\mathfrak{n}_1)$ and $(1+\pi)\varphi(\mathfrak{n}_2)$ form a Λ-basis of $(\varphi^*\mathbb{N}(\mathcal{T}))^{\psi=0}$.*

Proof   Let $\gamma \in \Gamma$ be a topological generator. Then (3.3) tells us that

$$\begin{pmatrix} \gamma \cdot \mathfrak{n}_1 & \gamma \cdot \mathfrak{n}_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \gamma(M).$$

This gives the equation

$$\begin{pmatrix} \gamma \cdot \mathfrak{n}_1 & \gamma \cdot \mathfrak{n}_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{n}_1 & \mathfrak{n}_2 \end{pmatrix} M^{-1} \cdot \gamma(M).$$

Hence, for both $i = 1, 2$, we have $(1-\gamma)\mathfrak{n}_i \in \pi^{k-1}\mathbb{N}(\mathcal{T})$ thanks to (3.4). As we assume that $k \geq 3$, we have, in particular,

$$(1-\gamma)\mathfrak{n}_i \in \pi^2\mathbb{N}(\mathcal{T}),$$

which is the condition required in Theorem 2.5[2]. Therefore, our result follows.   ∎

Recall from [LLZ10, Remark 3.4] that for all $z \in \mathbb{N}(\mathcal{T})^{\psi=1}$, we have $(1-\varphi)z \in (\varphi^*\mathbb{N}(\mathcal{T}))^{\psi=0}$. The latter is free of rank 2 over Λ, with basis $(1+\pi)\varphi(\mathfrak{n}_1), (1+\pi)\varphi(\mathfrak{n}_2)$ as given by Theorem 3.3. This allows us to define the Coleman maps (again, with respect to our chosen bases) as follows.

Definition 3.4   For $i \in \{1, 2\}$, we define the Λ-homomorphisms $\mathrm{Col}_i \colon \mathbb{N}(\mathcal{T})^{\psi=1} \to \Lambda$ given by the relation

$$(1-\varphi)z = \sum_{i=1}^{2} \mathrm{Col}_i(z) \cdot (1+\pi)\varphi(\mathfrak{n}_i) = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot M_{\log} \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix}.$$

Let $h_{\mathcal{T}}^1 \colon \mathbb{N}(\mathcal{T})^{\psi=1} \to H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$ be the Λ-isomorphism given by Theorem 2.6. By an abuse of notation, we shall write $\mathrm{Col}_1, \mathrm{Col}_2$ for the compositions $\mathrm{Col}_1 \circ (h_{\mathcal{T}}^1)^{-1}$ and $\mathrm{Col}_2 \circ (h_{\mathcal{T}}^1)^{-1}$ as well.

---

[2]This is the only place where we use the assumption that $k \geq 3$.

3.2 A Finite Projection of the Coleman Maps

Definition 3.5 For each $n \geq 1$, we define $H_n = \varphi^{n-1}(P^{-1})\cdots\varphi(P^{-1})$ and $\mathscr{H}_n = \mathfrak{M}^{-1}((1+\pi)H_n)$.

Remark 3.6 Note that $H_n \in \mathbb{A}_{\mathbb{Q}_p}^+$, and $\mathscr{H}_n \in \Lambda$; and $H_1 = \mathscr{H}_1 = 1$.

Lemma 3.7 *We have the congruence*

$$M_{\log} \equiv A^n \cdot \mathscr{H}_n \mod \widetilde{\omega}_{n-1,k-2}(X)\mathcal{H}.$$

Proof From (3.3), we have the relation $MP = A\varphi(M)$, which we can rewrite as $M = A\varphi(M)P^{-1}$. On iteration, we have

$$M = A^{n-1}\varphi^{n-1}(M)\varphi^{n-2}(P^{-1})\cdots\varphi(P^{-1})P^{-1}.$$

By (3.4), we have $\varphi^{n-1}(M) = 1 \mod \varphi^{n-1}(\pi^{k-1})$, so this implies that

$$M \equiv A^{n-1}\varphi^{n-2}(P^{-1})\cdots\varphi(P^{-1})P^{-1} \mod \varphi^{n-1}(\pi^{k-1}).$$

This implies that

$$\varphi(M) \equiv A^{n-1} \cdot H_n \mod \varphi^n(\pi^{k-1}).$$

Hence, we have the result by (2.2). ∎

Lemma 3.8 *For all $n \geq 1$ and $z \in \mathbb{N}(\mathcal{T})^{\psi=1}$, $(1 \otimes \varphi^{-n}) \circ (1-\varphi)z$ is congruent to an element in $\Lambda_{n,k-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ modulo $\widetilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$.*

Proof By Lemma 3.7 and the equation in Definition 3.4, we have the congruence

$$(1-\varphi)z \equiv \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot A^n \cdot \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix} \mod \widetilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}).$$

If we apply $(1 \otimes \varphi^{-n})$ to both sides, we obtain

$$(1 \otimes \varphi^{-n}) \circ (1-\varphi)z \equiv \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix} \mod \widetilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}).$$

As $\mathscr{H}_n$, $\mathrm{Col}_1(z)$ and $\mathrm{Col}_2(z)$ are all defined over Λ, we see that $(1 \otimes \varphi^{-n}) \circ (1-\varphi)z$ is indeed congruent to an element in $\Lambda_{n,p-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$. ∎

This allows us to give the following definition.

Definition 3.9 For $n \geq 1$, define

$$\underline{\mathrm{Col}}_n \colon H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \longrightarrow \Lambda_{n,k-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$$
$$z \longmapsto (1 \otimes \varphi^{-n}) \circ (1-\varphi) \circ (h^1_{\mathcal{T}})^{-1}(z) \mod \widetilde{\omega}_{n-1,k-2}(X).$$

We recall that $h^1_{\mathcal{T}}$ is an isomorphism by Theorem 2.6. Therefore, Lemma 3.8 tells us that the map $\underline{\mathrm{Col}}_n$ is well defined.

For an integer m, we define the twisting map

$$\mathrm{Tw}_m \coloneqq \mathrm{Tw}^{-m} \otimes t^{-m} e_m \colon \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \longrightarrow \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}(m)).$$

Consider the twisting map $\mathrm{Tw}^{k-j-1} \colon \sigma \mapsto \chi^{k-j-1}(\sigma)\sigma$ on Λ. Since $k - j - 1 \leq k - 1$, $\mathrm{Tw}^{k-j-1}(\widetilde{\omega}_{n-1,k-2}(X))$ is divisible by $\omega_{n-1}(X)$. Hence, Tw^{k-j-1} induces a natural map $\Lambda_{n,k-2} \to \Lambda_n$. Therefore, we can define

$$\underline{\mathrm{Col}}_{T,n} \colon H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \longrightarrow \Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T)$$
$$z \longmapsto \mathrm{Tw}_{-k+j+1} \circ \underline{\mathrm{Col}}_n(z \cdot e_{k-j-1}) \quad \mathrm{mod}\ \omega_{n-1}(X),$$

on identifying $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \cdot e_{k-j-1}$ with $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$.

Lemma 3.10   *The map $\underline{\mathrm{Col}}_{T,n}$ defines a Λ_n-homomorphism from $H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ to $\Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T)$.*

Proof   We note that $\underline{\mathrm{Col}}_{T,n}$ is a Λ-homomorphism, since both $\underline{\mathrm{Col}}_n$ and

$$x \mapsto \mathrm{Tw}^{k-j-1} \circ (x \cdot e_{k-j-1})$$

are Λ-linear. The fact that $\underline{\mathrm{Col}}_{T,n}$ factors through $H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ follows from the equation $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)_{\Gamma_n} = H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ (because of the vanishing of $H^2_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)$). ∎

We have the explicit formula

$$(3.5) \quad \underline{\mathrm{Col}}_{T,n}(z) \equiv \begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix} \cdot \mathrm{Tw}^{k-1-j}\left(\mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z \cdot e_{k-1-j}) \\ \mathrm{Col}_2(z \cdot e_{k-1-j}) \end{pmatrix} \right)$$
$$\mathrm{mod}\ \omega_{n-1}(X)\Lambda \otimes \mathbb{D}_{\mathrm{cris}}(T),$$

by Lemma 3.7 and the expansion of $1 - \varphi$ as given in Definition 3.4.

We now modify the definition of $\underline{\mathrm{Col}}_{T,n}$ to define a map that lands in Λ_n. For any $u \in \mathbb{Z}_p^\times$, we define $\underline{\mathrm{Col}}_{T,n,u} \colon H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \to \Lambda_n$ to be the composition of $\underline{\mathrm{Col}}_{T,n}$ and the linear functional on $\Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T) \to \Lambda_n$ given by $a \cdot \nu_1 + b \cdot \nu_2 \mapsto a + ub$. More explicitly, (3.5) tells us that $\underline{\mathrm{Col}}_{T,n,u}$ is given by

$$(3.6) \quad \underline{\mathrm{Col}}_{T,n,u}(z) \equiv \begin{pmatrix} 1 & u \end{pmatrix} \cdot \mathrm{Tw}^{k-1-j}\left(\mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z \cdot e_{k-1-j}) \\ \mathrm{Col}_2(z \cdot e_{k-1-j}) \end{pmatrix} \right) \quad \mathrm{mod}\ \omega_{n-1}(X)\Lambda.$$

Note that Lemma 3.10 tells us that $\underline{\mathrm{Col}}_{T,n,u}$ is Λ_n-linear.

3.3 Analysis of Bloch–Kato Subgroups Via Coleman Maps

If F is a finite extension of $\mathbb{Q}_p$, we write $H^1_f(F, T) \subset H^1(F, T)$ for the usual Bloch–Kato subgroup from [BK90] and $H^1_{/f}(F, T)$ denotes the quotient $H^1(F, T)/H^1_f(F, T)$. The goal of this section is to study $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T)$ via the map $\underline{\mathrm{Col}}_{T,n,u}$.

Let $\mathcal{T}^*$ be the $\mathcal{O}$-linear dual of $\mathcal{T}$. For each $n \geq 1$, we define the pairing

$$\langle \cdot, \cdot \rangle_n \colon H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}) \times H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}^*(1)\big) \longrightarrow \Lambda_n$$
$$(x, y) \longmapsto \sum_{\sigma \in \Gamma/\Gamma_n} [x, y^\sigma]_n \sigma,$$

where $[\cdot, \cdot]_n$ is the standard cup-product pairing

$$H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}) \times H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}^*(1)\big) \longrightarrow \mathcal{O}.$$

On taking inverse limits, this induces a pairing

$$\langle \cdot, \cdot \rangle : H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}\big) \times H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)\big) \longrightarrow \Lambda.$$

It is semi-linear over Λ with respect to the involution on Λ (which we denote by $\widetilde{\iota}$) in the following sense:

$$\langle \sigma x, y \rangle = \sigma \langle x, y \rangle, \quad \langle x, \sigma y \rangle = \sigma^{\widetilde{\iota}} \langle x, y \rangle$$

We can extend the pairing $\langle \cdot, \cdot \rangle$ by semi-linearity to

$$\big(\mathcal{H} \otimes_{\mathcal{O}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})\big) \times \big(\mathcal{H} \otimes_{\mathcal{O}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1))\big) \longrightarrow \mathcal{H},$$

which is again denoted by $\langle \cdot, \cdot \rangle$ by an abuse of notation.

Recall that in [PR94], Perrin-Riou defined the big exponential map

$$\Omega_{\mathcal{T}^*(1),1} : \big(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\big)^{\psi=0} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1)) \to \mathcal{H} \otimes H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)\big).$$

By [LLZ11, proof of Proposition 4.8], for all $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$,

$$(\mathfrak{M}^{-1} \otimes 1)(1 - \varphi)z = \sum_{i=1}^2 \big\langle z, \Omega_{\mathcal{T}^*(1),1}((1 + \pi) \otimes \mathfrak{v}'_i)\big\rangle \mathfrak{v}_i,$$

where $\mathfrak{v}'_1, \mathfrak{v}'_2$ is the dual basis of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1))$ to $\mathfrak{v}_1, \mathfrak{v}_2$ with respect to the natural pairing

$$[\cdot, \cdot] : \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \times \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1)) \longrightarrow \mathcal{O}.$$

Therefore,

$$\underline{\mathrm{Col}}_n(z) = \sum_{i=1}^2 \big\langle z, \Omega_{\mathcal{T}^*(1),1}\big((1 + \pi) \otimes \mathfrak{v}'_i\big)\big\rangle \varphi^{-n}(\mathfrak{v}_i) \mod \widetilde{\omega}_{n-1,k-2}$$

$$= \sum_{i=1}^2 \big\langle z, \Omega_{\mathcal{T}^*(1),1}\big((1 + \pi) \otimes (p\varphi)^n(\mathfrak{v}'_i)\big)\big\rangle \mathfrak{v}_i \mod \widetilde{\omega}_{n-1,k-2}$$

as the dual of φ^{-1} with respect to $[\cdot, \cdot]$ is $p\varphi$. This description allows us to make the following choice of u to describe the kernel of $\underline{\mathrm{Col}}_{T,n,u}$.

Proposition 3.11 *There exists $u \in \mathbb{Z}_p^\times$ such that $\ker(\underline{\mathrm{Col}}_{T,n,u}) = H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$.*

Proof Write

$$v' = (\mathfrak{v}'_1 + u\mathfrak{v}'_2) \cdot t^{-k+j+1} e_{k-j-1} \in \mathbb{D}_{\mathrm{cris}}(T^*(1))$$

and let $z \in H^1(\mathbb{Q}_p(\mu_{p^n}), T)$. If θ is a Dirichlet character of conductor $p^m > 1$, we have the interpolation formula of Perrin–Riou [PR94, §3.2.3] (see also [Lei11, §3.2])

$$(3.7) \qquad \frac{\theta(\underline{\mathrm{Col}}_{T,n,u}(z))}{(-1)^{k-j-1}(k-j-1)!} = \sum_{\sigma \in \Gamma/\Gamma_m} \frac{\theta^{-1}(\sigma)}{\tau(\theta)} \big[\exp^*_{T,m}(z^\sigma), p^n \varphi^{n-m}(v')\big],$$

where $\exp^*_{T,m} : H^1(\mathbb{Q}_p(\mu_{p^m}), T) \to \mathbb{Q}_p(\mu_{p^m}) \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$ is the Bloch–Kato dual exponential map and $\tau(\theta)$ is the Gauss sum of θ. There is a similar formula when θ is the trivial character on replacing φ^{-m} by $(1 - (\varphi^{-1})/p)(1 - \varphi)^{-1}$. We note that here $\exp^*_{T,m}(z)$ is the shorthand for $\exp^*_{T,m} \circ \mathrm{cor}_{n/m}(z)$, where $\mathrm{cor}_{n/m}$ denotes the corestriction map $H^1(\mathbb{Q}_p(\mu_{p^n}), T) \to H^1(\mathbb{Q}(\mu_{p^m}), T)$. Recall that $\exp^*_{T,n}$ factors

through $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T)$. Therefore, we see that $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$ is contained in $\ker(\underline{\mathrm{Col}}_{T,n,u})$.

We choose u so that $\varphi^{n-m}(v')$, $1 \le m \le n$ and $\varphi^n(1 - (\varphi^{-1})/p)(1 - \varphi)^{-1}(v')$ are not contained inside $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V)$. We note that such u exists, since all maps are surjective on $\mathbb{D}_{\mathrm{cris}}(V)$ and $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V)$ is of dimension one. Let v'' be any $\mathcal{O}$-basis of $\mathbb{D}_{\mathrm{cris}}(T)/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$. In particular, for each $m \ge 1$, there exists a non-zero constant $c_m \in \mathcal{O}$ such that $\varphi^{n-m}(v') \equiv c_m v''$ and $\varphi^n(1 - \frac{\varphi^{-1}}{p})(1 - \varphi)^{-1}(v') \equiv c_0 v''$ modulo $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$.

Suppose that $\underline{\mathrm{Col}}_{T,n,u}(z) = 0$. From (3.7), we deduce that

$$\sum_{\sigma \in \Gamma/\Gamma_n} \theta^{-1}(\sigma)[\exp^*_{T,n}(z^\sigma), v''] = 0$$

for all characters θ on Γ/Γ_n. By the independence of the characters, this implies that $[\exp^*_{T,n}(z^\sigma), v''] = 0$ for all σ. In particular, z is contained in the kernel of $\exp^*_{T,n}$, which is $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$. ∎

Corollary 3.12 *For any $u \in \mathbb{Z}_p^\times$ that satisfies the condition of Proposition 3.11, $\underline{\mathrm{Col}}_{T,n,u}$ induces an injection of Λ_n-modules, $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) \hookrightarrow \Lambda_n$, whose cokernel is finite.*

Proof The injectivity follows from Proposition 3.11. By [BK90, Theorem 4.1], $H^1_f(\mathbb{Q}_p(\mu_{p^n}), V)$ is isomorphic to $\mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\mu_{p^n})$. Hence, by duality $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), V)$ is isomorphic to $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\mu_{p^n})$. Therefore, the finiteness of the cokernel follows from the fact that the two sides have the same $\mathbb{Z}_p$-rank. ∎

We remark that our map $\underline{\mathrm{Col}}_{T,n,u}$ does depend on the choice of u, but it does not affect our calculations later; see the proof of Proposition 4.11.

4 Results on p-adic Valuations

4.1 Review of Kobayashi Rank

Given an $\mathcal{O}$-module N, we will write $\mathrm{len}(N)$ for the $\mathcal{O}$-length of N. We fix a family of primitive p^n-th root of unity ζ_{p^n} and write $\epsilon_n = \zeta_{p^n} - 1$.

Definition 4.1 Let $N = (N_n)$ be an inverse system of finitely generated $\mathcal{O}$-modules with transition maps $\pi_n \colon N_n \to N_{n-1}$. If π_n has finite kernel and cokernel, the Kobayashi rank ∇N_n is defined as

$$\nabla N_n \coloneqq \mathrm{len}(\ker \pi_n) - \mathrm{len}(\mathrm{coker}\, \pi_n) + \mathrm{rank}_{\mathcal{O}} N_{n-1}.$$

If L is an $\mathcal{O}[[X]]$-module, we define $\nabla_n L$ to be $\nabla(L/\omega_n(X)L)$, with the connecting map given by the natural projection $L/\omega_n(X)L \to L/\omega_{n-1}(X)L$, if its kernel and cokernel are finite.

Lemma 4.2 *Let $F \in \mathcal{O}[[X]]$ be a non-zero element. Let N be the inverse limit defined by $N_n = \mathcal{O}[[X]]/(F, \omega_n)$, where the the connecting maps are the natural projections.*

(i)   *Suppose that $F(\epsilon_n) \neq 0$; then ∇N_n is defined and is equal to $\mathrm{ord}_{\epsilon_n} F(\epsilon_n)$.*

(ii)   *When n is sufficiently large, ∇N_n is defined. Furthermore,*

$$\nabla N_n = e \times \mathrm{ord}_{\epsilon_n} F(\epsilon_n) = e\lambda(F) + (p^n - p^{n-1})\mu(F),$$

where e is the ramification index of $E/\mathbb{Q}_p$ and $\lambda(F)$, $\mu(F)$ are the Iwasawa invariants as defined in §2.3.

(iii)   *If L is a finitely generated torsion $\mathcal{O}[[X]]$-module, then $\nabla_n L$ is defined for $n \gg 0$ and its value is given by*

$$\lambda(L) + (p^n - p^{n-1})\mu(L),$$

where $\lambda(L)$ and $\mu(L)$ are the λ- and μ-invariants of a generator of the characteristic ideal of L.

Proof   This follows from the same proof as [Kob03, Lemma 10.5]. ∎

We write p^r for the size of the residue field of E. The following lemma allows us to relate the growth in the size of a tower of finite $\mathcal{O}$-modules and Kobayashi ranks.

Lemma 4.3   *Suppose that $N = (N_n)$ is an inverse limit of finite $\mathcal{O}$-modules such that $|N_n| = p^{s_n}$ for some integer $s_n \in r\mathbb{Z}$ for all $n \geq 1$. Then $r\nabla N_n = s_n - s_{n-1}$.*

Proof   Since N_{n-1} is finite, we have

$$\begin{aligned}
\nabla N_n &= \mathrm{len}(\ker \pi_n) - \mathrm{len}(\mathrm{coker}\, \pi_n) \\
&= \big(\mathrm{len}(N_n) - \mathrm{len}(\mathrm{Im}\,\pi_n)\big) - \big(\mathrm{len}(N_{n-1}) - \mathrm{len}(\mathrm{Im}\,\pi_n)\big) \\
&= \mathrm{len}(N_n) - \mathrm{len}(N_{n-1}).
\end{aligned}$$

In general, if L is a finite $\mathcal{O}$-module, then $|L| = p^{r\,\mathrm{len}(L)}$. Hence the result. ∎

Finally, we prove a lemma on p-adic valuations that will be needed later.

Lemma 4.4   *Let $F \in \mathcal{O}[[X]]$ be non-zero. Then for all sufficiently large integers n we have*

$$\mathrm{ord}_p F(\epsilon_n) = \mathrm{ord}_p \mathfrak{M}(F)(\epsilon_{n+1}).$$

Moreover, for $n \gg 0$ we also have

$$\mathrm{ord}_p F(\epsilon_n) = \mathrm{ord}_p \mathrm{Tw}(F)(\epsilon_n).$$

Proof   We can write $\mathfrak{M}(F) = (1 + \pi)\varphi(G)$ for some $G \in \mathbb{A}^+_{\mathbb{Q}_p}$. By Proposition 2.2, F and G have the same Iwasawa invariants, so $\mathrm{ord}_p F(\epsilon_n) = \mathrm{ord}_p G(\epsilon_n)$ for $n \gg 0$. This implies the first part of the lemma, since $(1+\pi)\varphi(G)(\epsilon_{n+1}) = \zeta_{p^{n+1}} G(\epsilon_n)$. The second part of the lemma follows from the fact that Tw preserves μ- and λ-invariants. ∎

4.2 Calculations on Evaluation Matrices

From now on, we will write $v = \operatorname{ord}_p(a_p)$, where a_p is the p-th Fourier coefficient of f. Following [Spr13, §4.1], given any 2×2 matrix $\phi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ defined over $\overline{\mathbb{Q}_p}$, we write

$$\operatorname{ord}_p(\phi) = \begin{pmatrix} \operatorname{ord}_p(a) & \operatorname{ord}_p(b) \\ \operatorname{ord}_p(c) & \operatorname{ord}_p(d) \end{pmatrix}.$$

Lemma 4.5 *Let* $1 \le i \le n - 2$; *then*

$$\operatorname{ord}_p(\varphi^i(P^{-1})(\epsilon_n)) = \begin{pmatrix} v & 0 \\ \frac{k-1}{p^{n-i-1}} & \infty \end{pmatrix}.$$

Proof Recall that

$$P = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{q^{k-1}} \\ \delta^{k-1} & \frac{a_p}{q^{k-1}} \end{pmatrix},$$

so its inverse is given by

$$P^{-1} = \begin{pmatrix} \frac{a_p}{\delta^{k-1}\varepsilon(p)} & \frac{1}{\delta^{k-1}} \\ -\frac{q^{k-1}}{\varepsilon(p)} & 0 \end{pmatrix}.$$

Therefore, our result follows from the fact that $\delta \in \mathbb{Z}_p^\times$, $\varepsilon(p) \in \mathcal{O}^\times$ and $\varphi^i(q)$ is equal to the p^{i+1}-cyclotomic polynomial, so $\varphi^i(q)(\epsilon_n) = (\zeta_{p^{n-i-1}} - 1)/(\zeta_{p^{n-i}} - 1)$ whose p-adic valuation is $1/p^{n-i-1}$. ∎

Proposition 4.6 *Assume that* $2v > \frac{k-1}{p}$. *For all* $n \ge 1$,

$$\operatorname{ord}_p(H_n(\epsilon_n)) = \begin{cases} \begin{pmatrix} v + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}} & \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix} & \text{if } n \text{ is odd,} \\[2ex] \begin{pmatrix} \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Proof By Lemma 4.5, we have

$$\operatorname{ord}_p\big(H_n(\epsilon_n)\big) = \begin{pmatrix} v & 0 \\ \infty & \infty \end{pmatrix} \begin{pmatrix} v & 0 \\ \frac{k-1}{p} & \infty \end{pmatrix} \cdots \begin{pmatrix} v & 0 \\ \frac{k-1}{p^{n-1}} & \infty \end{pmatrix}.$$

In particular,

$$(4.1) \qquad \operatorname{ord}_p\big(H_{n+1}(\epsilon_{n+1})\big) = \operatorname{ord}_p\big(H_n(\epsilon_n)\big) \begin{pmatrix} v & 0 \\ \frac{k-1}{p^n} & \infty \end{pmatrix}.$$

Therefore,

$$\operatorname{ord}_p(H_1(\epsilon_1)) = \begin{pmatrix} v & 0 \\ \infty & \infty \end{pmatrix} \quad \text{and} \quad \operatorname{ord}_p(H_2(\epsilon_2)) = \begin{pmatrix} \frac{k-1}{p} & v \\ \infty & \infty \end{pmatrix},$$

since $2v > \frac{k-1}{p}$ by our assumption.

Suppose that

$$\mathrm{ord}_p\big(H_{2\ell-1}(\epsilon_{2\ell-1})\big) = \begin{pmatrix} \nu + \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i}} & \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i-1}} \\ \infty & \infty \end{pmatrix},$$

$$\mathrm{ord}_p\big(H_{2\ell}(\epsilon_{2\ell})\big) = \begin{pmatrix} \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}} & \nu + \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix}$$

for some integer $\ell \geq 1$. By (4.1), we have first of all

$$\mathrm{ord}_p\big(H_{2\ell+1}(\epsilon_{2\ell+1})\big) = \begin{pmatrix} \nu + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} & \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}} \\ \infty & \infty \end{pmatrix},$$

because $\sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} < \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}}$. On applying (4.1) again, we have

$$\mathrm{ord}_p\big(H_{2\ell+2}(\epsilon_{2\ell+2})\big) = \begin{pmatrix} \sum_{i=1}^{\ell+1} \frac{k-1}{p^{2i-1}} & \nu + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix}$$

thanks to our assumption that $2\nu > \frac{k-1}{p}$, which implies that

$$2\nu + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} > \sum_{i=1}^{\ell+1} \frac{k-1}{p^{2i-1}}.$$

Therefore, our result follows by induction. ∎

For $i = 1, 2$, we fix two elements $F_1, F_2 \in \mathcal{O}[[X]]$ with μ_i and λ_i being its μ- and λ-invariants.

Corollary 4.7 *Under the condition that $2\nu > \frac{k-1}{p}$, for $n \gg 0$ we have the formulae*

$$\mathrm{ord}_{\epsilon_n}\big((\mathcal{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\big) = \begin{cases} \lambda_1 + (p^n - p^{n-1})\big(\frac{\mu_1}{e} + \nu + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}}\big) & n \text{ odd,} \\ \lambda_1 + (p^n - p^{n-1})\big(\frac{\mu_1}{e} + \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}}\big) & n \text{ even,} \end{cases}$$

$$\mathrm{ord}_{\epsilon_n}\big((\mathcal{H}_{n+1})_{1,2} \cdot F_2(\epsilon_n)\big) = \begin{cases} \lambda_2 + (p^n - p^{n-1})\big(\frac{\mu_2}{e} + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}}\big) & n \text{ odd,} \\ \lambda_2 + (p^n - p^{n-1})\big(\frac{\mu_2}{e} + \nu + \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}}\big) & n \text{ even.} \end{cases}$$

Proof By Lemma 4.4, $\mathrm{ord}_p \mathcal{H}_{n+1}(\epsilon_n) = \mathrm{ord}_p H_n(\epsilon_n)$. Hence, our result follows by combining Proposition 4.6 with Lemma 4.2(ii). ∎

Corollary 4.8 *Suppose that $2\nu > \frac{k-1}{p}$. For $n \gg 0$ and n odd, we have*

$$\mathrm{ord}_{\epsilon_n}\big((\mathcal{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\big) < \mathrm{ord}_{\epsilon_n}\big((\mathcal{H}_{n+1})_{1,2} \cdot F_2(\epsilon_n)\big) \quad \text{if } \frac{\mu_1}{e} + \nu + \frac{k-1}{p+1} \leq \frac{\mu_2}{e},$$

$$\mathrm{ord}_{\epsilon_n}\big((\mathcal{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\big) > \mathrm{ord}_{\epsilon_n}\big((\mathcal{H}_{n+1})_{1,2} \cdot F_2)(\epsilon_n)\big) \quad \text{if } \frac{\mu_1}{e} + \nu + \frac{k-1}{p+1} > \frac{\mu_2}{e}.$$

For $n \gg 0$ and n even, we have

$$\operatorname{ord}_{\epsilon_n}\big(\,(\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\big) < \operatorname{ord}_{\epsilon_n}\big(\,(\mathscr{H}_{n+1})_{1,2}\cdot F_2(\epsilon_n)\big) \quad \text{if } \frac{\mu_1}{e} < \frac{\mu_2}{e} + \nu + \frac{k-1}{p+1},$$

$$\operatorname{ord}_{\epsilon_n}\big(\,(\mathscr{H}_{n+1})_{1,1}\cdot F_1(\epsilon_n)\big) > \operatorname{ord}_{\epsilon_n}\big(\,(\mathscr{H}_{n+1})_{1,2}\cdot F_2(\epsilon_n)\big) \quad \text{if } \frac{\mu_1}{e} \geq \frac{\mu_2}{e} + \nu + \frac{k-1}{p+1}.$$

Proof Note that

$$\sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}} - \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}} > 0 \quad \text{and} \quad \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}} - \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}} > 0$$

and that both sequences are strictly increasing and tend to $\frac{k-1}{p+1}$ as $n \to \infty$. Hence, we have the result. ∎

4.3 Some Global Iwasawa Modules

For $n \geq 0$ let us write $K_n = \mathbb{Q}(\mu_{p^n})$.

Definition 4.9 (*cf.* [Kat04, §12.2])   For $m \geq 0$, we define

$$\mathbb{H}^m(T) \coloneqq \varprojlim_n H_{\text{ét}}^m\big(\operatorname{Spec} \mathcal{O}_{K_n}[1/p], j_* T\big),$$

where the inverse limit is respect to the corestriction maps, and j is the inclusion map $\operatorname{Spec} K_n \hookrightarrow \operatorname{Spec} \mathcal{O}_{K_n}[1/p]$.

By [Kat04, 12.4(3)], the modules $\mathbb{H}^m(T)$ are finitely-generated over Λ, and are zero unless $m \in \{1, 2\}$; and $\mathbb{H}^1(T)$ is free of rank 1 over Λ. We fix an element $\mathbf{z} \in \mathbb{H}^1(T)$ so that $\mathbb{H}^1(T) = \Lambda \cdot \mathbf{z}$. Tensoring with the basis vector e_{k-1-j} of $\mathcal{O}(k-1-j)$ gives a bijection

$$\mathbb{H}^1(T) \cong \mathbb{H}^1(\mathcal{T}),$$

and (in a slight abuse of notation) we will write $\operatorname{Col}_i(\mathbf{z})$ for the image of $\mathbf{z} \cdot e_{k-1-j}$ under Col_i composed with the localization map $\mathbb{H}^1(\mathcal{T}) \to H^1_{\text{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$.

Definition 4.10   For $i = 1, 2$ and η a Dirichlet character modulo p. Let μ_i^η be the μ-invariant of $\operatorname{Col}_i(\mathbf{z})^\eta$. For each $n \geq 1$, we define an integer $\tau(n, \eta) \in \{1, 2\}$ by

$$\begin{cases} 1 & \text{if } \frac{\mu_1^\eta}{e} + \nu + \frac{k-1}{p+1} \leq \frac{\mu_2^\eta}{e} \text{ and } n \text{ odd or } \frac{\mu_1^\eta}{e} < \frac{\mu_2^\eta}{e} + \nu + \frac{k-1}{p+1} \text{ and } n \text{ even,} \\ 2 & \text{otherwise.} \end{cases}$$

Furthermore, we write $q_n^* = \operatorname{ord}_{\epsilon_n}((\mathscr{H}_{n+1})_{1,\tau(n,\eta)}(\epsilon_n))$.

Note in particular that q_n^* is a sum of some powers of p, together with possibly ν, as given by Proposition 4.6. Furthermore, Corollary 4.8 tells us that

$$(4.2) \qquad \operatorname{ord}_{\epsilon_n}\Big(\sum_{i=1}^2 (\mathscr{H}_{n+1})_{1,i}\cdot \operatorname{Col}_i(\mathbf{z})^\eta(\epsilon_n)\Big) = q_n^* + \operatorname{ord}_{\epsilon_n} \operatorname{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta(\epsilon_n).$$

4.4 Analysis of Some Local Iwasawa Modules

For $n \geq 1$, we define

$$\mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^n})) = \mathrm{coker}\left(\mathbb{H}^1(T)_{\Gamma_n} \to H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T)\right),$$

which gives an inverse limit with the connecting maps given by the corestriction maps. We would like to study $\nabla \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ for a fixed Dirichlet character η modulo p.

Proposition 4.11 *Suppose that $\underline{\mathrm{Col}}_1(\mathbf{z})^\eta$ and $\underline{\mathrm{Col}}_2(\mathbf{z})^\eta$ are non-zero. For $n \gg 0$, $\nabla \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined, and its value is bounded above by*

$$\nabla_n \mathfrak{X}^\eta_{\mathrm{loc}} \leq e q^*_n + \nabla_n \left(\mathcal{O}[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta\right).$$

Proof Recall from Corollary 3.12, we have the injection

$$\underline{\mathrm{Col}}_{T,n+1,u} \colon H^1_{/f}(\mathbb{Q}_p(\mu_{p^{n+1}}), T) \hookrightarrow \Lambda_{n+1}.$$

On taking Γ_n-coinvariants, the *same* map (*not* $\underline{\mathrm{Col}}_{T,n,u}$) induces an injection

$$\underline{\mathrm{Col}}_{T,n+1,u} \colon H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) \hookrightarrow \Lambda_n,$$

which admits the same description as (3.6). We write coker_{n+1} and coker_n for the cokernels of these two maps, respectively. Then we have the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p(\mu_{p^{n+1}}), T) & \xrightarrow{\underline{\mathrm{Col}}_{T,n+1,u}} & \Lambda_{n+1} & \longrightarrow & \mathrm{coker}_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & {\scriptstyle \pi}\downarrow & & \\
0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) & \xrightarrow{\underline{\mathrm{Col}}_{T,n+1,u}} & \Lambda_n & \longrightarrow & \mathrm{coker}_n & \longrightarrow & 0,
\end{array}
$$

where the vertical maps are all natural projections. This gives

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^{n+1}})) & \longrightarrow & \Lambda_{n+1}/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})) & \longrightarrow & \mathrm{coker}_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & {\scriptstyle \pi}\downarrow & & \\
0 & \longrightarrow & \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^n})) & \longrightarrow & \Lambda_n/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})) & \longrightarrow & \mathrm{coker}_n & \longrightarrow & 0.
\end{array}
$$

Recall from Corollary 3.12 that coker_{n+1} is finite (in particular, coker_n too). Hence, on taking η-isotypic components, $\nabla \mathrm{coker}^\eta_{n+1}$ (with respect to π) is defined. In fact, it is given by $\mathrm{len}(\ker \pi^\eta)$, which is ≥ 0.

Furthermore, recall that we assume $\mathrm{Col}_i(\mathbf{z})^\eta \neq 0$ for $i = 1, 2$. Proposition 4.6 tells us that the second row of $\mathscr{H}_{n+1}(\epsilon_n)$ is 0. So, the formulae (3.6) and (4.2) imply that $\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})(\epsilon_n) \neq 0$ when $n \gg 0$. Hence, $\nabla(\Lambda_{n+1}/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})))^\eta = \nabla_n(\mathcal{O}[\![X]\!]/\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})^\eta)$ is defined. Its value is given by

$$e q^*_n + \nabla_n \left(\mathcal{O}[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta\right),$$

thanks to Lemma 4.2.

Therefore, the fact that the Kobayashi rank ∇ respects short exact sequences ([Kob03, Lemma 10.4]) tells us that $\nabla \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^{n+1}}))^\eta$ is defined and its value is equal to

$$\nabla_n \left(\mathcal{O}[\![X]\!]/\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})^\eta\right) - \mathrm{len}(\ker \pi^\eta).$$

Hence, we have the result.   ∎

This can be considered as a weakened version of the modesty proposition [Spr13, Proposition 3.10]. In the $k = 2$ case, equality holds because the projection π turns out to be an injection (see [Kob03, Lemma 10.7] and [Spr13, Lemma 4.12]).

5 Selmer Groups and Shafarevich–Tate Groups

5.1 Signed Selmer Groups

Let $T^\vee$ be the Pontryagin dual of T. As in [LLZ10], the Coleman maps allow us to define the Selmer groups

$$\mathrm{Sel}_i\big(T^\vee/\mathbb{Q}(\mu_{p^\infty})\big) = \ker\Big(\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^\infty})) \to \frac{H^1(\mathbb{Q}_p(\mu_{p^\infty}), T^\vee)}{\ker(\mathrm{Col}_i)^\perp}\Big)$$

for $i = 1, 2$. Here $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$ is the Bloch–Kato Selmer group from [BK90]. We shall write $\mathcal{X}(\mathbb{Q}(\mu_{p^n})) = \mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee$ for $n \geq 1$.

Let $\mathcal{X}_i$ be the Pontryagin dual of $\mathrm{Sel}_i(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$. We subsequently assume that for any Dirichlet character η that factors through $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, both $\mathcal{X}_1^\eta$ and $\mathcal{X}_2^\eta$ are $\mathcal{O}[[X]]$-torsion. Note that this is the case if either $k \geq 3$ or $a_p = 0$ by [LLZ10, Theorem 6.5]. In particular, $\nabla_n \mathcal{X}_i^\eta$ are defined for $n \gg 0$ by Lemma 4.2(iii).

We have the Poitou–Tate exact sequence (see for example [LLZ10, (61)])

$$(5.1) \qquad \mathbb{H}^1(T) \longrightarrow \mathrm{Im}\,\mathrm{Col}_i \longrightarrow \mathcal{X}_i \longrightarrow \mathcal{X}_0 \longrightarrow 0,$$

where $\mathcal{X}_0$ is $\mathbb{H}^2(T)$ and can be realized as the Pontryagin dual of the zero Selmer group $\mathrm{Sel}_0(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$, which is defined to be

$$\ker\Big(H^1(\mathbb{Q}(\mu_{p^\infty}), T^\vee) \longrightarrow \prod_v H^1(\mathbb{Q}(\mu_{p^\infty})_v, T^\vee)\Big),$$

where v runs through all places of $\mathbb{Q}(\mu_{p^\infty})$. Note that $\mathcal{X}_0$ is a torsion Λ-module by [Kat04, Theorem 12.4] and hence $\nabla_n \mathcal{X}_0^\eta$ is defined for $n \gg 0$ by Lemma 4.2(iii). Note that (5.1) gives the short exact sequence

$$0 \longrightarrow \frac{\mathrm{Im}\,\mathrm{Col}_i}{(\mathrm{Col}_i(\mathbf{z}))} \longrightarrow \mathcal{X}_i \longrightarrow \mathcal{X}_0 \longrightarrow 0.$$

Hence, our assumption that $\mathcal{X}_i^\eta$ be torsion implies that $\mathrm{Col}_i(\mathbf{z})^\eta \neq 0$. In particular, Proposition 4.11 applies.

Recall from [LLZ11, §5] that $\mathrm{Im}\,\mathrm{Col}_i^\eta$ is pseudo-isomorphic to

$$\prod_m (X - \chi(\gamma)^m + 1)\mathcal{O}[[X]],$$

where m runs through some subset of $\{0, 1, \dots, k-2\}$ depending on i and η. Let us write $\kappa_i(\eta)$ for the cardinality of this subset and write $\kappa(n, \eta) = \kappa_{\tau(n,\eta)}(\eta)$. We have the following generalization of [Spr13, Proposition 3.11].

Proposition 5.1 *For $i = 1, 2$, η any Dirichlet character modulo p and $n \gg 0$,*

$$\nabla_n \mathcal{X}_i^\eta = \nabla_n\big(\Lambda/\mathrm{Col}_i(\mathbf{z})\big)^\eta + \nabla_n \mathcal{X}_0^\eta - e\kappa_i(\eta).$$

Proof The following sequence

$$0 \to \Big(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})} \Big)^\eta \to \Big(\frac{\Lambda}{\mathrm{Col}_i(\mathbf{z})} \Big)^\eta \to \frac{\mathcal{O}[[X]]}{\prod_m (X - \chi(\gamma)^m + 1)\mathcal{O}[[X]]} \to G \to 0$$

is exact, where G is some finite subgroup. In particular, $\nabla_n G = 0$ for $n \gg 0$. We can work out the Kobayashi rank of the second last term using Lemma 4.2(ii). Recall from [Kob03, Lemma 10.4] that Kobayashi ranks respect exact sequences; therefore,

$$\nabla_n \Big(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})} \Big)^\eta + e\kappa_i(\eta) = \nabla_n \Big(\frac{\Lambda}{\mathrm{Col}_i(\mathbf{z})} \Big)^\eta.$$

Furthermore, from (5.1), we have the following exact sequence

$$0 \longrightarrow \frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})} \longrightarrow \mathfrak{X}_i \longrightarrow \mathfrak{X}_0 \longrightarrow 0,$$

which implies that

$$\nabla_n \Big(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})} \Big)^\eta + \nabla_n \mathfrak{X}_0^\eta = \nabla_n \mathfrak{X}_i^\eta.$$

Combining the two equations gives our result. ∎

Remark 5.2 Let μ_0^η be the μ-invariant of $\mathfrak{X}_0^\eta$. For $i = 1, 2$, let $\widetilde{\mu}_i^\eta$ be the μ-invariant of $\mathfrak{X}_i^\eta$. Then Proposition 5.1 implies that $\widetilde{\mu}_i^\eta = \mu_i^\eta - \mu_0^\eta$. In particular, $\mu_1^\eta - \mu_2^\eta = \widetilde{\mu}_1^\eta - \widetilde{\mu}_2^\eta$. Therefore, we can replace μ_1^η and μ_2^η by $\widetilde{\mu}_1^\eta$ and $\widetilde{\mu}_2^\eta$, respectively in Definition 4.10. In other words, we can define $\tau(n, \eta)$ using the μ-invariants of the dual Selmer groups $\mathfrak{X}_i$, instead of $\mathrm{Col}_i(\mathbf{z})$.

Corollary 5.3 *For $n \gg 0$, $\nabla \mathfrak{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined. Furthermore, its value is bounded above by*

$$eq_n^* + \nabla_n \mathfrak{X}_{\tau(n,\eta)}^\eta + e\kappa(n, \eta).$$

Proof Let

$$\mathcal{Y}(\mathbb{Q}(\mu_{p^n})) = \mathrm{coker}\big(H^1(G_{n,S}, T) \longrightarrow H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T)\big)$$

and $\mathfrak{X}_0(\mathbb{Q}(\mu_{p^n})) = \mathrm{Sel}_0(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee$. As a consequence of the Poitou–Tate exact sequence, we have the short exact sequence

$$0 \longrightarrow \mathcal{Y}(\mathbb{Q}(\mu_{p^n})) \longrightarrow \mathfrak{X}(\mathbb{Q}(\mu_{p^n})) \longrightarrow \mathfrak{X}_0(\mathbb{Q}(\mu_{p^n})) \longrightarrow 0$$

(*cf.* [Kob03, (10.35)]). But Proposition 10.6 in *op. cit.* says that

- $\nabla \mathcal{Y}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined for $n \gg 0$ and is equal to $\nabla \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$;
- $\nabla \mathfrak{X}_0(\mathbb{Q}(\mu_{p^{n+1}}))^\eta = \nabla_n \mathfrak{X}_0^\eta$.

Therefore,

$$\nabla \mathfrak{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta = \nabla \mathfrak{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta + \nabla_n \mathfrak{X}_0^\eta,$$

and our result follows from Propositions 4.11 and 5.1. ∎

5.2 Bloch–Kato–Shafarevich–Tate Groups

Let L be a number field. We recall that the Bloch–Kato–Shafarevich–Tate group of $T^\vee$ over L is defined to be

$$(5.2) \qquad \text{III}(L, T^\vee) = \frac{\text{Sel}(T^\vee/L)}{\text{Sel}(T^\vee/L)_{\text{div}}},$$

where $(\star)_{\text{div}}$ denotes the maximal divisible subgroup of $\star$. (See *e.g.*, [BK90, Remark 5.15.2]). If f corresponds to an elliptic curve $\mathcal{E}$ and the p-primary part of the classical Shafarevich–Tate group $\mathcal{E}$ is finite, then the two definitions of (p-primary) Shafarevich–Tate groups agree.

Proposition 5.4   *There exist integers $n_0^\eta, r_\infty^\eta \geq 0$ such that*

$$\text{corank}_{\mathcal{O}}\,\text{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta = r_\infty^\eta$$

for all $n \geq n_0^\eta$.

Proof   By Corollary 5.3, $\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined for $n \gg 0$. In particular, the kernel and cokernel of the connecting map

$$\text{Sel}\big(T^\vee/\mathbb{Q}(\mu_{p^{n+1}})\big)^\vee \longrightarrow \text{Sel}\big(T^\vee/\mathbb{Q}(\mu_{p^n})\big)^\vee$$

are finite for $n \gg 0$. In particular, $\text{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))$ and $\text{Sel}(T^\vee/\mathbb{Q}(\mu_{p^n}))$ must have the same $\mathbb{Z}_p$-corank. ∎

This implies that $\text{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta_{\text{div}} \cong (E/\mathcal{O})^{\oplus r_\infty^\eta}$ (as $\mathbb{Z}_p$-modules) for $n \gg 0$. Combined this with (5.2), we obtain the following short exact sequence of $\mathbb{Z}_p$-modules

$$0 \longrightarrow (E/\mathcal{O})^{\oplus r_\infty^\eta} \longrightarrow \text{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta \longrightarrow \text{III}\big(\mathbb{Q}(\mu_{p^{n+1}}), T^\vee\big)^\eta \longrightarrow 0.$$

Therefore, on taking Pontryagin duals, we deduce that

$$\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta = r_\infty^\eta + \nabla\text{III}(\mathbb{Q}(\mu_{p^{n+1}}), T^\vee)^\eta.$$

From Corollary 5.3, we deduce that

$$\nabla\text{III}\big(\mathbb{Q}(\mu_{p^{n+1}}), T^\vee\big)^\eta \leq eq_n^* + \nabla_n\mathcal{X}^\eta_{\tau(n,\eta)} + e\kappa(n,\eta) - r_\infty^\eta.$$

Therefore, we obtain the following theorem on applying Lemma 4.3.

Theorem 5.5   *Let $\#\text{III}(\mathbb{Q}(\mu_{p^n}), T^\vee)^\eta = p^{s_n^\eta}$. For $n \gg 0$,*

$$s_{n+1}^\eta - s_n^\eta \leq r\left(eq_n^* + \nabla_n\mathcal{X}^\eta_{\tau(n,\eta)} + e\kappa(n,\eta) - r_\infty^\eta\right),$$

where r is the integer so that the residue field of E has cardinality p^r.

Using Lemma 4.2, we can rewrite this formula as

$$s_{n+1}^\eta - s_n^\eta \leq d\left(q_n^* + \lambda_{\tau(n,\eta)} + (p^n - p^{n-1})\frac{\mu_{\tau(n,\eta)}}{e} + \kappa(n,\eta) - \frac{r_\infty^\eta}{e}\right),$$

where $d = [E:\mathbb{Q}_p]$.

A Growth of Tamagawa Numbers Over Cyclotomic Extensions

We let $T = T_f(j)$ and $\mathcal{T} = T_f(k-1)$ be the representations studied in the main part of the article. In particular, we assume all the previous hypotheses on T and $\mathcal{T}$ are satisfied throughout. Furthermore, we assume that the eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ are not integral powers of p. For notational simplicity, we will assume that the coefficient field E is $\mathbb{Q}_p$ throughout.

Recall the Perrin–Riou p-adic regulator

$$\mathcal{L}_{\mathcal{T}} \colon H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}\big) \longrightarrow \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$$

defined by $\mathfrak{M}^{-1} \circ (1-\varphi) \circ (h^1_{\mathcal{T}})^{-1}$, which is the map used to define the Coleman maps in Definition 3.4. We have the following interpolation formula.

Proposition A.1   *Let $n \geq 1$. For any $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$, $i \geq 0$ and a Dirichlet character δ of conductor p^n, we have*

$$\mathcal{L}_{\mathcal{T}}(z)(\chi^i \delta) = \begin{cases} i!(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1}\big(\exp^*(z_{0,-i})\big) \cdot t^{-i} e_i & \text{if } n = 0, \\ \frac{i! p^n}{\tau(\delta)} \varphi^n\big(\exp^*(\widetilde{e}_\delta \cdot z_{n,-i})\big) \cdot t^{-i} e_i & \text{otherwise,} \end{cases}$$

where $\tau(\delta)$ is the Gauss sum of δ, $z_{n,-i}$ is the projection of z in $H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}(-i))$ and $\widetilde{e}_\delta$ is the element $\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p)} \delta^{-1}(\sigma)\sigma$.

Proof   This is a slight reformulation of [LZ14, Theorem B.5], since we have the equation

$$\varphi(t^{-i} e_i) = p^{-i} \cdot t^{-i} e_i. \qquad \blacksquare$$

Corollary A.2   *Let $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathcal{T})$. Then $\mathcal{L}_{\mathcal{T}}(z)(\chi^i \delta) = 0$ if and only if $\widetilde{e}_\delta \cdot z_{n,-i} \in \widetilde{e}_\delta \cdot H^1_f(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}(-i))$.*

Proof   This is because our assumption on the eigenvalues of φ implies that $(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1}$ and φ^n are both invertible. $\qquad \blacksquare$

We write $K = \mathbb{Q}(\mu_{p^n})$ and $\Delta_K = \mathrm{Gal}(K/\mathbb{Q})$. For each character δ on Δ_K, we write p^{n_δ} for its conductor. Let K_p be the completion of K at the unique place above p (which may be identified with $\mathbb{Q}_p(\mu_{p^n})$). We fix a basis v for $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$ and its dual v' in $\mathbb{D}_{\mathrm{cris}}(T^*(1))/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T^*(1))$. We have the definition of the Tamagawa number as defined by Bloch–Kato [BK90]:

$$\mathrm{Tam}(T/K) = \big[H^1_f(K_p, T) \colon \mathcal{O}_{K_p} \cdot v \big] L_p(T, 1),$$

where $L_p(T, 1)$ is the Euler factor of the complex L-function $L_p(T, 1)$ at p and we identify $\mathcal{O}_{K_v} v$ with its image under the Bloch–Kato exponential map. We may decompose the Tamagawa number into isotypic components, namely

$$\mathrm{Tam}(T/K) = \prod_\eta \mathrm{Tam}(T/K)^\eta,$$

where the product runs through all the Dirichlet characters modulo p and $\mathrm{Tam}(T/K)^\eta$ is given by

$$\big[H^1_f(K_p, T)^\eta \colon (\mathcal{O}_{K_p} \cdot v)^\eta \big] L_p(T(\eta), 1),$$

which we may identify with $\mathrm{Tam}(T(\eta)/K^\Delta)$.

Lemma A.3   *Let d_K be the discriminant of K. Then we have the formula*

$$\mathrm{Tam}(T/K) = |d_K|_p^{-1}\big[\,\mathcal{O}_{K_p}\cdot v : H^1_{/f}(K_p, T)\big]L_p(T, 1),$$

where we identify $H^1_{/f}(K_p, T)$ with its image under the Bloch–Kato dual exponential map.

Proof   This follows from the commutative diagram

$$
\begin{array}{ccc}
\big(K_p \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)\big) \times \big(K_p \otimes \frac{\mathbb{D}_{\mathrm{cris}}(T^*(1))}{\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T^*(1))}\big) & \longrightarrow & K_p \\
\uparrow{\exp^*} \qquad\qquad \downarrow{\exp} & & \downarrow{\mathrm{Tr}_{K_p/\mathbb{Q}_p}} \quad\blacksquare \\
\big(\mathbb{Q}_p \otimes H^1_{/f}(K_p, T)\big) \times \big(\mathbb{Q}_p \otimes H^1_f(K_p, T^*(1))\big) & \longrightarrow & \mathbb{Q}_p.
\end{array}
$$

Take $\mathbf{z}$ to be a Λ-generator of $\mathbb{H}^1(T)$ as in the main part of the article. This gives a Λ-basis $\mathbf{z}\cdot e_{k-j-1}$ of $\mathbb{H}^1(\mathcal{T})$. We shall write $\mathcal{L}_T(\mathbf{z})$ for $\mathrm{Tw}_{-k+j+1}\circ\mathcal{L}_{\mathcal{T}}(\mathbf{z}\cdot e_{k-j-1})$ and

$$\widetilde{\nu}_K = \bigotimes_{\delta\in\widehat{\Delta}_K}\big(\varphi^{n_\delta}(1-\delta(p)\varphi)(1-p^{-1}\bar\delta(p)\varphi^{-1})^{-1}\nu\big).$$

Theorem A.4   *Suppose that $\mathcal{L}_T(\mathbf{z})(\delta) \neq 0$ for all $\delta\in\widehat{\Delta}_K$. Then,*

$$\bigotimes_{\delta\in\widehat{\Delta}_K}\mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/K)}{L_p(T, 1)}\prod_\delta[e_\delta H^1_{/f}(K_p, T) : e_\delta\mathbf{z}_K]\widetilde{\nu}_K.$$

Here, we write $a \sim_p b$ if a and b have the same p-adic valuation.

Proof   Let $\mathbf{z}_K$ be the projection of $\mathbf{z}$ in $H^1(K_p, T)$. For each character of Δ_K, we write $e_\delta = \sum_{\sigma\in\Delta_K}\delta^{-1}(\sigma)\sigma$ and let K_δ for the subfield of K defined by the kernel of δ. Our assumption means that $e_\delta\cdot\mathbf{z}_K \notin e_\delta\cdot H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$ for all δ by Corollary A.2. Note that $\sum e_\delta = [K:\mathbb{Q}]$. On applying Proposition A.1, we deduce that

$$\bigotimes_{\delta\in\widehat{\Delta}_K}\mathcal{L}_T(\mathbf{z})(\delta) \sim_p \prod_\delta\Big[\frac{e_\delta}{[K:\mathbb{Q}]}\mathcal{O}[\Delta_K]\nu : e_\delta\mathcal{O}[\Delta_K]\frac{p^{n_\delta}}{\tau(\delta)}\exp^*(\mathbf{z}_K)\Big]\widetilde{\nu}_K$$

$$\sim_p \prod_\delta p^{n_\delta}\Big[e_\delta\mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K:\mathbb{Q}]} : e_\delta\mathcal{O}[\Delta_K]\Big]$$

$$\times\Big[e_\delta\mathcal{O}[\Delta_K]\nu : e_\delta\mathcal{O}[\Delta_K]\exp^*(\mathbf{z}_K)\Big]\widetilde{\nu}_K.$$

Note that the factor $(k-j-1)!$ does not appear because of the Fontaine–Laffaille condition. Now, [Gil79, Proposition 1] tells us that

$$\Big[e_\delta\mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K:\mathbb{Q}]} : e_\delta\mathcal{O}[\Delta_K]\Big] = [K:K_\delta]\Big[e_\delta\mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K_\delta:\mathbb{Q}]} : e_\delta\mathcal{O}[\Delta_K]\Big] = 1.$$

Therefore, we deduce from the conductor-discriminant formula that

$$\bigotimes_{\delta\in\widehat{\Delta}_K}\mathcal{L}_T(\mathbf{z})(\delta) \sim_p |d_K|_p^{-1}\prod_\delta\big[e_\delta\mathcal{O}[\Delta_K]\nu : e_\delta\mathcal{O}[\Delta_K]\exp^*(\mathbf{z}_K)\big]\widetilde{\nu}_K.$$

Combining this with Lemma A.3 gives us the result. $\blacksquare$

Remark A.5 There is in fact a similar formula without assuming the non-vanishing of $\mathbb{I}_{\mathrm{arith}}(T)(\delta)$. It would involve Perrin-Riou's p-adic height. See [PR03, p.180].

Corollary A.6 *Let η be a Dirichlet character modulo p. Under the conditions of Theorem* A.4, *we have*

$$\nabla_n \mathcal{X}_{\mathrm{loc}}^{\eta} + b_{n+1}^{\eta} - b_n^{\eta} = q_n^* + \nabla_n(\mathbb{Z}_p[[X]]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^{\eta}) + p^{n-1}(p-1)n(k-j-1)$$

for $n \gg 0$, where $\tau(n, \eta)$ is as defined in Definition 4.10 *and b_i^{η} denotes the p-adic valuation of* $\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^i})^{\eta})$ *for $i = n, n+1$.*

Proof Let Δ_{n+1} be the set of Dirichlet characters of conductor p^{n+1} whose Δ-component is η. Its cardinality is given by $p^{n-1}(p-1)$. By Theorem A.4, we have

$$\bigotimes_{\delta \in \Delta_{n+1}} \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^{n+1}}))^{\eta}}{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^{\eta}} \prod_{\delta \in \Delta_{n+1}} \left[e_{\delta} H^1_{/f}(K_p, T) : e_{\delta} \mathbf{z}_K \right] \varphi^{n+1}(\nu)^{\otimes|\Delta_{n+1}|}.$$

This gives
(A.1)
$$\bigotimes_{\delta \in \Delta_{n+1}} \varphi^{-n-1} \circ \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^{n+1}}))^{\eta}}{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^{\eta}} \prod_{\delta \in \Delta_{n+1}} \left[e_{\delta} H^1_{/f}(K_p, T) : e_{\delta} \mathbf{z}_K \right] \nu^{\otimes|\Delta_{n+1}|}.$$

Note that $\varphi^{-n-1} \circ \mathrm{Tw}_{-k+j+1} = p^{(n+1)(k-j-1)} \mathrm{Tw}_{-k+j+1} \circ \varphi^{-n-1}$. The terms appearing on the left-hand side are therefore simply $p^{(n+1)(k-j-1)} \underline{\mathrm{Col}}_{T,n+1}(\mathbf{z})(\delta)$. Therefore, the p-adic valuation of the left-hand side of (A.1) is given by

$$p^{n-1}(p-1)(n+1)(k-j-1) + q_n^* + \mathrm{ord}_{\epsilon_n} \mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^{\eta}(\epsilon_n)$$

thanks to (4.2). Hence, we have the result. ∎

The proof of our Proposition 4.11 implies that the defect of our inequality in Theorem 5.5 is in fact given by the length of $\ker \pi^{\eta}$, where π is some projection map. We see here that we may in fact relate this defect to the Tamagawa numbers, namely,

$$\mathrm{len}_{\mathbb{Z}_p} \ker \pi^{\eta} = b_{n+1}^{\eta} - b_n^{\eta} - p^{n-1}(p-1)n(k-j-1).$$

Let t_n^{η} be the integer $s_n^{\eta} + b_n^{\eta}$, which is the p-adic valuation of $\#\mathrm{III}(\mathbb{Q}(\mu_{p^n}), T^{\vee})^{\eta} \times \mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^{\eta}$. The Bloch–Kato conjecture predicts that this quantity should be related to the leading coefficient of the complex L function of T at 1. Theorem 5.5 tells us that we have the equality

$$t_{n+1}^{\eta} - t_n^{\eta} = q_n^* + \nabla_n \mathcal{X}_{\tau(n,\eta)}^{\eta} + \kappa(n,\eta) - r_{\infty}^{\eta} + p^{n-1}(p-1)n(k-j-1)$$

for $n \gg 0$.

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