SOME PROPERTIES OF GRAPHS WITH MULTIPLE EDGES

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1. Introduction. In this paper we consider undirected graphs, with no edges joining a vertex to itself, but with possibly several edges joining pairs of vertices. The first part of the paper deals with the question of characterizing those sets of non-negative integers $d_1, d_2, \ldots, d_n$ and $\{c_{ij}\}$, $1 \leqslant i < j \leqslant n$, such that there exists a graph G with n vertices whose valences (degrees) are the numbers d_i, and with the additional property that the number of edges joining i and j is at most c_{ij}. This problem has been studied extensively, in the general case **(1, 2, 9, 11)**, in the case where the graph is bipartite **(3, 5, 7, 10)**, and in the case where the c_{ij} are all 1 **(6)**. A complete answer to this question has been given by Tutte in **(11)**. The existence conditions we obtain (Theorem 2.1) are simplifications of Tutte's conditions but are less general, being applicable only in case the graph G_c corresponding to positive c_{ij} satisfies a certain distance requirement on its odd cycles. Our primary interest in Theorem 2.1, however, attaches to the method of proof. For our proof depends on studying properties of certain systems of linear equations and inequalities, in a context which previously has been exploited only in the case when the matrix of the system is totally unimodular, i.e. when every square submatrix has determinant 0, 1, or -1 **(8)**. That similar results can be achieved when this is not so seems to us the principal point of interest of Theorem 2.1 and its proof.

In the second part of the paper we consider the question of performing certain simple transformations on a graph, called "interchanges," so that, by a sequence of interchanges one can pass from any graph in the class $\mathfrak{G}$ of all graphs with prescribed valences $d_1, d_2, \ldots, d_n$ and at most c_{ij} edges joining i and j, to any other graph in $\mathfrak{G}$. It is shown (Theorem 4.1) that if the graph G_c satisfies a certain cycle condition, this is always possible. The cycle condition required here is sufficiently general to include the case of the complete bipartite graph and hence Theorem 4.1 generalizes the interchange theorem of Ryser for (0, 1)-matrices having prescribed row and column sums **(10)**. The cycle condition also includes the case of an ordinary complete graph ($c_{ij} = 1$ for $1 \leqslant i < j \leqslant n$). Thus, following Ryser, one can deduce from Theorem 4.1 that, for any of the well-known integral-valued functions of a graph (such

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as the colouring number), the set of values attained by all graphs having prescribed valences is a consecutive set of integers.

The last part of the paper discusses other applications to the case in which all $c_{ij} = 1$. The existence conditions of Theorem 2.1 simplify considerably in this special case. They are stated explicitly in Theorem 5.1. It is also shown that one can transform an ordinary graph into a certain canonical form by interchanges. This result, suggested by a theorem of Hakimi **(6)** completes a lacuna in Hakimi's proof.

2. Graphs with prescribed valences. Let

$$d = (d_1, d_2, \ldots, d_n), \tag{2.1}$$

$$c = (c_{12}, c_{13}, \ldots, c_{1n}, c_{23}, c_{24}, \ldots, c_{2n}, \ldots, c_{n-1,n}) \tag{2.2}$$

be two vectors of non-negative integers, the vector c having $n(n-1)/2$ components. Denote by

$$\mathfrak{G} = \mathfrak{G}(d, c) \tag{2.3}$$

the class of all graphs on n vertices having the properties:

(a) the valence (degree) of vertex i is d_i, $1 \leqslant i \leqslant n$;

(b) the number of edges joining vertices i and j is at most c_{ij}, $1 \leqslant i < j \leqslant n$.

We call d the *valence vector* and c the *capacity vector.*

Throughout this paper we adopt the convention that $c_{ji} = c_{ij}$, $1 \leqslant i < j \leqslant n$, and $c_{ii} = 0$. This will simplify matters in writing sums. We also use this convention for other vectors whose components correspond to pairs (i, j), $1 \leqslant i < j \leqslant n$.

Let G_c denote the graph on n vertices in which there is an edge joining vertex i and vertex j if and only if $c_{ij} > 0$. We shall say that the capacity vector c satisfies the *odd-cycle condition* if the graph G_c has the property that any two of its odd (simple) cycles either have a common vertex, or there exists a pair of vertices, one from each cycle, which are joined by an edge. In other words, the distance between any two odd cycles of G_c is at most 1. In particular, if G_c is bipartite (has no odd cycles) or if G_c is complete (all $c_{ij} = 1$), then c obviously satisfies the odd-cycle condition.

THEOREM 2.1. *Assume that c satisfies the odd-cycle condition. Then $\mathfrak{G}(d, c)$ is non-empty if and only if*

(i) $\sum_{i=1}^{n} d_i$ *is even, and*

(ii) *for any three subsets S, T, U which partition $N = \{1, 2, \ldots, n\}$, we have*

$$\sum_{i \in S} d_i \leqslant \sum_{i \in T} d_i + \sum_{\substack{i \in S \\ j \in S \cup U}} c_{ij}. \tag{2.4}$$

(Empty sets are not excluded.)

Proof. The cases $n = 1$ and $n = 2$ can easily be handled separately, so in the course of this proof we shall assume that $n \geqslant 3$. Let A be the n by

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$n(n-1)/2$ incidence matrix of all pairs selected from $N = \{1, 2, \ldots, n\}$, Let

$$B = \begin{bmatrix} A & O \\ I & I \end{bmatrix},$$

where I is the identity matrix of order $n(n-1)/2$, and define the vector

$$b = \begin{pmatrix} d \\ c \end{pmatrix}.$$

Then $\mathfrak{G}$ is non-empty if and only if there is a non-negative integral vector z satisfying

$$Bz = b. \tag{2.5}$$

We now break the proof into a series of three lemmas.

LEMMA 2.2. *The equations* (2.5) *have an integral solution if and only if* (i) *holds.*

Lemma 2.2 does not require the non-negativity of b.

Assume first that the equations (2.5) have an integral solution z, and let x be the vector of the first $n(n-1)/2$ components of z. Let u be a vector with n components, each of which is 1. Then

$$u'Ax = 2 \sum_{i \leqslant j} x_{ij} = \sum_{i=1}^{n} d_i.$$

Since each x_{ij} is an integer, (i) follows.

To prove Lemma 2.2 in the reverse direction, we exhibit a specific integral solution of $Ax = d$. Clearly such a vector x can be extended to an integral vector z which is a solution of (2.5).

Let $s = \sum_i d_i$. Let

$$\begin{aligned} x_{12} &= d_1 + d_2 - \tfrac{1}{2}s, \\ x_{13} &= d_1 + d_3 - \tfrac{1}{2}s, \\ x_{23} &= \tfrac{1}{2}s - d_1, \\ x_{1j} &= d_j \qquad \text{for } 3 < j \leqslant n, \\ x_{ij} &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then this integral vector x clearly satisfies $Ax = d$.

LEMMA 2.3. *The equations* (2.5) *have a non-negative solution if and only if* (ii) *holds.*

It is a consequence of the duality theorem for linear equations and inequalities that (2.5) has a non-negative solution if and only if every vector y satisfying

$$y'B \geqslant 0 \tag{2.6}$$

also satisfies

$$(y, b) \geqslant 0. \tag{2.7}$$

Let C be the cone of all vectors y satisfying (2.6). In order to check (2.7), it suffices to look at the extreme rays of C. Let w be a vector on an extreme ray of C, so chosen that all its components are integers and have 1 as their greatest common divisor. Then it can be shown (we omit the details of the proof, since we shall give in §3 another proof of Lemma 2.3) that either every component of y is non-negative (in which case (2.7) is automatic), or else w has the following appearance. Denote the first n components of w by w_i and the last $n(n-1)/2$ components by w_{ij}, $1 \leqslant i < j \leqslant n$. Then there is a partition S, T, U of $N = \{1, 2, \ldots, n\}$ such that

$$w_i = \begin{cases} -1 & \text{for } i \in S, \\ 1 & \text{for } i \in T, \\ 0 & \text{for } i \in U, \end{cases}$$

$$w_{ij} = \begin{cases} 2 & \text{for } i \in S,\ j \in S, \\ 1 & \text{for } i \in S,\ j \in U, \\ 0 & \text{otherwise.} \end{cases}$$

If we take the inner product of w with b, then (2.7) is the same as (2.4).

Lemmas 2.2 and 2.3 make no use of the odd-cycle condition imposed on c. But this assumption is essential in Lemma 2.4.

LEMMA 2.4. *Let c satisfy the odd-cycle condition. If the equations* (2.5) *have both a non-negative solution and an integral solution, then they have a non-negative integral solution.*

Let $Ax = d$, $0 \leqslant x \leqslant c$. The proof proceeds constructively by reducing the number of non-integral components of x. Let G be the graph on n vertices in which an edge joins i and j if and only if x_{ij} is non-integral. Since each d_i is an integer, it follows that if G has edges, then it must contain a cycle, i.e. there is a sequence of distinct integers $i_1, i_2, \ldots, i_k$ such that $x_{i_1 i_2}, x_{i_2 i_3}, \ldots, x_{i_k i_1}$ are non-integral. We now consider cases.

Case 1. G contains an even cycle. Then we alter x by alternately adding and subtracting a real number ϵ around this cycle. This preserves the valence at each vertex, and ϵ can be selected so that (a) the bounds on components of x are not violated, and (b) at least one component of x corresponding to the even cycle has been made integral.

Case 2. G has only odd cycles. Let $1, 2, \ldots, k$, 1 represent an odd cycle of G. Suppose first that two components of x which are adjacent in this cycle have a non-integral sum, say x_{12}, x_{1k}. Then there is a j, distinct from 2 and k, such that x_{1j} is non-integral. It follows from this and the case assumption that G contains a subgraph which consists of two odd cycles joined by exactly one path (which may be of length 0). Let us denote the two odd cycles by $1, 2, \ldots, k, 1$ and $1', 2', \ldots, l', 1'$, and the path joining them by $1, j_1, j_2, \ldots, j_r, 1'$. (Thus $1 = 1'$ if the path has zero length.) Now consider the sequence

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(2.9) $$1, 2, \ldots, k, 1, j_1, j_2, \ldots, j_r, 1', 2', \ldots, l', 1', j_r, j_{r-1}, \ldots, j_1, 1$$

and the components of x corresponding to adjacent pairs of this sequence. Again we alter components of x corresponding to adjacent pairs of (2.9) by alternately adding and subtracting ϵ. This time components of x corresponding to the path joining the two odd cycles are alternately decreased and increased by 2ϵ, whereas components corresponding to the odd cycles are changed by ϵ. The valence at each vertex is preserved, and ϵ may be selected to decrease the number of non-integral components of x without violating $0 \leqslant x \leqslant c$.

It remains to consider the case in which each pair of components of x which are adjacent in the odd cycle $1, 2, \ldots, k, 1$ sum to an integer. Thus we have

(2.10) $$\begin{aligned} x_{12} + x_{23} \qquad\qquad &= d_2', \\ x_{23} + x_{34} \qquad &= d_3', \\ \ldots \qquad\qquad & \\ x_{k-1,k} + x_{1k} &= d_k', \\ x_{12} \qquad\qquad + x_{1k} &= d_1', \end{aligned}$$

for integers $d_1', d_2', \ldots, d_k'$. The system of equations (2.10) has a unique solution in which, for example,

$$x_{12} = \tfrac{1}{2}(d_2' - d_3' + d_4' - \ldots + d_1').$$

Thus, x_{12} is half of an odd integer, and similarly for other components of x corresponding to the odd cycle. Now, since $\sum^k{}_{i=1} d_i'$ is odd and $\sum^n{}_{i=1} d_i$ is even (by Lemma 2.2), the integer $\sum^n{}_{i=1} d_i - \sum^k{}_{i=1} d_i'$ is odd. Hence, there must be another component of x not yet accounted for which is also non-integral, and which is consequently contained in another cycle of G, having vertices $1', 2', \ldots, l'$, say. We may assume that this new cycle is odd, disjoint from the first, and that each component of x corresponding to the new cycle is half an odd integer, since otherwise we would be in a situation previously examined. Now, by the odd-cycle assumption on c, we may also assume that $c_{11'} > 0$. If $x_{11'}$ is non-integral, again we have a sequence of form (2.9). If $x_{11'} = 0$, change x as follows: add 1 to $x_{11'}$, subtract 1/2 from x_{12}, add 1/2 to $x_{23}, \ldots$, subtract 1/2 from $x_{1k'}$ subtract 1/2 from $x_{1'2'}$, add 1/2 to $x_{2'3'}, \ldots$, subtract 1/2 from $x_{1'l'}$. If $x_{11'}$ is a positive integer, reverse the alteration just described.

Hence, in all cases the number of non-integral components of x can be decreased. This proves Lemma 2.4 and hence Theorem 2.1.

It can be seen from examples that the odd-cycle assumption on c is essential for the sufficiency part of Theorem 2.1. For let $i_1, i_2, \ldots, i_k$ and $j_1, j_2, \ldots, j_l$ be two odd cycles of G_c violating the odd-cycle condition. Let

$$d_{i_1} = d_{i_2} = \ldots = d_{i_k} = d_{j_1} = d_{j_2} = \ldots = d_{j_l} = 1,$$

all other $d_i = 0$. Thus (i) holds. Moreover, taking components of x corresponding to the two cycles equal to 1/2 and all other components equal to 0 gives a solution of $Ax = d$, $0 \leqslant x \leqslant c$. Hence (ii) holds. But there is no integral solution to $Ax = d$, $0 \leqslant x \leqslant c$.

If each component of the valence vector d is 1, then an integral solution of $Ax = d$, $0 \leqslant x \leqslant c$, corresponds to a perfect matching (1-factor) of the graph G in which c_{ij} edges join i and j. Suppose that G is regular, having valence k at each vertex. Then taking $x_{ij} = c_{ij}/k$ yields a non-negative solution of equations (2.5). Hence Lemma 2.4 implies

THEOREM 2.5. *A regular graph on an even number of vertices which satisfies the odd-cycle condition contains a perfect matching.*

Theorem 2.5 is a generalization of a well-known theorem for bipartite graphs which, rephrased in terms of incidence matrices, asserts that an n by n (0, 1)-matrix having k 1's per row and column contains a permutation matrix.

3. Remarks on the connection with bipartite graphs. Let $d_1, d_2, \ldots, d_m$ and $d_{m+1}, d_{m+2}, \ldots, d_n$ be given non-negative integers such that

$$\sum_{i=1}^{m} d_i = \sum_{i=m+1}^{n} d_i, \tag{3.1}$$

and let s denote this common sum. Let

$$c_{ij} \geqslant 0, 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant n,$$

be given non-negative integers. Does there exist a bipartite graph such that the number of edges joining vertex i of $A = \{1, 2, \ldots, m\}$ and vertex j of $B = \{m+1, m+2, \ldots, n\}$ is at most c_{ij}, and such that the valence of vertex i is d_i, $1 \leqslant i \leqslant n$? It is well-known (7) that such a graph exists if and only if, for every $I \subseteq A$ and $J \subseteq B$ we have

$$\sum_{\substack{i \in I \\ j \in J}} c_{ij} \geqslant \sum_{i \in I} d_i + \sum_{j \in J} d_j - s. \tag{3.2}$$

Let us illustrate how this result is a consequence of Theorem 2.1. We only treat the sufficiency, since the necessity is, as usual, trivial. The cycle condition on c is, of course, satisfied, and (i) holds, since the sum of the valences is $2s$. We need only show that (3.2) implies (ii). Let S, T, U partition $\{1, 2, \ldots, n\}$. Let $S_1 = S \cap A$, $S_2 = S \cap B$, and similarly define T_1, T_2, U_1, U_2. Take $I = S_1$ and $J = S_2 \cup U_2$. Then, by (3.2), we have

$$\sum_{\substack{i \in S_1 \\ j \in S_2 \cup U_2}} c_{ij} \geqslant \sum_{i \in S_1} d_i + \sum_{j \in S_2 \cup U_2} d_j - s. \tag{3.3}$$

Now take $I = S_1 \cup U_1$, $J = S_2$. Then, by (3.2), we have

$$\sum_{\substack{i \in S_1 \cup U_1 \\ j \in S_2}} c_{ij} \geqslant \sum_{i \in S_1 \cup U_1} d_i + \sum_{j \in S_2} d_j - s. \tag{3.4}$$

Adding (3.3) and (3.4), we obtain

$$\sum_{\substack{i \in S \\ j \in S \cup U}} c_{ij} \geqslant 2 \sum_{i \in S_1} d_i + \sum_{i \in U_1} d_i + 2 \sum_{j \in S_2} d_j + \sum_{j \in U_2} d_j - 2s,$$

or

$$\sum_{\substack{i \in S \\ j \in S \cup U}} c_{ij} \geqslant \sum_{i \in S_1} d_i - \sum_{i \in T_1} d_i + \sum_{j \in S_2} d_j - \sum_{j \in T_2} d_j = \sum_{i \in S} d_i - \sum_{i \in T} d_i,$$

which is inequality (2.4).

On the other hand, we can show that (ii) is sufficient for the existence of a non-negative solution to (2.5), by using the sufficiency of (3.2) for bipartite graphs. Thus, let d and c be the given valence and capacity vector, respectively, for a graph on n vertices. Now consider the bipartite graph on $2n$ vertices, so paired that the ith vertex of part A and the ith vertex of part B are both required to have valence d_i, $1 \leqslant i \leqslant n$. For this bipartite graph, let y_{ij}, $1 \leqslant i, j \leqslant n$, be the number of edges joining vertex i of A and vertex j of B, and suppose that $y_{ij} \leqslant c_{ij}$. Then setting

$$x_{ij} = \tfrac{1}{2}(y_{ij} + y_{ji}), \qquad 1 \leqslant i < j \leqslant n, \tag{3.5}$$

yields a non-negative solution to (2.5). Hence, it suffices to show that (ii) implies (3.2). Let $I \subseteq \{1, 2, \ldots, n\}$, $J \subseteq \{1, 2, \ldots, n\}$ be given. Let $S = I \cap J$ and let $U = (I - S) \cup (J - S)$, $T = \overline{S \cup U}$. By (ii) we have

$$\sum_{\substack{i \in S \\ j \in S \cup U}} c_{ij} \geqslant \sum_{i \in S} d_i - \sum_{i \in T} d_i. \tag{3.6}$$

But

$$\sum_{\substack{i \in S \\ j \in S \cup U}} c_{ij} \leqslant \sum_{\substack{i \in I \\ j \in J}} c_{ij}$$

and

$$\sum_{i \in S} d_i - \sum_{i \in T} d_i = 2 \sum_{i \in S} d_i + \sum_{i \in U} d_i - \sum_{i=1}^{n} d_i = \sum_{i \in I} d_i + \sum_{j \in J} d_j - \sum_{i=1}^{n} d_i.$$

Thus, (3.6) implies (3.2).

This connection between Theorem 2.1 and bipartite subgraph theory shows, among other things, that an efficient construction is available for subgraphs, having prescribed valences, of a graph satisfying the odd-cycle condition. For, one can first construct the appropriate bipartite graph by methods known to be efficient **(3)**, and then apply the procedure outlined in the proof of Lemma 2.4 to remove any fractions resulting from (3.5). See also **(1, 2)**.

4. An interchange theorem. Our object in this section is to prove that if the capacity vector c satisfies a certain cycle condition, then for any two graphs $G_1, G_2 \in \mathfrak{G} = \mathfrak{G}(d, c)$, one can pass from G_1 to G_2 by a sequence of simple transformations, each of which produces a graph in $\mathfrak{G}$. These transformations we call "interchanges," following **(10)**, and they are defined as follows. For $G \in \mathfrak{G}$, let y_{ij} denote the number of edges joining i and j. If i, j, k, l are distinct vertices of G with $y_{ij} < c_{ij}$, $y_{jk} > 0$, $y_{kl} < c_{kl}$, and $y_{li} > 0$, an *interchange* adds 1 to y_{ij} and y_{kl}, and subtracts 1 from y_{jk} and y_{li}. Thus, an interchange is the simplest kind of transformation that can produce a new graph in $\mathfrak{G}$.

We now describe the condition to be imposed on the capacity vector c. Let us call a subgraph of G_c which is either an even cycle, or two odd cycles joined by exactly one path P (which may be of length zero), an *even set* of G_c. Observe that the latter kind of even set can be represented as a generalized even cycle, in which the vertices of P are repeated, as was done in the proof of Lemma 2.4. If the two odd cycles consist of vertices $1, 2, \ldots, k$ and $1', 2', \ldots, l'$ respectively, and the path, joining 1 and $1'$, has vertices $1, a_1, a_2, \ldots, a_m, 1'$, then a representation is

$$1, 2, \ldots, k, 1, a_1, a_2, \ldots, a_m, 1', 2', \ldots, l', 1', a_m, a_{m-1}, \ldots, a_1, 1. \tag{4.1}$$

We say that c satisfies the *even-set condition* if, for every even set E of G_c, there is a representation of the vertices of E as a generalized even cycle

$$b_1, b_2, \ldots, b_{2p}, b_1 \tag{4.2}$$

in which, for some i, b_i and b_{i+3} (the subscripts taken mod $2p$) are joined by an edge of G_c.

THEOREM 4.1. *Let c satisfy the even-set condition. If $G_1, G_2 \in \mathfrak{G}(d, c)$, then G_1 can be transformed into G_2 by a finite sequence of interchanges.*

Proof. We first introduce a distance between pairs of graphs in $\mathfrak{G}$. If x_{ij} is the number of edges joining i and j in one graph, y_{ij} the corresponding number in the other graph, then the *distance* between the graphs is

$$\sum_{i<j} |x_{ij} - y_{ij}|. \tag{4.3}$$

Let $\mathfrak{G}_1$ be the set of all graphs into which G_1 is transformable by finite sequences of interchanges, and let $\mathfrak{G}_2$ be the corresponding set arising from G_2. Let $H_1 \in \mathfrak{G}_1$ and $H_2 \in \mathfrak{G}_2$ be such that the distance between them is the minimum distance between graphs in $\mathfrak{G}_1$ and $\mathfrak{G}_2$. If the distance between H_1 and H_2 is zero, we are finished. Assume, therefore, that it is positive.

We now introduce some notation. If the number of edges joining i and j is greater in H_1 than in H_2, we shall write $(i, j)_1$. If the number is greater in H_2 than in H_1, we shall write $(i, j)_2$. Since H_1 and H_2 are not the same, there must exist at least one pair of vertices i and j such that $(i, j)_1$. Since the valence of j is the same in both graphs, there must exist a vertex k such that $(j, k)_2$. Continuing this way, we must finally obtain a cycle of distinct vertices

$$i_1, i_2, \ldots, i_k, i_1 \tag{4.4}$$

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such that

(4.5) $$(i_1, i_2)_1, \quad (i_2, i_3)_2, \quad (i_3, i_4)_1, \quad \ldots .$$

We now consider cases.

Case 1. In (4.4), k is even. We first examine the case $k = 4$. We then have

$$(i_1, i_2)_1, \quad (i_2, i_3)_2, \quad (i_3, i_4)_1, \quad (i_4, i_1)_2.$$

Hence, an interchange on H_1 involving the vertices i_1, i_2, i_3, i_4 yields a graph H_1' in $\mathfrak{G}_1$ which is closer to H_2, violating our assumption on the minimality of the distance between H_1 and H_2. Thus $k > 4$. Suppose now that we have established the impossibility of a cycle (4.4) of length l for all even $l < k$. We shall prove the impossibility of such a cycle of length k. Since c satisfies the even-set condition, and our cycle is an even set in G_c, we may assume without loss of generality that $c_{i_1 i_4} > 0$. Let $x_{i_1 i_4}$ be the number of edges in H_1 joining i_1 and i_4. If $x_{i_1 i_4} < c_{i_1 i_4}$, then we may perform an interchange on H_1 involving i_1, i_2, i_3, i_4 to produce a graph H_1' in $\mathfrak{G}_1$ which is closer to H_2. Hence $x_{i_1 i_4} = c_{i_1 i_4}$. Let $y_{i_1 i_4}$ be the number of edges joining i_1 and i_4 in H_2. An analogous argument shows that $y_{i_1 i_4} = 0$. Since $c_{i_1 i_4} > 0$, we have $(i_1, i_4)_1$. Now consider the sequence $i_1, i_4, i_5, \ldots, i_k, i_1$. This is an even cycle of form (4.4) with length less than k, a contradiction.

Case 2. In (4.4), k is odd. Then we have

$$(i_1, i_2)_1, \quad (i_2, i_3)_2, \quad \ldots, \quad (i_{k-1}, i_k)_2, \quad (i_k, i_1)_1.$$

Since the valence of i_1 is the same in both graphs, there must be a vertex j_1 such that $(i_1, j_1)_2$. If j_1 is i_r for some $r \neq 1$, then either $i_1, i_2, \ldots, i_r, i_1$ or $i_1, i_k, i_{k-1}, \ldots, i_r, i_1$ is an even alternating cycle which we have shown to be impossible. Similarly, we must have $(j_1, j_2)_1$, $(j_2, j_3)_2, \ldots$ for new vertices $j_2, j_3, \ldots$ until our sequence terminates with a vertex j_r which is either i_1 or j_t for $t < r$. If $j_r = i_1$, r even, or if $j_r = j_t$, $t < r$, $r - t$ even, again we have an even cycle. In the remaining cases $j_r = i_1$, r odd, or $j_r = j_t$, $t < r$, $r - t$ odd, we have an even set of G_c consisting of two odd cycles joined by just one path. Without loss of generality let

(4.6) $$i_1, \ldots, i_k, i_1, j_1, \ldots, j_t, j_{t+1}, \ldots, j_r = j_t, j_{t-1}, \ldots, j_1, i_1$$

be that representation of the set which exhibits the even-set condition. Again we shall proceed inductively to show the impossibility of (4.6). The smallest case to consider consists of five vertices arising in the order 1, 2, 3, 1, 4, 5, 1. The even-set condition implies that either $c_{24} > 0$ or $c_{35} > 0$. Without loss of generality, assume $c_{24} > 0$. Since $(2, 3)_2$, $(3, 1)_1$, $(1, 4)_2$, we conclude (reasoning as in Case 1) that $(2, 4)_2$. But then $(1, 2)_1$, $(2, 4)_2$, $(4, 5)_1$, and $(5, 1)_2$ form an even cycle, which we know to be impossible.

Next consider (4.6), assuming inductively that we have established the

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impossibility of sequences of this type having a smaller number of vertices. Using the even-set condition, the basic line of reasoning we have been following shows that a new even set with a smaller number of vertices in which edges are alternately $(\ )_1$ and $(\ )_2$ (which may or may not be an ordinary even cycle) would also exist, so that either the Case 1 argument applies or the induction assumption is violated.

This completes the proof of Theorem 4.1. We remark that, when G_c is a bipartite graph, if c does not satisfy the even-set condition, then there is a choice of $\{d_i\}$ so that interchanges are not possible. For the only even sets possible in the bipartite case are simple even cycles, and one can easily show by induction that if there is such a cycle $b_1, \ldots, b_{2k}, b_1$, with no edge in G_c joining b_i and b_{i+3} for any i, then there is an even cycle of length >4 for which G_c contains no edges joining vertices of the cycle except vertices adjacent in the cycle. Set $d_i = 1$ for all i in the latter cycle, 0 otherwise. The two graphs are possible, but one cannot reach either from the other by interchanges.

5. Applications to ordinary graphs. In this section we confine attention to the case in which all components of the capacity vector c are 1. Thus, G_c is the complete graph on n vertices. Since the odd-cycle condition and the even-set condition are both satisfied by c, Theorems 2.1 and 4.1 are applicable.

The existence conditions (ii) of Theorem 2.1 simplify enormously in this special case. For, arranging the components of the valence vector in monotonically decreasing order,

$$d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n, \tag{5.1}$$

it follows at once that all the inequalities (2.4) are equivalent to the $n(n+1)/2$ inequalities

$$\sum_{i=1}^{k} d_i \leqslant \sum_{i=l+1}^{n} d_i + k(l-1), \qquad 1 \leqslant k \leqslant l \leqslant n. \tag{5.2}$$

If we use the term "ordinary graph" to mean a graph in which at most one edge joins a pair of vertices, we then have

THEOREM 5.1. *There is an ordinary graph on n vertices having valences* (5.1) *if and only if* $\sum^n{}_{i=1} d_i$ *is even and the inequalities* (5.2) *hold.*

The inequalities (5.2) can be further simplified to a system of n inequalities, as follows. Represent the valences (5.1) by an n by n (0, 1)-matrix whose ith row contains d_i 1's, these being filled in consecutively from the left, except that a 0 is placed in the main diagonal position. Let $\bar{d}_i$, $1 \leqslant i \leqslant n$, be the column sums of this matrix. One can then show that

$$\sum_{i=1}^{k} \overline{d_i} = \operatorname*{Min}_{0 \leqslant l \leqslant n} \left\{ \sum_{i=l+1}^{n} d_i + kl - \operatorname{Min}(k, l) \right\}. \tag{5.3}$$

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On the other hand, (5.2) holds for all k, l in $1 \leqslant k \leqslant l \leqslant n$ if and only if the left side of (5.2) is at most the right side of (5.3) for all k in $1 \leqslant k \leqslant n$. Hence, inequalities (5.2) are equivalent to

$$\sum_{i=1}^{k} d_i \leqslant \sum_{i=1}^{k} \overline{d_i}, \qquad 1 \leqslant k \leqslant n. \tag{5.4}$$

We turn now to the notion of an interchange as applied to ordinary graphs. Here an interchange replaces edges (i, j) and (k, l) with (i, k) and (j, l), the latter pairs being non-edges originally. From Theorem 4.1 we have

THEOREM 5.2. *Let G_1 and G_2 be two ordinary graphs having the same valences. Then one can pass from G_1 to G_2 by a finite sequence of interchanges.*

In connection with Theorem 5.2, we note that an ordinary graph can be transformed by interchanges into a simple canonical form suggested by Hakimi **(6)**. This canonical form, which is the analogue of a similar one for the case of (0, 1)-matrices having prescribed row and column sums **(4, 5)**, can be described informally as follows. Assume (5.1). Then there will be edges from vertex 1 to vertices $2, 3, \ldots, d_1 + 1$. Reduce valences appropriately, arrange the new valences in decreasing order, and repeat the process. To prove that this canonical form can be realized, it is sufficient to carry out the first step of distributing the edges at vertex 1 to vertices $2, 3, \ldots, d_1 + 1$. Assume that, by interchanges, we have gone as far as possible in this direction, so there are edges from 1 to $2, \ldots, k$, $k < d_1 + 1$, and no edge from 1 to $k + 1$. Let t be any vertex other than $2, \ldots, k$ which is joined to 1 by an edge. Let u be any vertex joined to $k + 1$ by an edge. If t and u are not joined by an edge, an interchange involving $1, k + 1, u, t$, contradicts our assumption on k. Hence, t and u are joined by an edge. But since u was an arbitrary vertex joined to $k + 1$ by an edge and since t is joined to 1, it follows that the valence of t exceeds that of $k + 1$. This contradicts our scheme for numbering vertices, and hence proves the validity of the canonical form.

This argument provides another proof of Theorem 5.2, since any two ordinary graphs G_1 and G_2 having the same valences can be transformed into the canonical form by interchanges, and hence G_1 can be transformed into G_2.

We also observe that any vertex could play the role of vertex 1 in the construction of the canonical form outlined above, and hence there are a variety of "canonical forms," obtainable by selecting an arbitrary vertex, distributing its edges among other vertices having greatest valences, and repeating the procedure in the reduced problem.

A consequence of Theorem 5.2 is that, for any integer-valued function of a graph which changes by at most 1 under an interch
number), the val

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