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On the abundance of *k*-fold semi-monotone minimal sets in bimodal circle maps

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Abstract. Inspired by a twist map theorem of Mather. we study recurrent invariant sets that are ordered like rigid rotation under the action of the lift of a bimodal circle map g to the k-fold cover. For each irrational in the rotation set's interior, the collection of the k-fold ordered semi-Denjoy minimal sets with that rotation number contains a $(k-1)$-dimensional ball with the weak topology on their unique invariant measures. We also describe completely their periodic orbit analogs for rational rotation numbers. The main tool used is a generalization of a construction of Hedlund and Morse that generates symbolic analogs of these k-fold well-ordered invariant sets.

Key words: circle dynamics, Denjoy minimal sets, weak topology, circular ordered orbits
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1. *Introduction*

In a dynamical system a rotation number or vector measures the asymptotic speed and direction of an orbit. The rotation set collects all these together into a single invariant of the system. The natural question is how much this invariant tells you about the dynamics. Perhaps the first issue is whether for each rotation number there is a nice invariant set in which every point has that rotation number.

This question has been studied in a number of contexts, with the most complete answer available for maps of the circle, annulus, and two-dimensional torus. In these cases the basic question is enhanced by requiring that the invariant set of a given rotation vector has the same combinatorics as a rigid rotation. So, for example, for a continuous degree-one map g of the circle and a number ω in its rotation set, is there an invariant set Z_ω on which the action of g looks like the invariant set of rigid rotation of the circle by ω? This is made clearer and more precise by lifting the dynamics to the universal cover $\mathbb{R}$. The question then translates to whether the action of the lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ on the lift $\tilde{Z}_\omega$ is order-preserving. For this class of maps the answer is yes; such invariant sets always exist [18].

On the torus and annulus a general homeomorphism isotopic to the identity lacks the structure to force the desired invariant sets to be order-preserving, therefore topological analogs are used [11, 31, 41]. The monotone twist hypothesis provides the required additional structure in the annulus case. The celebrated Aubry–Mather theorem states that for each rational in the rotation set there is a periodic orbit and for each irrational a Denjoy minimal set, and the action of the map on these invariant sets is ordered in the circle factor like rigid rotation. These invariant sets are now called *Aubry–Mather sets*.

For an area-preserving monotone twist map the minimal set with a given irrational rotation number could be an invariant circle. When a parameter is altered and this circle breaks it is replaced by an invariant Denjoy minimal set. In [36] Mather investigated what additional dynamics this forces. He showed that in the absence of an invariant circle with a given irrational rotation number there are many other invariant minimal Cantor sets with the same rotation number and the dynamics on these sets is nicely ordered under the dynamics not in the base, but in finite covering spaces of the annulus.

More specifically, these invariant minimal sets are Denjoy minimal sets which are uniquely ergodic. Their collection is topologized using the weak topology on these measures. Mather showed that for a given irrational rotation number in the rotation set the collection of Denjoy minimal sets with that rotation number that are ordered in the k-fold cover contains a topological disk of dimension $k - 1$. In this paper we prove the analog of this result for a class $\mathcal{G}$ of bimodal degree-one maps of the circle. We also describe their periodic orbits which have nicely ordered lifts in the k-fold cover.

Mather's proof use variational methods. The main methods here come from symbolic dynamics and use a construction that generalizes one due to Hedlund and Morse ([38] and [24, p. 111]). Such generalizations are a common tool in topological dynamics ([35], [3, pp. 234–241], and [12]). This Hedlund–Morse (HM) construction, used here for a rotation number ω and number k, generates the itineraries under rigid rotation by ω with respect to an address systems made from $2k$ intervals on the circle. The closure of this set of itineraries yields the symbolic analog of an invariant set that is nicely ordered in the k-fold cover. These sets are termed *symbolic k-fold semi-monotone sets (symbolic kfsm sets)*. Varying the address system parameterizes the symbolic kfsm sets in both the Hausdorff and weak topologies.

A *physical kfsm set* (or just a kfsm set if the context is clear) is a g-invariant set Z that has a lift Z' to the k-fold cover on which the lift g_k of g acts like rigid rotation. Physical and symbolic kfsm sets are connected by a second tool.

The second tool uses addresses and itineraries, but this time to code orbits under the bimodal map g. Restricting to all orbits that land in the positive-slope region, we get an invariant set $\Lambda(g)$ which is coded by an order interval in the one-sided 2-shift Σ_2^+. Because we study invariant sets ordered in the k-fold cover, we lift this coding to the orbits which stay in the positive-slope region under g_k in the k-fold cover S_k. This yields a g_k-invariant set $\Lambda_k(g)$ which is then coded by a subshift $\hat{\Lambda}_k(g) \subset \Sigma_{2k}^+$.

This result connects the physical kfsm sets in $\Lambda_k(g)$, the symbolic kfsm sets in $\hat{\Lambda}_k(g)$, and the symbolic sets generated by the HM construction. Part (c) will be explained below.

THEOREM 1.1. *For $g \in \mathcal{G}$ the following assertions are equivalent.*

(a) *$Z \subset \Lambda(g)$ is a recurrent kfsm set for g.*

(b) *The symbolic coding of Z via the itinerary map is constructable via the HM process.*

(c) *Z is a recurrent set of an interpolated semi-monotone map $H_{\vec{c}}$ in the k-fold cover.*

Note that the result is restricted to recurrent kfsm sets. There are several reasons for this: first, the recurrent points are where the interesting dynamics occurs; second, invariant measures are always supported on recurrent sets; and finally, the HM construction produces recurrent sets. As is well known in Aubry–Mather theory, there are also non-recurrent kfsm sets which consist of a recurrent set and orbits homoclinic to that set. We also restrict to orbits that stay in the positive-slope region of g. Considering kfsm sets that also have points in the negative-slope region at most adds additional homoclinic orbits or shadow periodic orbits. See §13.2.

For each k, the HM construction depends on two parameters, a rotation number ω and a parameter $\vec{\nu}$ describing the address system on the circle. For a rational rotation number it produces a finite cluster of periodic orbits, while for irrationals it produces a semi-Denjoy minimal set. Since g is non-injective the analogs of Denjoy minimal sets have pairs of points that collapse in forward time, hence the 'semi' in their name.

Another main result is that the HM construction parameters $(\omega, \vec{\nu})$ yield a homeomorphic parameterization of the space of invariant measures on the recurrent symbolic kfsm sets with the weak topology. Via the itinerary map, this is pulled back to a parameterization of the space of invariant measures on the physical recurrent kfsm sets with the weak topology. It yields the following result in which $\rho(g)$ is the rotation interval of $g \in \mathcal{G}$.

THEOREM 1.2. *Assume $g \in \mathcal{G}$, $\alpha \notin \mathbb{Q}$, $\alpha \in \mathrm{Int}(\rho(g))$, and $k > 0$.*

(a) *In the weak topology there is a $(k-1)$-dimensional disk of kfsm semi-Denjoy minimal sets with rotation number α.*

(b) *If p_n/q_n is a sequence of rationals in lowest terms with $p_n/q_n \to \alpha$, then the number of distinct kfsm periodic orbits of g with rotation number p_n/q_n grows like q_n^{k-1}.*

Informally, a kfsm semi-Denjoy minimal set wraps k times around the circle with orbits moving at different average speeds in each loop. Lifting to the k-fold cover, these 'speeds' are given by the amount of the unique invariant measure present in a fundamental domain of S_k: more measure means slower speed. The k-dimensional vector of these measures is called the *skewness* of the minimal set. The sum of the skewness components is one, and thus the collection of possible skewnesses contains a $(k-1)$-dimensional ball. The skewness turns out to be an injective parameterization of the kfsm sets for a given irrational rotation number in the interior of the rotation set of a $g \in \mathcal{G}$ (see Remark 9.14).

The HM parametrization of kfsm sets with the Hausdorff topology is only lower semi-continuous. The points of discontinuity are given in Theorem 9.5.

We will on occasion use results derived from those of Aubry–Mather theory. While the context here is a bit different, the proofs are virtually identical and so are omitted. There are excellent expositions of Aubry–Mather theory; see, for example, [**33**], [**30**, Ch. 13], and [**23**, Ch. 2]. A version of Mather's theorem on Denjoy minimal sets is given in [**45**] for monotone recursion maps.

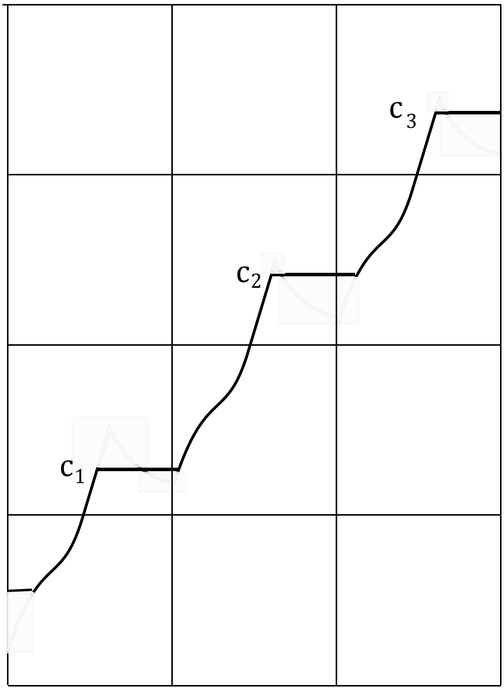

FIGURE 1. The lift of a $g \in \mathcal{G}$ to the 3-fold cover and an interpolated semi-monotone map.

We restrict attention here to a particular class of bimodal circle maps defined in §4.1. Using the Parry–Milnor–Thurston theorem for degree-one circle maps, the results can be transferred (with appropriate alterations) to general bimodal circle maps (see Remark 4.2).

Note that the results here immediately apply to a class of annulus homeomorphisms. This application can be done either via the Brown–Barge–Martin method using the inverse limit of $g \in \mathcal{G}$ [**5**, **14**] or via the symbolic dynamics in annulus maps with good coding like a rotary horseshoe, for example, [**12**, **20**, **25**, **32**].

Figure 1 illustrates the conceptual framework that inspired the results here. It shows the graph of a map $g \in \mathcal{G}$ lifted to the 3-fold cover. At three heights $(c_1, c_2, c_3) = \vec{c}$ the graph is cut off, yielding a semi-monotone circle map $H_{\vec{c}}$. Such maps have a unique rotation number and well-understood recurrent sets which are of necessity semi-monotone sets. As $\vec{c}$ is varied, the rotation number $\rho(H_{\vec{c}}) = \rho(\vec{c})$ varies continuously. Thus one would expect that the level sets $\rho^{-1}(\omega)$ provide a parameterization of the kfsm sets with rotation number ω. In particular, for irrational ω this level set should be a $(k-1)$-dimensional disk as in Theorem 1.2(a). This is true for $g \in \mathcal{G}$. Figure 5 below shows some level sets. It is worth noting that this figure is not a bifurcation diagram, but rather a detailed analysis of the dynamics present in a single map.

While providing a valuable heuristic, this point of view is not as technically tractable as the HM construction and we content ourselves with just a few comments on it in §13. One of these adds item (c) to the list of equivalent conditions in Theorem 1.1.

The literature on bimodal circle map dynamics is vast and we briefly mention only two threads here. Symbolic dynamics for degree-one bimodal circle maps goes back at least to [6, 26, 27]. The interpolated 'flat spot' map trick for finding 1-fold semi-monotone sets was discovered and used by many people in the early 1980s; references include [10, 18, 29, 37, 40, 42–44]. The author learned the trick from G.R. Hall in spring 1983 and the idea of applying it in finite covers emerged in conversations with him.

There are many questions raised by this work; here we mention three. As is well known, the 1-fold symbolic semi-monotone sets generated by the HM construction are the much-studied Sturmian sequences. The general symbolic kfsm sets are thus a generalization of one property of the Sturmians to more symbols (there are many other generalizations). The Sturmians have many marvelous properties such as their connection to the Farey tree and substitutions: which properties are shared by symbolic kfsm sets?

The HM construction is an explicit parameterized way of getting well-controlled orbits that do not preserve the cyclic order in the base and thus in most cases force positive entropy as well as additional orbits. A second question is how the parameterization given by the the HM construction interacts with the forcing orders on orbits in dimension one and two (see, for example, [1, 13]).

A final question relates to the global parameterization of kfsm sets by the HM construction. Each bimodal map $\in \mathcal{G}$ corresponds to a specific set of parameters, namely, those that generate symbolic kfsm sets whose physical counterparts are present in g. What is the shape of this set of parameters?

After this work was completed the author became aware of the considerable literature on sets in the circle that are invariant and nicely ordered under the action of $z \mapsto z^d$ [8, 9, 17, 21, 22, 34]. While the exact relationship of that theory and what is contained in this paper is not clear, it is clear that the two areas share many basic ideas and methods. These include families of interpolated semi-monotone circle maps with flat spots, tight and loose gaps in invariant Cantor sets, parameterization of the sets using the position of the flat spots, and parameterization of the sets with irrational rotation number by an analog of skewness. Section 14 contains a few more comments on the relationship of the problems.

2. *Preliminaries*

2.1. *Dynamics.* Throughout this section X is a metric space and $g : X \to X$ is a continuous, onto map. Since the maps g we will be considering will usually not be injective, we will just be considering forward orbits, so $o(x, g) = \{x, g(x), g^2(x), \ldots\}$.

A point x is *recurrent* if there exists a sequence $n_i \to \infty$ with $g^{n_i}(x) \to x$. A g-invariant set Z is called *recurrent* if every point $z \in Z$ is recurrent. Note that *a recurrent* subset is usually different than *the recurrent set*, with the latter being the closure of all recurrent points. A compact invariant set Z is called *minimal* if every point $z \in Z$ has a forward orbit that is dense in Z.

The one-sided shift space on n symbols is $\Sigma_n^+ = \{0, \ldots, n-1\}^{\mathbb{N}}$ and that on $\mathbb{Z}$ symbols is $\Sigma_{\mathbb{Z}}^+ = \mathbb{Z}^{\mathbb{N}}$. Occasionally we will write Σ_∞^+ for $\Sigma_{\mathbb{Z}}^+$. For clarity we note that in this paper $0 \in \mathbb{N}$. In every case we give one-sided shift spaces the lexicographic order and the left shift map is denoted by σ, perhaps with a subscript to indicate the shift space upon

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which it acts. Maps between shifts and subshifts here will always be defined by their action on individual symbols, so, for example, $\varphi : \Sigma_n^+ \to \Sigma_n^+$ defined on symbols by $s \mapsto \varphi(s)$ means that $\varphi(s_0 s_1 s_2 \ldots) = \varphi(s_0)\varphi(s_1)\varphi(s_2) \ldots$. For a block $B = b_0 \ldots b_{N-1}$ in Σ_n^+ its cylinder set is $[B] = \{\underline{s} \in \Sigma_n^+ : s_i = b_i, i = 0, \ldots, N-1\}$. Note that all our cylinder sets start with index 0.

The space–map pairs (X, f) and (Y, g) are said to be *topologically conjugate* by h if h is a homeomorphism from X onto Y and $hf = gh$.

We will frequently use the standard dynamical tool of addresses and itineraries. Assume $X = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_{n-1}$, with $\sqcup$ denoting disjoint union. Define the *address map* A as $A(x) = j$ when $x \in X_j$ and the *itinerary map* $\iota : X \to \Sigma_n^+$ by $\iota(x)_i = A(g^i(x))$. It is immediate that $\sigma \circ \iota = \iota \circ g$. In many cases here, ι will be continuous and injective, yielding a topological conjugacy from (X, g) to a subset of (Σ_n^+, σ).

We will also encounter the situation where the X_j are not disjoint, but intersect only in their frontiers $\mathrm{Fr}(X_j)$. In this case we define a 'good set' $G = \{x : o(x, g) \cap (\cup \mathrm{Fr}(X_j)) = \emptyset\}$. In this case the itinerary map is defined as $\iota : G \to \Sigma_n^+$.

For $Z \subset X$, its interior, closure and frontier are denoted by $\mathrm{Int}(Z)$, $\mathrm{Cl}(Z)$, and $\mathrm{Fr}(X)$, respectively. The ϵ-ball about x is $N_\epsilon(x)$. The Hausdorff distance between two sets is denoted $\mathrm{HD}(X, Y)$. For an interval I in $\mathbb{R}$, $|I|$ denotes it length, and for a finite set Z, $\#Z$ is its cardinality. On an ordered k-tuple the map τ is the left cyclic shift, so $\tau(a_1, a_2, \ldots, a_k) = (a_2, a_3, \ldots, a_k, a_1)$. On the circle S^1, $\theta_1 < \theta_2$ is defined as long as $|\theta_1 - \theta_2| < 1/2$.

If $g : X \to X$ and $Y \subset X$, then $h : X \to X$ is said to *interpolate* g on Y if $h(y) = g(y)$ for all $y \in Y$, or in symbols, $g_{|Y} = h_{|Y}$.

2.2. *The circle, finite covers and degree-one circle maps.* While the only compact manifold here will be a circle, it will clarify matters to use the language of covering spaces.

In general, if $\pi : \tilde{Y} \to Y$ is a covering space, and $Z \subset Y$, *a lift* of Z is any set $Z' \subset \tilde{Y}$ with $\pi(Z') = Z$. The *full lift* of Z is $\tilde{Z} = \pi^{-1}(Z)$. If $g : Y \to Y$ lifts to $\tilde{g} : \tilde{Y} \to \tilde{Y}$ and $Z \subset Y$ is g-invariant then the full lift $\tilde{Z}$ is $\tilde{g}$-invariant, a property that is usually not shared by a lift Z'.

The universal cover of the circle is $\mathbb{R}$, with $T(x) = x + 1$ generating the deck group. Thus the covering space is $\pi : \mathbb{R} \to S^1 = \mathbb{R}/T = \mathbb{R}/\mathbb{Z} = [0, 1]/\sim$. We will only study maps $g : S^1 \to S^1$ whose lifts $\tilde{g}$ commute with the deck transformation, $\tilde{g}T = T\tilde{g}$, or $\tilde{g}(x + 1) = \tilde{g}(x) + 1$. These circle maps are commonly termed *degree one*. Our given g will usually have a preferred lift $\tilde{g}$ and so all other lifts are obtained as $T^n\tilde{g}$ or $\tilde{g} + n$.

Central to our study are the finite k-fold covers of the circle for each $k > 0$, $S_k = \mathbb{R}/T^k = \mathbb{R}/k\mathbb{Z} = [0, k]/\sim$. A generator of the deck group is $T_k : S_k \to S_k$ induced by T on $\mathbb{R}$ and the covering space is $\pi_k : S_k \to S^1$. A preferred lift $\tilde{g}$ of g to $\mathbb{R}$ induces a preferred lift $\tilde{g}_k : S_k \to S_k$ that commutes with T_k. We also need the map from the universal cover to the k-fold cover treating S_k as the base space $p_k : \mathbb{R} \to S_k$.

A g-periodic point x is said to have *rotation type* (p, q) with respect to the preferred lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ if x has period q and for some lift $x' \in \mathbb{R}$, $\tilde{g}^q(x') = T^p x'$. Note that there is no requirement here that p and q are relatively prime.

A central concern in this paper is how g-minimal sets in S^1 lift to k-fold covers.

THEOREM 2.1. *Let $g : S^1 \to S^1$ be degree one and fix $1 < k < \infty$.*

(a) *If $Z \subset S^1$ is a minimal set, then there exists an m which divides k so that the full lift of Z to S_k satisfies*

$$\tilde{Z} = \bigsqcup_{j=1}^{m} Z'_j \tag{2.1}$$

with each Z'_j minimal under $\tilde{g}_k$, $\pi_k(\tilde{Z}'_j) = Z$ and $T_k(Z'_j) = Z'_{j+1}$ with indices mod k.

(b) *If $Z', Z'' \subset S_k$ are $\tilde{g}_k$ minimal sets, we have $\pi_k(Z') = \pi_k(Z'')$ if and only if $T_k^p(Z') = Z''$ for some p.*

(c) *If $x \in S^1$ is a periodic point with rotation type (p, q), let $m = \gcd(k, p)$. There exist $x'_j \in \pi_k^{-1}(x) \subset S_k$ with $1 \leq j \leq m$ and*

$$\pi_k^{-1}(o(x, g)) = \bigsqcup_{j=1}^{m} o(x'_j, \tilde{g}_k), \tag{2.2}$$

the period of each x'_j under $\tilde{g}_k$ equal to kq/m, and $T_k(x'_j) = x'_{j+1}$ with indices mod k.

Proof. To prove (a) we begin with two preliminary facts with similar proofs. First, we show that for any $z' \in \tilde{Z}$, $\pi_k(\mathrm{Cl}(o(z', \tilde{g}_k))) = Z$. Let $z = \pi_k(z')$ and pick $y \in Z$. By minimality there exists $g^{n_i}(z) \to y$. Lifting and using the compactness of S_k, there are a subsequence $n_{i'}$ and a $y' \in S_k$ with $\tilde{g}_k^{n_{i'}}(z') \to y'$. Thus $g_k^{n_{i'}}(z) = \pi_k(\tilde{g}_k^{n_{i'}}(z')) \to \pi_k(y')$ and so $y = \pi_k(y')$.

Second, we show that for any $z', y' \in \tilde{Z}$, there exists a p with $T^p(y') \in \mathrm{Cl}(o(z', \tilde{g}_k))$. Let $z = \pi_k(z')$ and $y = \pi_k(y')$. By minimality again, we have $g^{n_i}(z) \to y$. Lifting and passing to a subsequence, there are a subsequence $n_{i'}$ and a $y'' \in S_k$ with $\tilde{g}_k^{n_{i'}}(z') \to y''$. Thus $\pi_k(y'') = y$ also, so there is a p with $y'' = T_k^p(y')$ and so $T_k^p(y') \in \mathrm{Cl}(o(z', \tilde{g}_k))$.

Now for the main proof, pick $z' \in \tilde{Z}$ and let $Z'_1 = \mathrm{Cl}(o(z', \tilde{g}_k))$, so by the first fact, $\pi_k(Z'_1) = Z$. We now show Z'_1 is minimal under $\tilde{g}_k$. If not, there is a $y' \in Z'_1$ with $\mathrm{Cl}(o(y', \tilde{g}_k)) \subsetneq \mathrm{Cl}(o(z', \tilde{g}_k))$. By the second preliminary fact, there is some p with

$$\mathrm{Cl}(o(T_k^p(z'), \tilde{g}_k)) \subset \mathrm{Cl}(o(y', \tilde{g}_k)) \subsetneq \mathrm{Cl}(o(z', \tilde{g}_k)).$$

Acting by the homeomorphism T_k^p and iterating the strict inclusions, we have

$$\begin{aligned}\mathrm{Cl}(o(z', \tilde{g}_k)) = \mathrm{Cl}(o(T_k^{pk}(z'), \tilde{g}_k)) &\subsetneq \mathrm{Cl}(o(T_k^{p(k-1)}(z'), \tilde{g}_k)) \subsetneq \cdots \subsetneq \mathrm{Cl}(o(T_k^p(z'), \tilde{g}_k)) \\ &\subsetneq \mathrm{Cl}(o(z', \tilde{g}_k)),\end{aligned}$$

a contradiction, so $\tilde{g}_k$ acting on Z'_1 is minimal. Thus since $\tilde{g}_k T_k = T_k \tilde{g}_k$, $\tilde{g}_k$ acting on each $Z'_j := \tilde{g}_k^j(Z'_1)$ is minimal. Now minimal sets either coincide or are disjoint, so there is a least m with $T_k^{m+1} Z'_1 = Z'_1$.

For the proof of (b), assume $\pi_k(Z') = \pi_k(Z'')$. Now $Z := \pi_k(Z')$ is minimal under g and thus since $Z' \subset \pi_k^{-1}(Z)$ and minimal sets are always disjoint or equal, using (2.1) we have that $Z' = Z'_j$ for some j. Similarly, $Z'' = Z'_{j'}$ for some j', and thus $Z' = T^p Z'_1$ and $Z'' = T^{p'} Z'_1$, and so $Z'' = T^{p'-p}(Z')$ as required.

Turning to (c), since the deck group of S_k is $\mathbb{Z}_k$ there is a natural identification of $\pi_k^{-1}(x)$ with $\mathbb{Z}_k$, with x'_1 identified with zero. Since $\tilde{g}_k^q(\tilde{x}) = T^p\tilde{x}$ in $\mathbb{R}$ the induced action of $\tilde{g}_k^q$ on $\mathbb{Z}_k$ is $n \mapsto n + p$ mod k. An easy elementary number theory argument yields that this action has exactly $\gcd(p, k)$ distinct orbits. Thus $\tilde{g}_k^q$ has exactly $\gcd(p, k)$ distinct orbits when acting on $\pi_k^{-1}(x)$. But $x'_i, x'_j \in \pi_k^{-1}(x)$ are on the same $\tilde{g}_k^q$ orbit if and only if they are on the same $\tilde{g}_k$ orbit and each orbit in $\pi_k^{-1}(o(x, g))$ contains at least one point from $\pi_k^{-1}(x)$. Thus $\tilde{g}_k$ acting on $\pi_k^{-1}(o(x, g))$ has exactly $\gcd(p, k)$ orbits. The rest of the form of (2.2) follows from part (a). $\square$

While it is not used here, a similar result holds for $\mathbb{Z}$-covers and their cyclic quotients when the map on the base has a lift that commutes with the deck transformations.

Remark 2.2. Some special cases of (c) are worth noting. If $\gcd(p, k) = 1$, then x lifts to a single period-qk orbit in S_k. If $p = k$, then x lifts to k different period-q orbits in S_k. When $k = 2$, there is a simple dichotomy. When p is odd, x lifts to one period-$2q$ orbit; when p is even, x lifts to a pair of period-q orbits.

2.3. *Rotation number and interval.* For $\tilde{g} : \mathbb{R} \to \mathbb{R}$ a fixed lift of a degree-one $g : S^1 \to S^1$, define the *rotation number* of $x' \in \mathbb{R}$ as

$$\rho(x', \tilde{g}) = \lim_{n\to\infty} \frac{\tilde{g}^n(x') - x'}{n} \tag{2.3}$$

when the limit exists. Note that this value depend in a simple way on the choice of lift $\tilde{g}$ of g, namely, $\rho(x', \tilde{g} + m) = \rho(x', \tilde{g}) + m$. In most cases below there will be a preferred lift of a given g that will be used in (2.3) and we define $\rho(x, g) = \rho(x', \tilde{g})$ where x' is a lift of x. When g is understood we will just write $\rho(x)$. If x is a periodic point of rotation type (p, q) then $\rho(x) = p/q$.

If Z is a g-invariant set, let

$$\rho(Z) = \{\rho(x, g) : x \in Z\}$$

and $\rho(g) = \rho(S^1, g)$. The latter set has been proved to be a closed interval [**28**, **39**] and thus it is called the *rotation interval* of the map. We shall also have occasion to use $\rho(\tilde{g})$ with the obvious meaning.

There is a alternative way of computing the rotation interval using upper and lower maps that is now standard ([**10**, **18**, **29**, **37**] and elsewhere). Given a lift of a degree-one circle map $\tilde{g} : \mathbb{R} \to \mathbb{R}$, let $\tilde{g}_u(x) = \sup\{\tilde{g}(y) \colon y \le x\}$ and $\tilde{g}_\ell(x) = \inf\{\tilde{g}(y) \colon y \ge x\}$. If g_u and g_ℓ are their descents to S^1 they are both semi-monotone maps and so each of their rotation sets is a single point (see Lemma 3.1 below). The rotation interval of g is

$$\rho(g) = [\rho(g_\ell), \rho(g_u)]. \tag{2.4}$$

To define the rotation number of a g-invariant Borel probability measure μ, start by letting $\Delta_g : S^1 \to \mathbb{R}$ be $\Delta_g(x) = \tilde{g}(x') - x'$ which is independent of the choice of lift x'. Then

$$\rho(\mu) = \int \Delta_g \, d\mu. \tag{2.5}$$

Note that when μ is ergodic by the pointwise ergodic theorem for μ-almost every x, $\rho(x, g) = \rho(\mu)$.

For points, invariant sets and measures in the cyclic cover S_k under the preferred lift $\tilde{g}_k$, there are two ways to consider the rotation number. The most common will be to project to the base and define

$$\rho_k(x, g) = \rho(\pi_k(x), g). \tag{2.6}$$

For μ a $\tilde{g}_k$-invariant measure, let

$$\rho_k(\mu) = \rho((\pi_k)_*\mu). \tag{2.7}$$

Remark 2.3. The other way to work with rotation numbers in S_k is to consider $\tilde{g}_k$ as a map of the circle itself. To work on the standard circle we first rescale S_k via $D_k : S_k \to S^1$ via $D_k(\theta) = \theta/k$. Note that D_k is not a covering map but rather a coordinate rescaling homeomorphism. For $Z \subset S_k$, then $\rho(D_k Z, D_k \circ \tilde{g}_k \circ D_k^{-1})$ is the desired rotation number. These two methods are related simply by $\rho_k(x, g) = k\rho(D_k Z, D_k \circ \tilde{g}_k \circ D_k^{-1})$.

3. *Semi-monotone degree-one maps*

3.1. *Definition and basic properties.*

In this section we describe the basics of a small but crucial expansion of the class of circle homeomorphisms, namely, continuous maps whose lifts are *semi-monotone*. They share many of the properties of circle homeomorphisms and are a standard and important tool in circle dynamics.

Thus we consider continuous, degree-one $h : S^1 \to S^1$ whose lifts $\tilde{h}$ to $\mathbb{R}$ satisfy that $x_1' < x_2'$ implies $\tilde{h}(x_1') \le \tilde{h}(x_2')$. Note that this is independent of the choice of lift $\tilde{h}$ of h. We shall also call such maps *weakly order-preserving*. Note that in topology a monotone map is one with connected point inverses. In this sense a semi-monotone map is monotone. On the other hand, considering the point of view of order relations, semi-monotone is contrasted with monotone. We adapt the latter viewpoint. Let $\mathcal{H}$ be the collection of all such maps with the C^0-topology, and let $\tilde{\mathcal{H}}$ denote all their lifts.

A *flat spot* for a $h \in \mathcal{H}$ is a non-trivial closed interval J where $h(J)$ is a constant and for which there is no larger interval containing J on which h is constant. A given h can have at most a countable number of flat spots J_i and we define the 'positive-slope region' of h as $P(h) = S^1 \setminus (\cup \operatorname{Int} J_i)$. The proof of the next result is standard. For the irrational case, see [4].

LEMMA 3.1. *Assume $h \in \mathcal{H}$ with preferred lift $\tilde{h}$.*

(a) *The rotation number $\rho(x, h)$ exists and is the same for all $x \in S^1$ and so $\rho(h)$ is a single number.*
(b) *The map $\rho : \tilde{\mathcal{H}} \to \mathbb{R}$ is continuous.*
(c) *If $\tilde{h}, \tilde{h}_1 \in \tilde{\mathcal{H}}$ and $\tilde{h}_1 \le \tilde{h}$ then $\rho(\tilde{h}_1) \le \rho(\tilde{h})$.*
(d) *If $\rho(h) = p/q$ in lowest terms then all periodic orbits have rotation type (p, q) and the recurrent set of h consists of a union of such periodic orbits.*
(e) *If $\rho(h) = \alpha \notin \mathbb{Q}$ then h has exactly one recurrent set which is a minimal set Z and it is wholly contained in $P(h)$. Further, h is uniquely ergodic with the unique invariant measure supported on Z.*

Definition 3.2. The minimal set in (e) above is called a *semi-Denjoy minimal set*. More generally, an abstract minimal set is called semi-Denjoy if it is topologically conjugate to the semi-Denjoy minimal set in a semi-monotone degree-one circle map.

Remark 3.3. A semi-Denjoy minimal set looks like a usual Denjoy minimal set, with the added feature that endpoints of gaps can collapse to a point under forward iteration. It is clear that any $h \in \mathcal{H}$ is a near-homeomorphism (the uniform limit of homeomorphisms). Thus, following from a theorem of Brown [15], the inverse limit $\underleftarrow{\lim}(h, S^1)$ is a circle and the natural extension is a circle homeomorphism. In particular, the inverse limit of a semi-Denjoy minimal set is a Denjoy minimal set. For example, in case of a single flat spot, the two endpoints of the flat spot form a gap in the minimal set and they have same forward orbit. Taking the inverse limit splits open this orbit into a forward invariant gap.

3.2. *Finitely many flat spots.* We next introduce a subclass of $\mathcal{H}$ which includes the semi-monotone maps considered in this paper. Let $\mathcal{H}_\ell$ consist of those $h \in \mathcal{H}$ that have exactly ℓ flat spots. In $P(h)$ we require that h is C^1 and $h' > 1$, where we have used a one-sided derivative at the end points of the flat spots.

Definition 3.4. Assume $h \in \mathcal{H}_\ell$ and $\rho(h) \notin \mathbb{Q}$. Thus h has a semi-Denjoy minimal set Z, and since $Z \subset P(h)$, for any flat spot J, $Z \cap \mathrm{Int}(J) = \emptyset$. The flat spot J is called tight for h if $\mathrm{Fr}(J) \subset Z$, and otherwise the flat spot is called loose.

LEMMA 3.5. *Assume $h \in \mathcal{H}_\ell$.*

(a) *If $\rho(h) = p/q$ in lowest terms then the number of (p, q)-periodic orbits wholly contained in $P(h)$ is at least one and at most ℓ.*
(b) *If $\rho(h) \notin \mathbb{Q}$, a flat spot J_i is loose if and only if there are an $n > 0$ and an i' with $h^n(J_i) \in J_{i'}$. In particular, there is always at least one tight flat spot.*
(c) *If Z is the maximal recurrent set of h in $P(h)$, then*

$$Z = S^1 \setminus \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{\ell} h^{-n}(\mathrm{Int}(J_i)), \tag{3.1}$$

and so if $o(x, h) \subset P(h)$ then $h^n(x) \in Z$ for some $n \geq 0$.

Proof. For part (a), since $\rho(h) = p/q$ in lowest terms, every periodic point has period q. By the conditions on the derivatives of $h \in \mathcal{H}_\ell$, there are four classes of periodic points.

(1) x is unstable with $Dh^q(x) > 1$ and $o(x, h) \subset \mathrm{Int}(P(h))$.
(2) x is superstable with $Dh^q(x) = 0$ and $o(x, h) \cap (\bigcup_{i=1}^{\ell} \mathrm{Int}(J_i)) \neq \emptyset$ while $o(x, h) \cap (\bigcup_{i=1}^{\ell} \mathrm{Fr}(J_i)) = \emptyset$.
(3) x is superstable with $Dh^q(x) = 0$ and $o(x, h) \cap (\bigcup_{i=1}^{\ell} \mathrm{Int}(J_i)) \neq \emptyset$ while $o(x, h) \cap (\bigcup_{i=1}^{\ell} \mathrm{Fr}(J_i)) \neq \emptyset$ and $o(x, h)$ contains both left and right endpoints of flat spots.
(4) x is semi-stable with $Dh^q(x) = 0$ from one side and $Dh^q(x) > 1$ from the other and $o(x, h) \cap (\bigcup_{i=1}^{\ell} \mathrm{Int}(J_i)) = \emptyset$ while $o(x, h) \cap (\bigcup_{i=1}^{\ell} \mathrm{Fr}(J_i)) \neq \emptyset$ and $o(x, h)$ contains only left or only right endpoints of flat spots.

This implies that all periodic points are isolated so there are finitely many of them.

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For $i = 1, \ldots, 4$ let n_i be the number of periodic orbits of type (1). Using the fixed point index on h^q, $n_1 = n_2 + n_3$. Each orbit of type (3) hits two flat spots and each of type (2) and (4) at least one and a flat spot cannot contain multiple periodic orbits, and so $n_1 + 2n_3 + n_4 \leq \ell$. Thus the total number of periodic orbits wholly contained on $P(h)$ is $n_1 + n_3 + n_4 = n_2 + 2n_3 + n_4 \leq \ell$

For part (b) assume first that $h^n(J_i) \cap J_{i'} = \emptyset$ for all $n > 0$ and i'. If J_i was loose, there would exist $z_1, z_2 \in Z$ with $z_1 \leq J_i \leq z_2$ with at least one inequality strict and $Z \cap (z_1, z_2) = \emptyset$. Thus $h((z_1, z_2))$ is a non-trivial interval with $h^n((z_1, z_2)) \subset P(h)$ for all $n > 0$. This is impossible since h is expanding in $P(h)$ and so J_i must not be loose.

For the converse, say $h^n(J_i) \in J_{i'}$ for some $n > 0$ and first note $i = i'$ is impossible since that would imply h has a periodic point. Since $h^n(J_i)$ is a point there exists a non-trivial interval $[z_1, z_2]$ properly containing J_i with $h^n([z_1, z_2]) = J_{i'}$ and so $(z_1, z_2) \cap Z = \emptyset$, and so J_i is a loose flat spot.

Finally, since $h^n(J_i) \cap J_i = \emptyset$ and there are finitely many flat spots there is at least one J_i with $h^n(J_i) \cap J_{i'} = \emptyset$ for all $n > 0$ and i'.

For (c), assume y is such that $o(y, h) \subset P(h)$ and $o(y, h) \cap Z = \emptyset$. Let $x, x' \in Z$ with $y \in (x, x')$ and $(x, x') \cap Z = \emptyset$. Because of the uniform expansion in $P(h)$ there are a flat spot J and an $n \geq 0$ so that $J \subset h^n([x, x'])$. If $\rho(h) \notin \mathbb{Q}$, then by (c) for some n', $h^{n+n'}([x, x'])$ is a tight flat spot and so $h^{n+n'+1}(y) \in Z$, where Z is the semi-Denjoy minimal set given in Lemma 3.1(c).

Now assume $\rho(h) = p/q$. In this case x and x' are periodic orbits and so $J \subset h^{n+wq}([x, x'])$ for all $w \geq 0$, and so $h^{n+wq}(y) \in h^{n+wq}([x, x']) \setminus \mathrm{Int}(J)$ using the assumption that $o(y, h) \subset P(h)$. But from (a), $h^{n+wq}(J) \subset J$. Thus by monotonicity, $h^{n+wq}(y)$ is either always in the left component of $[x, x'] \setminus \mathrm{Int}(J)$ or in the right component. This violates the expansion in $P(h)$ and so, for some $j > 0$, $h^j(y) \in Z$, which yields (3.1). □

4. *A class of bimodal circle maps and their positive-slope orbits*

4.1. *The class $\mathcal{G}$.* We introduce the class of bimodal, degree-one maps of the circle that will be the focus here. The class is defined using properties of their lifts. We say that a lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is piecewise smooth if it is continuous and there are $0 \leq x_0 \leq \cdots \leq x_n \leq 1$ so that $\tilde{g}$ is C^2 in each interval (x_i, x_{i+1}) and the right- and left-hand derivatives exist at each x_i.

Definition 4.1. Let $\tilde{\mathcal{G}}$ be the class of all $\tilde{g} : \mathbb{R} \to \mathbb{R}$ with the following properties.

(a) $\tilde{g}$ is piecewise smooth and $\tilde{g}(x' + 1) = \tilde{g}(x') + 1$.

(b) There are a pair of points $0 = x_{\min} < x_{\max} < 1$ so that $\tilde{g}' > 1$ in $[x_{\min}, x_{\max}]$ and $\tilde{g}$ is monotone decreasing in $[x_{\max}, x_{\min} + 1]$.

(c) $x_{\min} \leq \tilde{g}(x_{\min}) < \tilde{g}(x_{\max}) \leq x_{\max} + 1$.

The class $\mathcal{G}$ consists of all $g : S^1 \to S^1$ which have a lift in $\tilde{\mathcal{G}}$.

Note that without loss of generality we have assumed that $x_{\min} = 0$. Also by assumption, $x_{\min}$ and $x_{\max}$ are a non-smooth local minimum and maximum, respectively. It follows from (2.4) that $g \in \mathcal{G}$ implies $\rho(g) \subset [0, 1]$.

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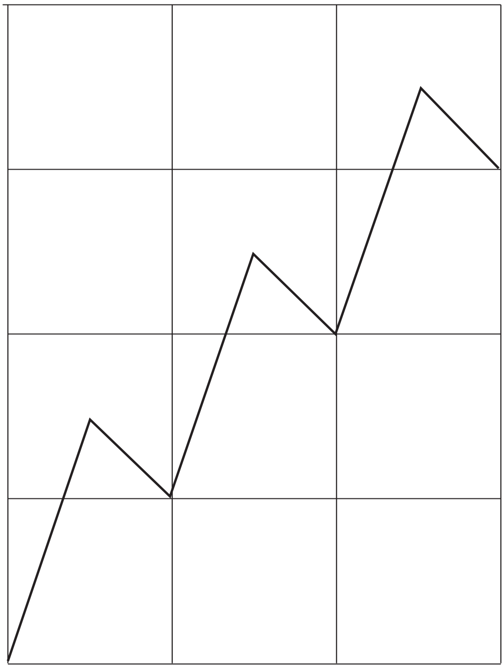

FIGURE 2. The model map f_m in the 3-fold cover.

Standing assumption. From this point on g denotes a given element of $\mathcal{G}$ and its preferred lift is the one with $\tilde{g} \in \tilde{\mathcal{G}}$.

Remark 4.2. Assume h is a general bimodal map with $\rho(h) \subset (0, 1)$ and $\rho(h)$ not equal to a single point. The Parry–Milnor–Thurston theorem for degree-one circle maps yields a semi-conjugacy to a map $g \in \mathcal{G}$ which is PL†. Point inverses of the semi-conjugacy are either points or a closed interval. Thus, using standard results from one-dimensional dynamics and under various hypotheses, most of the results of the paper can be transferred with appropriate alterations to a general bimodal map.

4.2. *The model map.* We will use a model map f_m as a specific example throughout the paper. We shall see that, in a sense, it is the largest map in the class $\mathcal{G}$ and all other maps $g \in \mathcal{G}$ may be considered subsystems.

Define $\tilde{f}_m : \mathbb{R} \to \mathbb{R}$ on [0, 1] as

$$\tilde{f}_m(x) = \begin{cases} 3x & \text{for } 0 \leq x \leq 1/2, \\ -x + 2 & \text{for } 1/2 \leq x \leq 1, \end{cases}$$

and extend it to $\mathbb{R}$ to satisfy $\tilde{f}_m(x + 1) = \tilde{f}_m(x) + 1$. Let f_m be the projection of $\tilde{f}_m$ to S^1. See Figure 2. Thus, $x_{\min} = 0$, $x_{\max} = 1/2$, and $\rho(f_m) = [0, 1]$.

† The result for circles does not seem to be stated and proved anywhere in the literature, but as noted in [16] the proof in [1] works for the circle with minimal alteration

4.3. *Positive-slope orbits.* Given $g \in \mathcal{G}$ with preferred lift $\tilde{g}$, let $\Lambda_\infty(g)$ be the points $x' \in \mathbb{R}$ whose orbits under $\tilde{g}$ stay in the closed region where $\tilde{g}$ has positive slope, so

$$\Lambda_\infty(g) = \left\{ x' \in \mathbb{R} : o(x', \tilde{g}) \subset \bigcup_{j=-\infty}^{\infty} [j, j + x_{\max}] \right\}.$$

We give $\Lambda_\infty(g)$ the total order coming from its embedding in $\mathbb{R}$. Note that it is both $\tilde{g}$- and T-invariant.

Now we treat the k-fold cover as $S_k = [0, k]/\sim$ and let $\Lambda_k(g)$ be the orbits that stay in the positive-slope region of $\tilde{g}_k : S_k \to S_k$, so

$$\Lambda_k(g) = \left\{ x' \in S_k : o(x', \tilde{g}_k) \subset \bigcup_{j=0}^{k-1} [j, j + x_{\max}] \right\}.$$

Alternatively, $\Lambda_k(g) = p_k(\Lambda_\infty(g))$ or $\Lambda_k(g) = \pi_k^{-1}(\Lambda_1(g))$.

We discuss the restriction to positive-slope orbits in §13.2.

Standing assumption. Unless otherwise specified, the terminology 'physical kfsm set' or just 'kfsm set' carries the additional restriction that it is contained in the positive-slope region of some $g \in \mathcal{G}$.

5. *Symbolic description of positive-slope orbits*

For a map $g \in \mathcal{G}$ we develop in this section a symbolic coding for the orbits in Λ_k for $k = 1, \ldots, \infty$.

5.1. *The itinerary maps.* We work first in the universal cover or $k = \infty$. Since $g \in \mathcal{G}$, we may find points $z_{\max}$ and $z_{\min}$ with $0 = x_{\min} < z_{\max} < z_{\min} < x_{\max}$ and $\tilde{g}(z_{\max}) = x_{\max}$ and $\tilde{g}(z_{\min}) = x_{\min} + 1$. For $j \in \mathbb{Z}$ define a collection of intervals $\{I_j\}$ on $\mathbb{R}$ by

$$\begin{aligned} I_{2j} &= [j, z_{\max} + j], \\ I_{2j+1} &= [z_{\min} + j, x_{\max} + j]. \end{aligned} \tag{5.1}$$

See Figure 3. Note that since $\tilde{g}([z_{\max}, z_{\min}]) = [x_{\max}, x_{\max} + 1]$ we have that

$$\Lambda_\infty(g) = \left\{ x' \in \mathbb{R} : o(x', \tilde{g}) \subset \bigcup_{j=-\infty}^{\infty} I_j \right\}.$$

Using $\{I_j\}$ as an address system with the dynamics $\tilde{g}$, let the itinerary map be $\iota_\infty : \Lambda_\infty(g) \to \Sigma_{\mathbb{Z}}^+$. Note that Λ_∞ is the good set and, using expansion and the disjointness of the address intervals, ι_∞ is a homeomorphism onto its image.

Now passing to the k-fold cover, to code the positive-slope orbits $\Lambda_k(g)$, treat $S_k = [0, k]/\sim$ and use the dynamics $\tilde{g}_k$ with the address system $\{I_0, I_1, \ldots, I_{2k-2}, I_{2k-1}\}$. This yields an itinerary map $\iota_k : \Lambda_k \to \Sigma_{2k}^+$ which is also a homeomorphism onto its image.

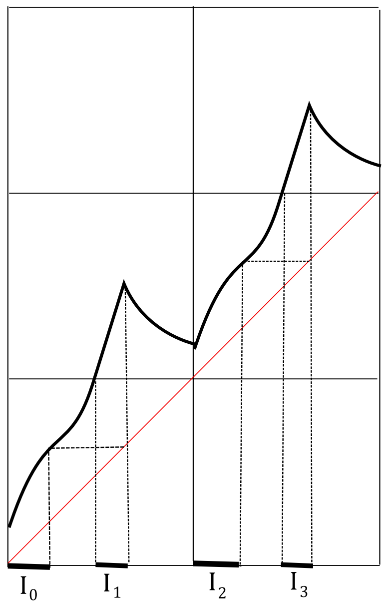

FIGURE 3. The address intervals in the 2-fold cover.

Example: The model map. For the model map f_m we have $z_{\max} = 1/6$ and $z_{\min} = 1/3$ and so $I_{2j} = [j, 1/6 + j]$ and $I_{2j+1} = [1/3 + j, 1/2 + j]$

5.2. *Symbolic analogs of covering spaces.* This section develops the necessary machinery for the complete description of the image of the various itinerary maps. We will need the symbolic analogs of the covering spaces and maps described in §2.2.

Definition 5.1. Define a subshift $\Omega_\infty \subset \Sigma^+_{\mathbb{Z}}$ by its allowable transitions

$$2j \to 2j,\ 2j \to 2j+1,\ 2j+1 \to 2j+2,\ 2j+1 \to 2j+3. \tag{5.2}$$

For $k < \infty$ let Ω_k be the subshift of Σ^+_{2k} with allowable transitions as in (5.2) for $j = 0, \ldots, 2k-1$ and indices reduced mod $2k$.

Since for $g \in \mathcal{G}$ we have $\tilde{g}(I_{2j}) \subset I_{2j} \cup I_{2j+1}$ and $\tilde{g}(I_{2j+1}) \subset I_{2j+2} \cup I_{2j+3}$, we have the following lemma.

LEMMA 5.2. *For $g \in \mathcal{G}$ and $k = 1, \ldots, \infty$, $\iota_k(\Lambda_k(g)) \subset \Omega_k$.*

Under the itinerary maps the spaces $\mathbb{R}$, S_k, and S^1 will correspond to the shift spaces Ω_∞, Ω_k, and $\Omega_1 = \Sigma^+_2$. The dynamics on the 'physical spaces' induced by g will correspond to left shifts on the symbol spaces. The shift spaces will also have the analogs of the covering projections and deck transformations. These maps will be indicated by a hat and defined using the action on individual symbols as follows.

The analogs of the generator of the group of covering translations are $\hat{T}_\infty : \Omega_\infty \to \Omega_\infty$ given by $s \mapsto s + 2$ for all $s \in \mathbb{Z}$ and $\hat{T}_k : \Omega_k \to \Omega_k$ given by $s \mapsto s + 2 \bmod 2k$ for all

$s \in \mathbb{Z}$, while the analogs of the covering maps are $\hat{p}_k : \Omega_\infty \to \Omega_k$ by $s \mapsto s \bmod 2k$ and $\hat{\pi}_k : \Omega_k \to \Sigma_2^+$ by $s \mapsto s \bmod 2$. In the latter we allow $k = \infty$ under the convention that $2\infty = \mathbb{Z}$, yielding $\hat{\pi}_\infty : \Omega_\infty \to \Sigma_2^+$. Note then that $\hat{\pi}_\infty = \hat{p}_1$. A lift and the full lift are defined as usual with, for example, a lift of $Y \subset \Omega_1 = \Sigma_2^+$ to Ω_k being a set $Y' \subset \Omega_k$ with $\hat{\pi}_k(Y') = Y$. Note that $\hat{T}_k$, $\hat{\pi}_k$, and p_k are all continuous.

The roles of the maps g, $\tilde{g}_k$, and $\tilde{g}$ in §2.2 are played by the various shift maps on the sequence spaces. For clarity we use a subscript to indicate which space the shift is acting on: $\sigma_k : \Omega_k \to \Omega_k$. We again allow $k = \infty$. All the various maps satisfy the same commutativity relations as their unhatted analogs. So, for example, $\hat{\pi}_k \hat{T}_k = \pi_k$, $\sigma_k \hat{T}_k = \hat{T}_k \sigma_k$, and $\hat{\pi}_k \sigma_k = \sigma_1 \hat{\pi}_k$. The itinerary maps $\iota_k : \Lambda_k(g) \to \Omega_k$ act naturally by transforming the spaces and maps of §2.2 to their symbolic analogs as in part (b) of the next lemma.

LEMMA 5.3. *For $k = 1, \ldots, \infty$, the following assertions hold.*

(a) $\Omega_k = \hat{p}_k(\Omega_\infty)$.

(b) $\hat{\pi}_k \iota_k = \iota_1 \pi_k$.

(c) *If $\underline{s}, \underline{t} \in \Omega_k$ and $\hat{\pi}_k(\underline{s}) = \hat{\pi}_k(\underline{t})$, then there exists an n with $\underline{s} = \hat{T}_k^n \underline{t}$.*

Proof. Parts (a) and (b) are easy to verify. For (c) we prove the case $k = \infty$ which implies the $k < \infty$ cases. Assume $\hat{\pi}_\infty(\underline{s}) = \underline{w}$. The transitions in (5.2) coupled with the structure of $\underline{w}$ imply that once s_0 is determined the parity structure of $\underline{s}$ determines all of $\underline{w}$. Similarly, once t_0 is determined all of $\underline{t}$ is determined. Once again (5.2) implies that if $s_0 - t_0 = 2n$ then for all i, $s_i = t_i + 2n$. □

Remark 5.4. It would perhaps seem more natural that $\Sigma_\mathbb{Z}^+$ should act as the symbolic universal cover of Σ_2^+, but the crucial covering space property expressed by (c) would not hold in this case. For example, if $\underline{s} = .131^\infty$ and $\underline{t} = .151^\infty$ then $\hat{\pi}_\infty(\underline{s}) = \hat{\pi}_\infty(\underline{t})$ but $\hat{T}^n(\underline{s}) \neq \underline{t}$ for all n.

5.3. *Rotation numbers and sets.* We give the analogs of the definitions in §2.3 for the symbolic case. For $\underline{s} \in \Sigma_2^+$ let

$$\hat{\rho}(\underline{s}) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} s_i \tag{5.3}$$

when the limit exists. For $\hat{\mu}$ a shift-invariant measure on Σ_2^+, let $\hat{\rho}(\hat{\mu}) = \hat{\mu}([1])$. When $\hat{\mu}$ is ergodic, by the pointwise ergodic theorem, for $\hat{\mu}$-almost every $\underline{s}$, $\hat{\rho}(\underline{s}) = \hat{\rho}(\hat{\mu})$

For $\hat{Z} \subset \Omega_k$ let $\hat{\rho}_k(\hat{Z}) = \hat{\rho}(\hat{\pi}_k(\hat{Z}))$, and for $\hat{\mu}$ a σ_k-invariant measure on Ω_k let $\hat{\rho}_k(\hat{\mu}) = \hat{\rho}((\hat{\pi}_k)_*(\hat{\mu}))$.

6. *Topological conjugacies and the image of the itinerary maps*

In this section we develop the analog of kneading invariants for the symbolic coding of the positive-slope orbits for $g \in \mathcal{G}$.

Recall that Σ_2^+ is given the lexicographic order. Assume $\underline{\kappa}_0, \underline{\kappa}_1 \in \Sigma_2^+$ satisfy

$$\underline{\kappa}_0 \leq o(\underline{\kappa}_i, \sigma) \leq \underline{\kappa}_1 \tag{6.1}$$

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for $i = 0, 1$. The corresponding *dynamical order interval* is

$$\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle = \{\underline{s} \colon \underline{\kappa}_0 \leq o(\underline{s}, \sigma) \leq \underline{\kappa}_1\}.$$

Returning to $g \in \mathcal{G}$, note that $\tilde{g}(I_0) \subset [x_{\min}, x_{\max}]$ and $\tilde{g}(I_1) \subset [x_{\min}, x_{\max}]$, while $\tilde{g}([z_{\max}, z_{\min}]) = [x_{\max}, x_{\min} + 1]$. This implies that $\Lambda_1(g) \subset [x_{\min}, x_{\max}]$. Since $\Lambda_1(g)$ is compact we may define $\underline{\kappa}_0 = \underline{\kappa}_0(g) = \iota_1(\min(\Lambda_1))$ and $\underline{\kappa}_1 = \underline{\kappa}_1(g) = \iota_1(\max(\Lambda_1))$. By construction these $\underline{\kappa}$s satisfy (6.1).

We showed above that $\iota_k(\Lambda_k(g)) \subset \Omega_k$. The next theorem says that the image is constrained by the dynamical order interval $\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$. Accordingly, for $k = 1, \ldots, \infty$ we define $\hat{\Lambda}_k(g) = \Omega_k \cap \hat{\pi}_k^{-1}(\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle)$ and note that this is a σ_k-invariant set.

THEOREM 6.1. *Assume $g \in \mathcal{G}$ and construct κ_0 and κ_1 from g as above. Then for $k = 1, \ldots, \infty$ the itinerary map ι_k is a topological conjugacy from $(\Lambda_k(g), (\tilde{g}_k)_{|\Lambda_k(g)})$ to $(\hat{\Lambda}_k(g), \sigma_k)$. Further, ι_∞ is order-preserving.*

Proof. We first prove the first assertion for $k = 1$ or that $\iota_1(\Lambda_1(g)) = \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$. Let $*$ be an arbitrary symbol and define a map $\chi : [0, x_{\max}] \sqcup \{*\} \to [0, x_{\max}] \sqcup \{*\}$ by

$$\chi(x) = \begin{cases} \tilde{g}(x) & \text{for } x \in I_0, \\ * & \text{for } x \in (z_{\min}, z_{\max}) \sqcup \{*\}, \\ \tilde{g}(x) - 1 & \text{for } x \in I_1. \end{cases}$$

It easily follows that

$$\Lambda_1(g) = \{x \in [0, x_{\max}] \colon \chi^n(x) \neq * \text{ for all } n > 0\},$$

and if we use the dynamics of χ with the address system I_0, I_1 the resulting itinerary map $\Lambda_1 \to \Sigma_2^+$ is exactly ι_1. Now since g is expanding on $I_0 \cup I_1$ and $I_0 \cap I_1 = \emptyset$, ι_1 is an order-preserving conjugacy from (Λ_1, g) to $(\iota_1(\Lambda_1), \sigma_1)$. Finally, since $\min \Lambda_1(g) \leq o(x, g) \leq \max \Lambda_1(g)$ for all $x \in \Lambda_1(g)$ we have that $\kappa_0 \leq o(\underline{s}, \sigma) \leq \kappa_1$ for all $\underline{s} \in \iota_1(\Lambda_1)$, and further that for any such $\underline{s}$ there is an $x \in \Lambda_1(g)$ with $\iota_1(x) = \underline{s}$. Thus $\iota_1(\Lambda_1(g)) = \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$.

We now show that

$$\iota_k(\Lambda_k(g)) = \Omega_k \cap \hat{\pi}_k^{-1}(\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle). \tag{6.2}$$

We already know from Lemma 5.2 that the left-hand side is in Ω_k. Next, since $\pi_k(\Lambda_k(g)) = \Lambda_1(g)$ using Lemma 5.3(b) and the first paragraph of the proof, we have

$$\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle = \iota_1(\Lambda_1(g)) = \iota_1(\pi_k(\Lambda_k(g))) = \hat{\pi}_k \iota_k(\Lambda_k(g)) \tag{6.3}$$

so the left-hand side of (6.2) is also in $\hat{\pi}_k^{-1}(\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle)$.

Now assume that $\underline{s}$ is in the right-hand side of (6.2). Certainly then $\hat{\pi}_k(\underline{s}) \in \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$ and so there is an $x \in \Lambda_1(g)$ with $\iota_1(x) = \hat{\pi}_k(\underline{s})$. Pick a lift $x' \in \Lambda_k(g)$ with $\pi_k(x') = x$. Again using Lemma 5.3(b),

$$\hat{\pi}_k(\underline{s}) = \iota_1(x) = \iota_1 \pi_k(x') = \hat{\pi}_k \iota_k(x'). \tag{6.4}$$

Thus, using Lemma 5.3(c), there is an n with $\iota_k(x') = \hat{T}_k^n(\underline{s})$ and so

$$\iota_k \hat{T}_k^{-n} x' = \hat{T}_k^{-n} \iota_k x' = \underline{s}$$

and $\hat{T}_k^{-n} x' \in \Lambda_k$. Thus $\underline{s} \in \iota_k(\Lambda_k)$ as required.

For ι_∞ as with ι_1, since the I_j are disjoint and the $\tilde{g}_{|I_j}$ are expanding, we have that ι_∞ is an order-preserving homeomorphism onto its image. The fact that it is a semi-conjugacy follows because it is an itinerary map. □

Example. (The model map). For the model map f_m we have $\kappa_0 = .0^\infty$ and $\kappa_1 = .1^\infty$ and so in this case $\hat{\Lambda}_k(\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle)$ is the entire subshift Ω_k.

Remark 6.2

(a) $\hat{\rho} \circ \iota_k = \rho$ (when defined) and $\hat{\rho}_k \circ \iota_k = \rho_k$

(b) When μ a g-invariant measure supported in $\Lambda_1(g)$, we have $\rho(\mu) = \mu(I_1)$.

7. *k-fold semi-monotone sets*

While our eventual interest is in invariant sets in the circle, it is convenient to first give definitions in the universal cover $\mathbb{R}$ and the cyclic covers S_k.

7.1. *Definitions.*

The next definition makes sense for any degree-one map, but for concreteness we restrict to $g \in \mathcal{G}$.

Definition 7.1. Let $g \in \mathcal{G}$ have preferred lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$.

(a) A $\tilde{g}$-invariant set $Z' \subset \mathbb{R}$ is *kfsm* if $T^k(Z') = Z'$ and $\tilde{g}$ restricted to Z' is weakly order-preserving, or for $z_1', z_2' \in Z'$,

$$z_1' < z_2' \quad \text{implies} \quad \tilde{g}(z_1') \leq \tilde{g}(z_2')$$

(b) A $\tilde{g}_k$-invariant set $Z \subset S_k$ is *kfsm* if it has a $\tilde{g}$-invariant lift $Z' \subset \mathbb{R}$ which is.

These definitions are independent of the choice of lift $\tilde{g}$. Note that the same terminology is used for sets in the universal and cyclic covers and that implicit in being a kfsm set is the fact that the set is invariant.

When $k = 1$ the lift Z' in the definition must satisfy $T(Z') = Z'$ and $\pi(Z') = Z$ and so $Z' = \pi^{-1}(Z)$, the full lift to $\mathbb{R}$.

7.2. *Interpolation.*

To say that $Z \subset S_k$ is kfsm means roughly that it is semi-monotone, treating S_k as the usual circle. To formalize this as in Remark 2.3 it will be useful to rescale S_k to S^1 using $D_k : S_k \to S^1$ and consider the map $D_k \circ \tilde{g}_k \circ D_k^{-1}$.

LEMMA 7.2. *The following assertions are equivalent.*

(a) *The $\tilde{g}_k$-invariant set $Z \subset S_k$ is kfsm.*

(b) *$D_k Z$ is 1-fold semi-monotone under $D_k \circ \tilde{g}_k \circ D_k^{-1}$ and there exists a semi-monotone circle map h defined on S_k which interpolates $\tilde{g}_k$ acting on Z.*

(c) *The lift $Z' \subset \mathbb{R}$ of Z in Definition 7.1(b) has the property that there is a continuous $H : \mathbb{R} \to \mathbb{R}$ that interpolates $\tilde{g}$ acting on $\tilde{Z}^*$, is weakly order-preserving, and satisfies $H(x + k) = H(x) + k$.*

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We now restrict to positive-slope orbits as in §4.3 and collect together kfsm invariant sets in S_k and their invariant measures. We will comment on kfsm sets which intersect the negative-slope region in §13.2. We also restrict attention to invariant sets that are recurrent.

Definition 7.3. Given $g \in \mathcal{G}$, let $\mathcal{B}_k(g)$ be all compact, recurrent kfsm sets in $\Lambda_k(g) \subset S_k$ with the Hausdorff topology and $\mathcal{N}_k(g)$ be all $\tilde{g}_k$-invariant, Borel probability measures whose support is a $Z \in \mathcal{B}_k(g)$ with the weak topology.

Remark 7.4. A standard argument from Aubry–Mather theory yields that the collection of all kfsm sets is compact in the Hausdorff topology. Since $\Lambda_k(g)$ is compact, the collection of positive-slope kfsm sets is also compact. However, since $\mathcal{B}_k(g)$ contains just the recurrent kfsm sets, it is not compact (see §§9.1 and 13.2). We show shortly that $\mathcal{N}_k(g)$ is compact.

7.3. *Symbolic k-fold semi-monotone sets and the map g.* As with kfsm sets in the 'physical' spaces S_k and $\mathbb{R}$, we define their symbolic analogs in the symbol spaces Ω_k and Ω_∞ where we give the symbol spaces the lexicographic order.

Definition 7.5

(1) A σ_∞-invariant set $\hat{Z}' \subset \Omega_\infty$ is *symbolic k-fold semi-monotone* if $\hat{T}^k_\infty(\hat{Z}') = \hat{Z}'$ and σ_∞ restricted to $\hat{Z}'$ is weakly order-preserving, or for $\underline{s}, \underline{t} \in \hat{Z}'$,

$$\underline{s} < \underline{t} \quad \text{implies} \quad \sigma_\infty(\underline{s}) \leq \sigma_\infty(\underline{t}).$$

(2) A σ_k-invariant set $\hat{Z} \subset \Omega_k$ is *symbolic k-fold semi-monotone* if there is a σ_∞-invariant lift $\hat{Z}'$ to Ω_∞ (that is, $\hat{p}_k(\hat{Z}') = \hat{Z}$) which is kfsm.

Everything has been organized thus far to ensure that k-fold semi-monotone sets are preserved under the itinerary maps.

THEOREM 7.6. *Given $g \in \mathcal{G}$, for $k = 1, 2, \ldots, \infty$, a $\tilde{g}_k$-invariant set $Z \subset \Lambda_k(g)$ is kfsm if and only if $\iota_k(Z) \subset \hat{\Lambda}_k(g)$ is.*

Proof. We prove the $k = \infty$ case; the $k < \infty$ case follows. Theorem 6.1 shows that ι_∞ is an order-preserving bijection. Since $\iota_\infty T^k_\infty = \hat{T}^k_\infty \iota_\infty$, we have that $T^k_\infty(Z) = Z$ if and only if $\hat{T}^k_\infty \iota_\infty(Z) = \iota_\infty(Z)$. Using the additional fact that $\iota_\infty \tilde{g} = \sigma_\infty \iota_\infty$ we have that $\tilde{g}$ is weakly order-preserving on Z if and only if σ_∞ is weakly order-preserving on $\iota_\infty(Z)$ □

In analogy with Definition 7.3 we collect together the various symbolic kfsm sets and their invariant measures.

Definition 7.7. For $k < \infty$, given $g \in \mathcal{G}$, let $\hat{\mathcal{B}}_k(g)$ be all compact, invariant, recurrent symbolic kfsm sets in $\hat{\Lambda}_k(g)$ with the Hausdorff topology and $\hat{\mathcal{N}}_k(g)$ be all g-invariant, Borel probability measures with the weak topology whose support is a $\hat{Z} \in \hat{\mathcal{B}}_k(g)$.

LEMMA 7.8. *For $k < \infty$, the following assertions hold.*

(a) *The map $\iota_k : \Lambda_k(g) \to \hat{\Lambda}_k(g)$ induces homeomorphisms $\mathcal{B}_k(g) \to \hat{\mathcal{B}}_k(g)$ and $\mathcal{N}_k(g) \to \hat{\mathcal{N}}_k(g)$.*

(b) *The spaces $\mathcal{N}_k(g)$ and $\hat{\mathcal{N}}_k(g)$ are compact.*

Proof. For part (a) we know that ι_k is a conjugacy that takes kfsm sets to kfsm sets, which yields that $\mathcal{N}_k(g) \to \hat{\mathcal{N}}_k(g)$ is a homeomorphism. By hypothesis any $g \in \mathcal{G}$ is C^2 in $P(g)$ and so there is some $M > 1$ with $g' < M$ on $P(g)$ and thus on all address intervals I_j. It is standard that this implies that ι_k is Hölder with exponent $\nu = \log 2k/\log M$. This then implies that ι_k preserves Hausdorff convergence and so $\mathcal{B}_k(g) \to \hat{\mathcal{B}}_k(g)$ is a homeomorphism.

For part (b), since the space of all $\tilde{g}_k$-invariant Borel probability measures is compact metric, it suffices to show that $\mathcal{N}_k(g)$ is closed, and so assume $\mu_n \in \mathcal{N}_k(g)$ and $\mu_n \to \mu$ weakly with $X_n := \operatorname{supp}(\mu_n)$ a recurrent kfsm set.

As noted in Remark 7.4 the collection of all kfsm sets in Λ_k is compact in the Hausdorff topology and so there exist a kfsm set X and $n_i \to \infty$ with $X_{n_i} \to X$. A standard argument which we give here shows that $\operatorname{supp}(\mu) \subset X$. If this inclusion does not hold, there exists an $x \in \operatorname{supp}(\mu) \cap X^c$, then let $\epsilon = d(x, X)$. Since the atoms of μ are countable, we may find an $\epsilon_1 < \epsilon/4$ so that, letting $U = N_{\epsilon_1}(x)$, we have that $\mu(\operatorname{Fr}(U)) = 0$. Thus U is a continuity set for μ. It then follows, using the fact that $x \in \operatorname{supp}(\mu)$ and a standard result (pp. 16–17 of [7]), that $\mu_{n_i}(U) \to \mu(U) > 0$. Thus for large enough i, with $m = n_i$ we have $X_m \subset N_{\epsilon/4}(X)$ and so $\emptyset = U \cap X_m = U \cap \operatorname{supp}(\mu_m)$ with $\mu_m(U) > 0$ a contradiction. Thus $\operatorname{supp}(\mu) \subset X$. Now any invariant measure supported on X must be supported on its recurrent set and so $\mu \in \mathcal{N}_k(g)$, as required. The compactness of $\hat{\mathcal{N}}_k(g)$ follows from part(a). □

Example: The model map. For the model map f_m, $\hat{\Lambda}_k(f_m) = \Omega_k$, and so the set $\hat{\mathcal{B}}_k(f_m)$ is the collection of all symbolic recurrent kfsm sets in Ω_k. Thus while the definition of symbolic kfsm set is abstract and general by Theorems 7.6 and 6.1, symbolic kfsm sets share all the properties of 'physical' kfsm sets.

7.4. *Rotation numbers and sets.* For $Z \in \mathcal{B}_k(g)$ recall from §2.3 that $\rho_k(Z) = \rho(\pi_k(Z), g)$.

LEMMA 7.9. *Assume $Z \in \mathcal{B}_k(g)$. Then the following assertions hold.*

(a) *$\rho_k(Z)$ exists and is a single number.*
(b) *If $\rho_k(Z) = \omega \notin \mathbb{Q}$ then Z is a semi-Denjoy minimal set.*
(c) *If $\rho_k(Z) = p/q$ with $\gcd(p, q) = 1$, then Z consists of at least one and at most k periodic orbits all with the same rotation number and period equal to $qk/\gcd(p, k)$.*
(d) *$\rho_k : \mathcal{B}_k(g) \to \mathbb{R}$ and $\hat{\rho}_k : \hat{\mathcal{B}}_k(g) \to \mathbb{R}$ are continuous*

Proof. By Theorem 7.2 there exists a continuous, semi-monotone $H : S_k \to S_k$ which interpolates the action of $\tilde{g}_k$ on Z. Rescaling to the standard circle, let $H_k : S^1 \to S^1$ be defined as $H_k := D_k \circ H \circ D_k^{-1}$. By Lemma 3.1(a), $\rho(H_k) = \omega$ is a single number and, since $\rho_k(Z) = k\rho(DZ, H_k)$, (a) follows. If $\rho_k(Z) \notin \mathbb{Q}$ then $\rho(DZ, H_k) \notin \mathbb{Q}$ and so, by Lemma 3.1(e), $D_k Z$ and thus Z is a semi-Denjoy minimal set, yielding (b).

Now assume $\rho_k(Z) = p/q$ in lowest terms and so $\rho(D_k Z, H_k) = p/(qk)$. Written in lowest terms,

$$\frac{p}{qk} = \frac{p/\gcd(p, k)}{kq/\gcd(p, k)}.$$

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But since H_k is semi-monotone, its recurrent set is a collection of periodic orbits and its rotation number in lowest terms has their period as its denominator which is thus $qk/\gcd(p, k)$. Since by assumption, $Z \subset \Lambda_k(g)$ we may choose H to have k flat spots. Thus, using Lemma 3.5, Z consists of at least one and at most k periodic orbits, finishing (c).

It is standard from Aubry–Mather theory that ρ is continuous on the collection of all kfsm sets and thus it is continuous restricted to the recurrent kfsm sets. As for measures, since $\rho(\mu) = \int \Delta_g \, d\mu$ using definition (2.5) with Δ_g continuous, continuity of $\hat{\rho}_k$ follows from the definition of weak convergence. □

Definition 7.10. If $Z \in \mathcal{B}_k$ and it consists of a finite collection of periodic orbits, then it is called a cluster.

Remark 7.11

(a) For the case of general recurrent symbolic kfsm $\hat{Z}$, as we commented at the end of the last subsection, we may consider $\hat{Z} \in \hat{\Lambda}_k(f_m) = \Omega_k$ with f_m the model map. Using the itinerary map $\iota_k : \Lambda_k(f_m) \to \hat{\Lambda}_k(f_m)$ we have from Theorem 7.6 that $(\iota_k)^{-1}(\hat{Z})$ is kfsm for f_m and then all the conclusions of the previous theorem hold for it. Then using Theorem 6.1, the conclusions of the previous theorem hold with the obvious addition of hats in the appropriate places.

(b) We shall need this implication of the symbolic case below. If $\hat{Z} \subset \Omega_k$ with $\rho_k(\hat{Z}) = \alpha \notin \mathbb{Q}$, then there exists a continuous, onto $\phi : \hat{Z} \to S_k$ which is weakly order-preserving, $\phi\sigma_k = R_\alpha \phi$, and $\#\phi^{-1}(x) = 1$ for all but a countable number of R_α-orbits on which $\#\phi^{-1}(x) = 2$.

(c) Using Lemma 3.1, a measure in $\mathcal{N}_k(g)$ is either the unique measure on a semi-Denjoy minimal set or a convex combination of measures supported on the periodic orbits in a cluster.

(d) A $Z \in \mathcal{B}_k(g)$ is minimal if and only if it is uniquely ergodic, and similarly for $Z \in \hat{\mathcal{B}}_k(g)$

8. *The HM construction*

At this point for a given $g \in \mathcal{G}$ we have reduced the identification of its positive-slope kfsm sets to a question in symbolic dynamics. In this section we answer this symbolic question via a generalization of the HM procedure. The generalization constructs all symbolic kfsm recurrent sets for each k.

Since a linear order is essential to the notion of semi-monotonicity we will again begin working on the line and then project to cyclic covers.

8.1. *Definition and basic properties.*

Fix an integer $k > 0$, a real number $\omega \in (0, 1)$, and a vector $\vec{\nu} = (\nu_1, \ldots, \nu_k)$ with $\nu_i \geq 0$ and $\sum \nu_i = k - k\omega$. Such a pair $(\omega, \vec{\nu})$ is called *allowable*. Start with the intervals defined for $0 \leq j \leq k-1$ by

$$
\begin{aligned}
X_{2j} &= \left(\sum_{i=1}^{j} \nu_i + j\omega, \sum_{i=1}^{j+1} \nu_i + j\omega \right), \\
X_{2j+1} &= \left(\sum_{i=1}^{j+1} \nu_i + j\omega, \sum_{i=1}^{j+1} \nu_i + (j+1)\omega \right),
\end{aligned}
\tag{8.1}
$$

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and then extend for $\ell \in \mathbb{Z}$ and $0 \leq m \leq 2k-1$ as $X_{\ell k+m} = X_m + \ell k$. Thus each X_{2j} has width ν_{j+1} and each X_{2j+1} has width ω and the entire structure yields a T^k-invariant address system under the dynamics $R_\omega(x) = x + \omega$ on $\mathbb{R}$

The good set G depends on k, ω, and $\vec{\nu}$ and is given by

$$G = \{x' \in \mathbb{R} : o(x', R_\omega) \cap \partial X_i = \emptyset \text{ for all } i\}.$$

Note that G is dense, G_δ and has full Lebesgue measure. The itinerary map with respect to the given address system is denoted by $\zeta_\infty : G \to \Sigma^+_{\mathbb{Z}}$.

Definition 8.1. Let $A_k(\omega, \vec{\nu}) = \mathrm{Cl}(\zeta_\infty(G))$.

Remark 8.2. By construction, $A_k(\omega, \vec{\nu})$ is σ_∞- and $\hat{T}^k_\infty$-invariant. In addition, since for all j, $R_\omega(X_{2j}) \subset X_{2j} \cup X_{2j+1}$ and $R_\omega(X_{2j+1}) \subset X_{2j+2} \cup X_{2j+3}$, we have $\zeta_\infty(G_\infty) \subset \Omega_\infty$.

8.2. *Cyclic covers.* We now return to the compact quotients where the recurrent dynamics takes place and introduce measures into the HM construction.

For fixed $k > 0$ and allowable $(\omega, \vec{\nu})$ treat $\{X_0, \ldots, X_{2k-1}\} \subset S_k = [0, k]/\sim$ as an address system under the dynamics given by $R_\omega(x) = x + \omega \bmod k$. Define the good set $G_{k\omega\vec{\nu}}$ and on it the itinerary map $\zeta_{k\omega\vec{\nu}}$. We will often suppress the dependence of these quantities on various of the subscripted variables when they are clear from the context.

Definition 8.3. Given k and an allowable $(\omega, \vec{\nu})$, define the itinerary map $\zeta_k : G_k \to \Sigma^+_{2k}$ as above. Let $B_k(\omega, \vec{\nu}) = \mathrm{Cl}(\zeta_k(G_k)) \subset \Sigma^+_{2k}$ and $\lambda_k(\omega, \vec{\nu}) = (\zeta_k)_*(m)/k$ where m is the measure on S_k induced by Lebesgue measure on $\mathbb{R}$.

Remark 8.4

(1) By construction, $\hat{p}_k(A_k(\omega, \vec{\nu})) = B_k(\omega, \vec{\nu}) \subset \Omega_k$, and so $\rho_k(B_k(\omega, \vec{\nu})) = \omega$.
(2) Let $W_k = \{(x, \omega, \vec{\nu}) \colon x \in G_{k\omega\vec{\nu}}\}$. It is easy to check that the map $(x, \omega, \vec{\nu}) \mapsto \zeta_{k\omega\vec{\nu}}(x)$ is continuous on W_k.

The next theorem describes the structure of the $B_k(\omega, \vec{\nu})$ and shows that all symbolic kfsm sets are constructed by the HM procedure with ω equal to their rotation number.

THEOREM 8.5

(a) *For $\alpha \notin \mathbb{Q}$, $B_k(\alpha, \vec{\nu})$ is a semi-Denjoy minimal set with unique invariant probability measure $\lambda_k(\omega, \vec{\nu})$.*
(b) *For $p/q \in \mathbb{Q}$, $B_k(p/q, \vec{\nu})$ is a finite collection of periodic orbits each with rotation number p/q and period $qk/\gcd(p, k)$, and $\lambda_k(p/q, \vec{\nu})$ is a convex combination of the measures supported on the periodic orbits.*
(c) *A $\hat{Z} \subset \Omega_k$ is a recurrent symbolic kfsm set with $\rho_k(Z) = \omega$ if and only if $\hat{Z} = B_k(\omega, \vec{\nu})$ for some allowable $\vec{\nu}$. Thus the collection of invariant probability measures supported on symbolic recurrent kfsm sets is exactly the collection of $\lambda_k(\omega, \vec{\nu})$ for all allowable $(\omega, \vec{\nu})$.*

Proof. We begin by proving portions of (a) and (b). For part (a) we first show that $B_k(\alpha, \vec{\nu})$ is minimal using a characterization usually attributed to Birkhoff. If $f : X \to X$ is a

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continuous function of a compact metric space and $x \in X$, then $\mathrm{Cl}(o(x, f))$ is a minimal set if and only if for all $\epsilon > 0$ there exists an N so that for all $n \in \mathbb{N}$ there is a $0 < i \le N$ with $d(f^{n+i}, x) < \epsilon$. Pick x in the good set G. Since (S^1, R_α) is minimal, $o(x, R_\alpha)$ has the given property. Since ζ_k restricted to G is a homeomorphism and $\zeta_k R_\alpha = \sigma_k \zeta_k$, $o(\zeta_k(x), R_\alpha)$ has the desired property and, further, $o(\zeta_k(x), R_\alpha)$ is dense in $\zeta_k(G)$ and thus in $B_k(\alpha, \vec{\nu}) = \mathrm{Cl}(\zeta_k(G))$. Thus $B_k(\alpha, \vec{\nu})$ is minimal under σ_k.

For part (b) note first that since $R_{p/q}$ is of finite order and there are finitely many address intervals, $B_k(p/q, \vec{\nu})$ must consist of finitely many periodic orbits. The other properties in (a) and (b) will follow from (c) (proved using just these two partial results on (a) and (b)) and Theorem 7.9 using Remark 7.11(a).

For part (c), we first show that $B_k(\omega, \vec{\nu})$ is a recurrent symbolic kfsm set. By parts (a) and (b) we know that $B_k(\omega, \vec{\nu})$ is recurrent, and by Remark 8.4 that $B_k(\omega, \vec{\nu}) \subset \Omega_k$ and $\rho_k(B_k(\omega, \vec{\nu}) = \omega$. We show that $B_k(\omega, \vec{\nu})$ is a symbolic kfsm set by showing that its full lift $A_k(\omega, \vec{\nu})$ to Ω_∞ is as required by Definition 8.1. As noted in Remark 2.2, $\hat{T}^k_\infty(A_k(\omega, \vec{\nu})) = A_k(\omega, \vec{\nu})$ so we need to show that $\hat{\sigma}_\infty$ is semi-monotone on $A_k(\omega, \vec{\nu})$.

The first step is to show that ζ_∞ is weakly order-preserving. Assume $x'_1, x'_2 \in G$ with $x'_1 < x'_2$. It could happen (when ω is rational) that $\zeta_\infty(x'_1) = \zeta_\infty(x'_2)$, but if there exists a least n with $(\zeta_\infty(x'_1))_n \neq (\zeta_\infty(x'_2))_n$, then since $I_m < I_{m+1}$ for all m and R_ω is order-preserving, certainly $(\zeta_\infty(x'_1))_n < (\zeta_\infty(x'_2))_n$, and so ζ_∞ is weakly order-preserving.

We now show that $\hat{\sigma}_\infty$ is semi-monotone on $A_k(\omega, \vec{\nu})$. Let G be the good set for ζ_∞ and assume $\underline{s}, \underline{t} \in \zeta_\infty(G)$ with $\underline{s} < \underline{t}$. Then there exist $x'_1, x'_2 \in G$ with $\zeta_\infty(x'_1) = \underline{s}$ and $\zeta_\infty(x'_2) = \underline{t}$ and, of necessity, $x'_1 < x'_2$ and so $R_\omega(x'_1) < R_\omega(x'_2)$. Since $\zeta_\infty R_\omega = \sigma_\infty \zeta_\infty$, we have

$$\sigma_\infty(\underline{s}) = \sigma_\infty \zeta_\infty(x'_1) = \zeta_\infty R_\omega(x_1) \le \zeta_\infty R_\omega(x_2) = \sigma_\infty \zeta_\infty(x'_2) = \sigma_\infty(\underline{t}).$$

Thus σ_∞ is weakly order-preserving on $\zeta_\infty(G)$ and so on $A_k(\omega, \vec{\nu})$. We have that $A_k(\omega, \vec{\nu})$ satisfies all the conditions of the lift in Definition 7.5 and thus $B_k(\omega, \vec{\nu})$ is symbolic kfsm.

Now for the converse assume that $\hat{Z} \subset \Omega_k$ is symbolic recurrent kfsm with $\hat{\rho}(\hat{Z}) = \omega$. Let $\hat{Z}' \subset \Omega_\infty$ be the lift that satisfies Definition 7.5. The proof splits into the two cases when ω is rational and irrational.

First assume $\omega = p/q$ with $\gcd(p, q) = 1$. We know from Lemma 7.9 and Remark 7.11 that $\hat{Z}$ consists of at most k distinct periodic orbits each with period kq/d with $d = \gcd(p, k)$. We assume for simplicity that $\hat{Z}$ is a single periodic orbit. The case of multiple periodic orbits is similar but with more elaborate indexing.

For $i = 0, \ldots, kq/d - 1$ let $P_i = (2i+1)d/2q \subset S_k$ and $\mathcal{P} = \{P_i\}$. Since $\hat{Z}$ is a kfsm periodic orbit with ρ_k-rotation number p/q we may find an order-preserving bijection $\phi : \hat{Z} \to \mathcal{P}$ with $\phi\sigma_k = R_{p/q}\phi$ on $\hat{Z}$. Thus $\phi\sigma_k\phi^{-1}$ acts on $\mathcal{P}$ as $P_i \mapsto P_{i+p/d}$ reducing indices mod kq/d.

For $j = 0, \ldots, k-1$, let $X'_j = \phi(\hat{Z} \cap [j])$ where recall that $[j]$ is the length-one cylinder set in Σ^+_{2k}. Since ϕ is order-preserving, each X'_j consists of a collection of adjacent points from $\mathcal{P}$. If $X'_j = \{P_{n(j)}, \ldots, P_{m(j)}\} \neq \emptyset$, let $X_j = [P_{n(j)} - d/(2q), P_{m(j)} + d/(2q)]$ and when $X'_j = \emptyset$ let $X_j = \emptyset$. We now claim that $\{X_j\}$ is an address system as used in the HM construction where $\vec{\nu}$ is defined by $\nu_{j+1} = |X_{2j}|$ and that $|X_{2j+1}| = p/q$ for $j = 0, \ldots, k-1$, yielding $\hat{Z}' = B_k(p/q, \vec{\nu})$.

Letting ζ be the itinerary map for the address system $\{X_j\}$, for $\underline{s} \in \hat{Z}$ we have by construction that $\zeta\phi(\underline{s}) = \underline{s}$. In addition, for all $x \in [\phi(\underline{s}) - d/(2q), \phi(\underline{s}) + d/(2q)]$ we also have $\zeta(x) = \zeta\phi(\underline{s}) = \underline{s}$. Thus for any point x in the good set G, $\zeta(x) = \underline{s}$ for some $\underline{s} \in \hat{Z}$. This shows that $\hat{Z} = \mathrm{Cl}(\zeta(G))$. The last step needed to show that $\hat{Z} = B_k(p/q, \vec{\nu})$ is to check that the address system is of the type used in the HM construction.

We need only check that $|X_{2j+1}| = p/q$, and for this it suffices to show that $\#X_{2j+1} = p/d$. Assume first that $\#X_{2j+1} < p/d$. Recalling that $\phi\sigma_k\phi^{-1}$ acts on the X_i' like $i \mapsto i + p/d$, we see that there will be some $P_m \in X_{2j}'$ and $P_{m+p/d} \in X_{2j+2}'$. Thus using ϕ^{-1} there is a $\underline{s} \in \hat{Z}$ with $s_0 = 2j$ and $s_1 = 2j+2$, a contradiction to the fact that $\hat{Z} \subset \Omega_k$ and thus its allowable transitions are given by (5.1). On the other hand, if $\#X_{2j+1} > p/d$ we have some $P_m \in X_{2j+1}'$ and $P_{m+p} \in X_{2j+1}'$, again yielding a contradiction to $\hat{Z} \subset \Omega_k$.

The irrational case is basically a continuous version of the rational one. By Remark 7.11(b) we have a continuous, onto $\phi : \hat{Z} \to S_k$ which is weakly order-preserving, $\phi\sigma_\infty = R_\alpha\phi$, and $\#\phi^{-1}(x) = 1$ for all but a countable number of R_α-orbits on which $\#\phi^{-1}(x) = 2$.

For $j = 0, \ldots, k-1$, let $X_j = \phi([j])$. Thus X_j is a closed interval (perhaps empty) with $\cup X_j = \mathbb{R}$, $X_j \le X_{j+1}$ and adjacent intervals intersect only in their single common boundary point. We use $\{X_j\}$ as an address system with dynamics R_α, good set G, and itinerary map ζ. By construction, if $\underline{s} \in \hat{Z}$ with $\phi(\underline{s}) \in G$, then $\underline{s} = \zeta\phi(\underline{s})$ and so $\phi^{-1}(G) = \zeta\phi(\phi^{-1}(G)) = \zeta(G)$. Since $\hat{Z}$ is a Cantor set and $\phi^{-1}(G)$ is $\hat{Z}$ minus a countable set of σ_k-orbits we have that $\phi^{-1}(G)$ is dense $\hat{Z}$. Thus, taking closures, $\hat{Z} = \mathrm{Cl}(\zeta(G))$.

To finish we must show that $\{X_j\}$ is the type of address system allowable in the HM construction. We just need $|X_{2j+1}| = \alpha$ for all j. The proof is similar to the rational case. If $|X_{2j+1}| < \alpha$ then $\hat{Z}$ has a transition $2j \to 2j+2$, and if $|X_{2j+1}| > \alpha$ then $\hat{Z}$ has a transition $2j+1 \to 2j+1$. Either is a contradiction to $\hat{Z} \subset \Omega_k$. Thus letting $\nu_{j+1} = |X_{2j}|$ for $j = 0, \ldots, k-1$, we have $\hat{Z} = \mathrm{Cl}(\zeta(G)) = B_k(\alpha, \vec{\nu})$.

The last sentence in (c) follows from the construction of $\lambda_k(\omega, \vec{\nu})$. □

Remark 8.6. In §9.3 below we shall see that for the irrational case $\rho(\hat{Z}) = \omega \notin \mathbb{Q}$ there is a unique $\vec{\nu}$ with $\hat{Z} = B_k(\omega, \vec{\nu})$ and for rational p/q there are, in general, many $\vec{\nu}$ with $Z = B_k(p/q, \vec{\nu})$. But note that if $\hat{Z}$ is a single periodic orbit then the proof above produces what we show is the unique $\vec{\nu}$ with $\hat{Z} = B_k(p/q, \vec{\nu})$.

9. *Parameterization of $\mathcal{B}_k(g)$ and $\mathcal{N}_k(g)$ by the HM construction*

We know from Theorem 8.5(c) that the HM construction yields a correspondence between sets $B_k(\omega, \vec{\nu})$ and symbolic kfsm sets in Ω_k. In addition, for a map $g \in \mathcal{G}$, using Theorem 7.6, we get a bijection from kfsm sets in $\Lambda_k(g)$ to those in $\hat{\Lambda}_k(g) \subset \Omega_k$. Thus the HM construction provides a parameterization of $\mathcal{B}_k(g)$. In this section we examine this parameterization in detail as well as that of $\mathcal{N}_k(g)$.

9.1. *Resonance and holes.* As remarked above, the collection of all kfsm sets is closed in the compact metric space consisting of all compact g-invariant sets with the Hausdorff topology. Thus the collection of all kfsm sets is complete. We have restricted attention here to recurrent kfsm sets or $\mathcal{B}_k(g)$. This is because the recurrent ones are the most dynamically

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interesting and carry the invariant measures, but also, as shown in Theorem 8.5, they are what is parameterized by the HM construction. As a consequence our primary space of interest $\mathcal{B}_k(g)$ is not complete, but rather has holes at points to be specified. What happens roughly is that as one takes the Hausdorff limit of recurrent kfsm sets the resulting kfsm set can have homoclinic points that are not recurrent and so the limit is not recurrent and thus not any $B_k(\omega, \vec{\nu})$. This is a phenomenon well known in Aubry–Mather theory. Another point of view on these 'holes' is given in §13.2 using the family of interpolated semi-monotone maps.

In the HM construction fix $0 < k < \infty$. For a given allowable $(\omega, \vec{\nu})$, recall that the address intervals are $X_j = X_{j\vec{\nu}}$ for $j = 0, \ldots, 2k-1$. Define $\ell_j = \ell_{j\vec{\nu}}$ and $r_j = r_{j\vec{\nu}}$ by $[\ell_j, r_j] := X_j$. Note that $r_{j+1} = \ell_j$ with indices reduced mod $2k$.

Definition 9.1. The pair $(\omega, \vec{\nu})$ is called *resonant* if $R^n_\omega(\ell_j) = \ell_{j'}$ for some $n > 1$ and j, j'. A pair that is not resonant is called *non-resonant*.

Remark 9.2. Note that for a rational $\omega = p/q$ all $(p/q, \vec{\nu})$ are resonant, as are all $(\omega, \vec{\nu})$ when some $\nu_i = 0$. Also, for all $(\omega, \vec{\nu})$ and j,

$$R_\omega(\ell_{2j-1}) = \ell_{2j}, \tag{9.1}$$

which is the reason why n is restricted to $n > 1$ in the definition.

The next lemma locates the 'holes' in the space of all symbolic kfsm sets and thus in any $\hat{\mathcal{B}}(g)$.

LEMMA 9.3

(a) *Assume parameters $(\alpha, \vec{\nu})$ with $\alpha \notin \mathbb{Q}$ resonant. There exist a sequence $\vec{\nu}^{(i)} \to \vec{\nu}$ and a non-recurrent kfsm Z with $B_k(\alpha, \vec{\nu}^{(i)}) \to Z$ in the Hausdorff topology on all compact subsets of Σ^+_{2k}.*

(b) *Assume parameters $(p/q, \vec{\nu})$ with $p/q \in \mathbb{Q}$. There exists a sequence $\omega^{(i)} \to p/q$ and a non-recurrent kfsm Z with $B_k(\omega^{(i)}, \vec{\nu}) \to Z$ in the Hausdorff topology on all compact subsets of Σ^+_{2k}.*

Proof. We suppress the dependence on k to simplify notation. For (a), the resonance hypothesis implies that there are odd a and b with $R^n_\alpha(X_{a\vec{\nu}}) = X_{b\vec{\nu}}$ for some $n > 0$ where we may assume $a < b$. Since $R^n_\alpha(r_{a\vec{\nu}}) = r_{b\vec{\nu}}$, by shrinking some ν_j for $a < j < b$ we obtain a $\vec{\nu}\,'$ and $x < r_{a\vec{\nu}}$ and arbitrarily close to it with $x \in G_{\alpha\vec{\nu}\,'}$ and $R^n_\alpha(x) \in X_{b+1,\vec{\nu}\,'}$. In this way we can obtain sequences $\vec{\nu}^{(i)} \to \vec{\nu}$ and $x_i \nearrow r_{a\vec{\nu}}$ with $x_i \in G_{\alpha,\vec{\nu}^{(i)}}$ and $R^n_\alpha(x_i) \in X_{b+1,\vec{\nu}^{(i)}}$. Thus

$$\zeta_{\alpha,\vec{\nu}^{(i)}}(x_i) = .a \ldots (b+1)\zeta_{\alpha,\vec{\nu}^{(i)}}(R^{n+1}_\alpha(x_i)).$$

To simplify matters, assume that $R_\alpha(r_{b\vec{\nu}}) \in G_{\alpha,\vec{\nu}}$; more complicated resonances are similar. Since $R^{n+1}_\alpha(x_i) \to R^{n+1}_\alpha(r_{a\vec{\nu}}) = R_\alpha(r_{b\vec{\nu}})$ using Remark 8.4(b),

$$\zeta_{\alpha,\vec{\nu}^{(i)}}(x_i) \to .a \ldots (b+1)\zeta_{\alpha,\vec{\nu}}(R_\alpha(r_{b\vec{\nu}}) := \underline{s}.$$

Passing to a subsequence if necessary, by the compactness of the collection of symbolic kfsm sets there is a kfsm Z with $B_k(\alpha, \vec{\nu}^{(i)}) \to Z$ in the Hausdorff topology, and by its

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construction $\underline{s} \in Z$. But $\underline{s}$ cannot be recurrent since by the resonance any length-$(n+1)$ block in $\zeta_{\alpha,\vec{\nu}}(R_\alpha(r_{b\vec{\nu}})$ must start with a and end in b.

The argument for (b) is similar, but now the perturbation must be in the parameter ω. This is because if $\omega = p/q$ is fixed, then $R_\omega^n(X_a) = X_a$ with $n = qk/\gcd(p,k)$ for all $\vec{\nu}$. Fix an a and so $R_{p/q}^n(r_a) = r_a$. By increasing ω incrementally we may find sequences $\omega^{(i)} \searrow p/q$ and $x_i \nearrow r_a$ with $x_i \in G_{\omega^{(i)},\vec{\nu}}$ so that the initial length-$(n+1)$ block of $\zeta_{\omega^{(i)}\vec{\nu}}(x_i)$ is $a \ldots a+1$. Thus if $\zeta_{p/q\vec{\nu}}(r_a + \epsilon) = P^\infty$ for small ϵ then

$$\zeta_{\omega^{(i)},\vec{\nu}}(x_i) \to .a \ldots (a+1)P_2P_3 \ldots P_{n-1}P^\infty := \underline{t}$$

where $P = (a+1)P_2P_3 \ldots P_{n-1}$. As in the proof of (a) passing to a subsequence if necessary, there is a kfsm Z with $B_k(\omega^{(i)}, \vec{\nu}) \to Z$ in the Hausdorff topology, and by its construction $\underline{t} \in Z$. But $\underline{t}$ cannot be recurrent since any length-$(n+1)$ block in P^∞ must start and end with a. □

9.2. *Continuity and injectivity.* In the HM construction the explicit dependence of A_k and B_k on the pair $(\omega, \vec{\nu})$ was included. However, note that the elements of the pair have the interdependence $\sum \nu_i = k(1-\omega)$. Thus when we treat A_k and B_k as functions it is sometimes better to eliminate the interdependence and treat them as functions of $\vec{\nu}$ alone. Nonetheless, the two variable version will also continue to be useful. Thus we sometimes overload the function A_k and write

$$A_k(\vec{\nu}) = A_k\left(1 - \sum \nu_i/k, \vec{\nu}\right),$$

and similarly for B_k and the measure-valued map λ_k. The collection of allowable parameters for each k is then

$$\mathcal{D}_k = \left\{\vec{\nu} \in \mathbb{R}^k : \nu_i \geq 0, \sum_{i=1}^k \nu_i \leq k\right\}.$$

The set of HM parameters corresponding to symbolic kfsm sets for $g \in \mathcal{G}$ is defined as

$$\mathrm{HM}_k(g) = \{\vec{\nu} \in \mathcal{D}_k : B_k(\vec{\nu}) \subset \hat{\mathcal{B}}_k(g)\}.$$

Remark 9.4. By Theorem 8.5, $B_k : \mathrm{HM}(g) \to \hat{\mathcal{B}}_k(g)$ is surjective and so $\iota_k^{-1}B_k$: $\mathrm{HM}(g) \to \mathcal{B}_k(g)$ provides a parameterization of the positive-slope kfsm recurrent sets of $g \in \mathcal{G}$ and $(\iota_k^{-1})_*\lambda_k : \mathrm{HM}(g) \to \mathcal{N}_k(g)$ their invariant measures.

Example: The model map. For the model map f_m, $\mathrm{HM}_k(f_m) = \mathcal{D}_k$ since $\hat{\Lambda}_k(f) = \Omega_k$.

The first issue in what the HM construction tells us about $\mathcal{B}_k(g)$ and $\mathcal{N}_k(g)$ is to understand the nature of the maps B_k and λ_k. Lemma 9.3 showed that for B_k there is an essential distinction between the resonance and non-resonance cases.

THEOREM 9.5. *Assume $g \in \mathcal{G}$, for each $k > 0$,*

(a) *The map $(\iota_k)^{-1} \circ B_k : \mathrm{HM}_k(g) \to \mathcal{B}_k(g)$ is onto, and further it is continuous at non-resonant values and discontinuous at resonant values.*

(b) *The map $(\iota_k)_*^{-1} \circ \lambda_k : \mathrm{HM}_k(g) \to \mathcal{N}_k(g)$ is a homeomorphism and thus $\mathrm{HM}_k(g)$ is compact.*

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Proof. Since we know from Lemma 7.8 that ι_k and $(\iota_k)_*$ are homeomorphisms we only consider B_k and λ_k. While these are functions of $\vec{\nu}$ alone, for the proof it is clearer to resort to the two-variable versions with the proviso that $\omega = 1 - \sum \nu_i/k$. Note that we have already shown in Theorem 8.5 that λ_k and B_k are onto $\hat{\mathcal{N}}_k(g)$ and $\hat{\mathcal{B}}_k(g)$, respectively. We will often need to include the explicit dependence of various objects on the variables, for example, $\ell_j(\omega, \vec{\nu})$, and we often suppress the dependence on k.

We prove (b) first. We first show that λ_k is continuous. For each $j = 1, \ldots, 2k-1$ and $i \in \mathbb{N}$, let $\ell_j^{(i)}(\omega, \vec{\nu}) = R_\omega^{-i}(\ell_j(\omega, \vec{\nu}))$. The first observation from the HM construction is that

$$|\ell_j^{(i)}(\omega, \vec{\nu}) - \ell_j^{(i)}(\omega_0, \vec{\nu}_0)| \leq \|(\omega, \vec{\nu}) - (\omega_0, \vec{\nu}_0)\|_1. \tag{9.2}$$

For a length-N block $B = b_0 \ldots b_{N-1}$ in Ω_k, let

$$Y_B(\omega, \vec{\nu}) = \bigcap_{i=0}^{N-1} R_\omega^{-i}(\mathrm{Int}(X_{b_i}(\vec{\nu}))),$$

and so $x \in G_{\omega,\vec{\nu}} \cap Y_B(\omega, \vec{\nu})$ implies that $\zeta_{\omega\vec{\nu}}(x)$ begins with the block B. Also by the HM construction, $\lambda_k(\omega, \vec{\nu})([B]) = m(Y_B(\omega, \vec{\nu}))$, with m being Lebesgue measure on the circle.

Recall that the weak topology on Σ_{2k}^+ is generated by the metric

$$d(\mu, \mu') = \sum_{i=1}^{\infty} \frac{|\mu([B_i]) - \mu'([B_i])|}{2^i}$$

where $\{B_i\}$ is some enumeration of the blocks in Σ_{2k}^+. Since each $Y_B(\omega, \vec{\nu})$ is a (perhaps empty) interval with endpoints some $\ell_i^{(j)}(\omega, \vec{\nu})$, inequality (9.2) implies that

$$|m(Y_B(\omega, \vec{\nu})) - m(Y_B(\omega_0, \vec{\nu}_0))| \leq 2\|(\omega, \vec{\nu}) - (\omega_0, \vec{\nu}_0)\|_1.$$

Thus, summing over blocks,

$$d(\lambda_k(\omega, \vec{\nu}), \lambda_k(\omega_0, \vec{\nu}_0)| \leq 2\|(\omega, \vec{\nu}) - (\omega_0, \vec{\nu}_0)\|_1,$$

so λ_k is continuous.

Since by definition in the HM construction, $\lambda_k(\omega, \vec{\nu})([2j]) = \nu_{j+1}$, λ_k is injective. Recall now that for the model map, $\mathrm{HM}(f_m) = \mathcal{D}_k$ which is compact. So $\lambda_k : \mathrm{HM}(f_m) \to \hat{\mathcal{N}}_k(f_m)$ is a homeomorphism whose image is the set of all measures on recurrent symbolic kfsm sets in Ω_k. Thus, since $\mathrm{HM}_k(g) \subset \mathcal{D}_k$ we have that $\lambda_k : \mathrm{HM}_k(g) \to \hat{\mathcal{N}}_k(g)$ is also a homeomorphism. The compactness of $\hat{\mathcal{N}}_k(g)$ was proved in Lemma 7.8.

The proof of (a) is based on the following claim: B_k is continuous at $(\omega_0, \vec{\nu}_0)$ if and only if for all N there exists $\delta > 0$ so that $\|(\omega, \vec{\nu}) - (\omega_0, \vec{\nu}_0)\| < \delta$ implies that for all blocks B of length up to N we have $Y_B(\omega_0, \vec{\nu}_0)$ non-empty exactly when $Y_B(\omega, \vec{\nu})$ is non-empty.

To prove the claim, first note that continuity is equivalent to the following: given $\epsilon > 0$, there exists $\delta > 0$ so that $\|(\omega, \vec{\nu}) - (\omega_0, \nu_0)\| < \delta$ implies that for each $\underline{s} \in \zeta_{\omega\vec{\nu}}(G_{\omega\vec{\nu}})$ there is a $\underline{t} \in \zeta_{\omega_0\vec{\nu}_0}(G_{\omega_0\vec{\nu}_0})$ with $d(\underline{s}, \underline{t}) < \epsilon/2$ and for each $\underline{t} \in \zeta_{\omega_0\vec{\nu}_0}(G_{\omega_0\vec{\nu}_0})$ there is an $\underline{s} \in \zeta_{\omega\vec{\nu}}(G_{\omega\vec{\nu}})$ with $d(\underline{s}, \underline{t}) < \epsilon/2$. This implies that $\mathrm{HD}(\zeta_{\omega\vec{\nu}}(G_{\omega\vec{\nu}}), \zeta_{\omega_0\vec{\nu}_0}(G_{\omega_0\vec{\nu}_0})) < \epsilon/2$ and thus $\mathrm{HD}(B_k(\omega, \vec{\nu}), B_k(\omega_0, \vec{\nu}_0)) < \epsilon$. Since $d(\underline{s}, \underline{t})$ is small exactly when $\underline{s}$ and $\underline{t}$ agree in a long prefix block B and $Y_B(\omega, \vec{\nu}) = \zeta_{\omega\vec{\nu}}^{-1}([B])$, the claim follows.

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We show that B_k satisfies the condition in the claim when $(\omega, \vec{\nu})$ is non-resonant. Given N for $j = 1, \ldots, 2k-1$ and $i = 0, \ldots, N$, consider again $\ell_j^{(i)}(\omega, \vec{\nu}) = R_\omega^{-i}(\ell_j(\omega, \vec{\nu}))$. By the HM construction we have $\ell_{2m}^{(n+1)}(\omega, \vec{\nu}) = \ell_{2m-1}^{(n)}(\omega, \vec{\nu})$ for all n, m, and $(\omega, \vec{\nu})$. By non-resonance at $(\omega_0, \vec{\nu}_0)$, all the other $\ell_j^{(i)}(\omega_0, \vec{\nu}_0)$ are disjoint. Since by (9.2) each $\ell_j^{(i)}(\omega, \vec{\nu})$ depends continuously on $(\omega, \vec{\nu})$ and the endpoints of each $Y_B(\omega, \vec{\nu})$ are some $\ell_j^{(i)}(\omega, \vec{\nu})$, we may find a δ so that $\|(\omega, \vec{\nu}) - (\omega_0, \vec{\nu}_0)\| < \delta$ implies that the $\ell_j^{(i)}(\omega, \vec{\nu})$ are ordered around S_k in the same way and with the same gaps between them as the $\ell_j^{(i)}(\omega_0, \vec{\nu}_0)$. This implies that for each block B of length $B \le N$, $Y_B(\omega_0, \vec{\nu}_0)$ is non-empty exactly when $Y_B(\omega, \vec{\nu})$ is non-empty and so B_k is continuous.

For the discontinuity, since the sets Z in Lemma 9.3(ab) are not recurrent, they are not equal to $B_k(\omega_0, \vec{\nu}_0)$. □

Remark 9.6

(a) The parameter space $\mathcal{D}_k$ is $(k-1)$-dimensional. Assuming $\omega \notin \mathbb{Q}$, for a fixed $n > 1$ and j, j', the collection of all $\vec{\nu} \in \mathcal{D}_k$ which yield $R_\omega^n(\ell_j) = \ell_{j'}$ is a $(k-2)$-dimensional affine subspace. Thus the set of resonance parameters is a countable dense collection of codimension- one affine subspaces and so the non-resonance case is a full measure and dense G_δ set.

(b) One can show that B_k is lower semi-continuous [**12**]; in particular, if $(\omega^{(i)}, \vec{\nu}^{(i)}) \to (\omega, \vec{\nu})$ and some subsequence of $B_k(\omega^{(i)}, \vec{\nu}^{(i)})$ converges to Z in the Hausdorff topology, then $B_k(\omega, \vec{\nu}) \subset Z$. The semi-continuity lemma (see p. 114 of [**19**]) yields that a lower semi-continuous set-valued function is continuous on a dense G_δ set. In the case of $(\iota_k)^{-1} \circ B_k$ the last theorem exactly identifies this continuity set as the non-resonant $(\omega, \vec{\nu})$.

9.3. *Slices and skewness.* Recall that the rotation number functions $\rho : \mathcal{B}_k(g) \to \mathbb{R}$ and $\hat{\rho} : \mathcal{B}_k(g) \to \mathbb{R}$ are continuous on the various spaces, as are their measure-theoretic analogs. Thus we may define closed slices with a given rotation number as follows.

Definition 9.7. For $g \in \mathcal{G}$ let $\mathcal{B}_{k\omega}(g) = \{Z \in \mathcal{B}_k(g) \colon \rho(Z) = \omega\}$ and $\hat{\mathcal{B}}_{k\omega}(g) = \{Z \in \hat{\mathcal{B}}_k(g) \colon \hat{\rho}(Z) = \omega\}$; the restriction of ι_k to $\mathcal{B}_{k\omega}(g)$ is denoted by $\iota_{k\omega}$. The slices of invariant measures $\mathcal{N}_{k\omega}(g)$ and $\hat{\mathcal{N}}_{k\omega}(g)$ are defined similarly. The ω-slice of HM parameters is $\mathrm{HM}_{k\omega}(g) = B_k^{-1}(\mathcal{B}_{k\omega}(g)) = \lambda_k^{-1}(\mathcal{N}_{k\omega}(g))$.

Definition 9.8. For $p/q \in \mathbb{Q}$ an allowable parameter $\vec{\nu}$ is called pure if $B_k(p/q, \vec{\nu})$ consists of a single periodic orbit. The collection of p/q pure parameters is denoted by $\mathrm{Pure}_{k,p/q} \subset \mathcal{D}_{k,p/q}$ and it will be shown in Lemma 12.4 to be an affine lattice. For a $g \in \mathcal{G}$ its pure parameters are $\mathrm{Pure}_{k,p/q}(g) = \mathrm{HM}_{k,p/q}(g) \cap \mathrm{Pure}_{k,p/q}$.

Remark 9.9. For a given symbolic kfsm p/q-periodic orbit P, by Theorem 8.5(c) there is some $\vec{\nu}$ with $B_k(p/q, \vec{\nu}) = P$. Since a periodic orbit is uniquely ergodic and λ_k is injective this $\vec{\nu}$ is unique. Thus there is a bijection between symbolic kfsm p/q-periodic orbits and $\mathrm{Pure}_{k,p/q}$.

LEMMA 9.10. *Assume $g \in \mathcal{G}$. Then the following assertions hold.*

(a) *For all ω, $(\iota_{k\omega}^{-1})_* \circ \lambda_{k\omega} : \mathrm{HM}_{k\omega}(g) \to \mathcal{N}_{k\omega}(g)$ is a homeomorphism.*

(b) *When* $\alpha \notin \mathbb{Q}$, $\iota_{k\alpha}^{-1} \circ B_{k\alpha} : \mathrm{HM}_{k\alpha}(g) \to \mathcal{B}_{k\alpha}(g)$ *is injective as well as continuous at non-resonant* $(\alpha, \vec{\nu})$ *and discontinuous at resonant* $(\alpha, \vec{\nu})$.

(c) *When* $p/q \in \mathbb{Q}$, $\iota_{kp/q}^{-1} B_{kp/q} : \mathrm{HM}_{kp/q}(g) \to \mathcal{B}_{kp/q}(g)$ *is injective on* $\mathrm{Pure}_{k,p/q}$.

Proof. Since ι_k restricts to a homeomorphism on slices we only consider $B_{k\omega}$ and $\lambda_{k\omega}$. Part (a) follows immediately from Theorem 9.5.

For (b), when $\alpha \notin \mathbb{Q}$ the assignment of a semi-Denjoy kfsm set with rotation number α to its unique invariant measure yields a bijection $\mathcal{B}_{k\alpha}(g) \to \mathcal{N}_{k\alpha}(g)$ and $\hat{\mathcal{B}}_{k\alpha}(g) \to \hat{\mathcal{N}}_{k\alpha}(g)$. Since, by (a), $\lambda_{k\alpha}$ is injective, we have that $B_{k\alpha}$ is also. Continuity of $B_{k\alpha}$ at non-resonant values on irrational slices follows directly from (a). Discontinuity at resonant values on irrational slices follows from Lemma 9.3(a).

For (c), when $p/q \in \mathbb{Q}$ the assignment of the single periodic orbit $B_k(p/q, \vec{\nu})$ to its unique invariant measure yields the injectivity using (a) as in the proof of (b). □

Remark 9.11. Since $\mathcal{B}_{kp/q}(g)$ is a finite set, the continuity of $\iota_{kp/q}^{-1} B_{kp/q} : \mathrm{HM}_{kp/q}(g) \to \mathcal{B}_{kp/q}(g)$ is not particularly interesting, but we will remark on it in §12.4.

The *skewness* $\gamma(\mu)$ of a $\tilde{g}_k$-invariant measure in S_k equals the amount of measure in each fundamental domain. When its *j*th component is large, its $\tilde{g}_k$-orbits are moving slowly through $[j-1, j)$. When we project to the base S^1 in the next section the skewness thus indicates how quickly orbits are moving in the *j*th loop of the kfsm set.

Definition 9.12. Assume $g \in \mathcal{G}$. Then the following assertions hold.

(a) For $\eta \in \mathcal{N}_k(g)$, $\gamma(\eta) = (\eta([0, 1)), \eta([1, 2)), \ldots, \eta([k-1, k)))$.

(b) For $\hat{\eta} \in \hat{\mathcal{N}}_k(g)$, $\hat{\gamma}(\eta) = (\hat{\eta}([0] \cup [1]), \hat{\eta}([2] \cup [3]) \ldots, \hat{\eta}([2k-2] \cup [2k-1]))$.

Note that the skewness takes values in the unit simplex $\sum a_i = 1$, $a_i \geq 0$, and contains no information about the rotation number.

LEMMA 9.13. *Assume* $g \in \mathcal{G}$. *Then the following assertions hold.*

(a) $\hat{\gamma} \circ (\iota_k)_* = \gamma$.

(b) $\gamma(\lambda_k(\omega, \vec{\nu})) = (\omega + \nu_1, \omega + \nu_2, \ldots, \omega + \nu_k)/k$.

(c) *For* $\eta \in \mathcal{N}_{k\omega}$, $\gamma_1(\eta) = k\gamma(\eta) - \omega\mathbb{1}$ *is inverse to* $(\iota_k)_*^{-1} \circ \lambda_k$ *and so it is a homeomorphism.*

(d) γ *is a homeomorphism from* $\mathcal{N}_{k\omega}(g)$ *onto its image. as is* $\hat{\gamma}$ *from* $\hat{\mathcal{N}}_{k\omega}(g)$ *onto its image.*

Remark 9.14. The last lemma formalizes the description in the Introduction of the parametrization of the weak disks of semi-Denjoy minimal sets by their speed in each 'loop' around the circle. For rational pure parameters the skewness counts the number of elements in each fundamental domain and this thus yields a discrete parametrization of the kfsm p/q-periodic orbits.

10. *kfsm sets in* S^1 *and* Ω_1

10.1. *In* S^1. We now return to our central concern, g-invariant sets in S^1 that have a lift to S_k that is semi-monotone. Once again the definition makes sense for any degree-one circle map, but we restrict to the class $\mathcal{G}$.

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Definition 10.1. Given $g \in \mathcal{G}$, a compact g-invariant set $Z \subset S^1$ is kfsm if it has a $\tilde{g}$-invariant lift $Z' \subset \mathbb{R}$ which is kfsm, or equivalently, Z has a $\tilde{g}_k$-invariant lift $Z^* \subset S_k$ which is kfsm. Let $\mathcal{C}_k(g)$ be all compact, invariant, recurrent kfsm sets in $\Lambda_1(g)$ with the Hausdorff topology and $\mathcal{O}_k(g)$ be all g-invariant, Borel probability measures supported on $Z \in \mathcal{C}_k(g)$ with the weak topology

Thus when Z is kfsm, it has a lift to S_k which is semi-monotone under the action of $\tilde{g}_k$ on its lift.

To make contact with the usual definitions in Aubry–Mather theory, assume that $x \in S^1$ is such that $o(x, f)$ is kfsm. This happens exactly when there is a point $x' \in \mathbb{R}$ with $\pi_\infty(x') = x$ and for all positive integers ℓ, m, n,

$$\tilde{g}^{\ell}(x') < T^{km}\tilde{g}^{n}(x') \quad \text{implies} \quad \tilde{g}^{\ell+1}(x') \leq T^{m}\tilde{g}^{n+1}(x')$$

In Aubry–Mather theory one would write $x_j = \tilde{g}^j(x')$.

Remark 10.2

(a) $\pi_k : S_k \to S^1$ induces continuous onto maps $\mathcal{B}_k(g) \to \mathcal{C}_k(g)$ and $\mathcal{N}_k(g) \to \mathcal{O}_k(g)$.

(b) $Z^* \subset S_k$ is kfsm if and only if $\pi_k(Z^*) \subset S^1$ is.

(c) If $Z \subset S^1$ is k-fold semi-monotone then it is also ℓk-fold semi-monotone for any $\ell > 0$.

(d) If P is a periodic orbit of g of type (p, q) (which are perhaps not relatively prime) then P has a lift P' to $\mathbb{R}$ with $T^p(P') = P'$ and is monotone since $g \in \mathcal{G}$ implies $\tilde{g}(x') \geq x'$ and so P is automatically p-fold semi-monotone.

(e) Using Lemma 7.6 a recurrent kfsm set in S^1 is either a collection of periodic orbits all with the same rotation number (a cluster) or else a semi-Denjoy minimal set. A minimal kfsm set in S^1 is either a single periodic orbit or else a semi-Denjoy minimal set.

(f) A collection of periodic orbits all with the same rotation number that individually are kfsm when considered as a set is not of necessity a kfsm set (that is, a cluster).

10.2. *Symbolic kfsm sets in* Ω_1. We now consider symbolic kfsm sets in the symbolic base $\Omega_1 = \Sigma_2^+$.

Definition 10.3. A σ_1-invariant set $\hat{Z} \subset \Omega_1 = \Sigma_2$ is kfsm if there is a σ_∞-invariant lift $\hat{Z}'$ (that is, $\hat{p}_\infty(\hat{Z}') = \hat{Z}$) which is kfsm or equivalently, $\hat{Z}$ has a σ_k-invariant lift $\hat{Z}^* \subset \Omega_k$ which is kfsm. Given $g \in \mathcal{G}$, let $\hat{\mathcal{C}}_k(g)$ be all recurrent kfsm sets in $\hat{\Lambda}_1(g)$ with the Hausdorff topology and $\hat{\mathcal{O}}_k(g)$ be all g-invariant, Borel probability measures supported on $\hat{Z} \in \hat{\mathcal{C}}_k(g)$ with the weak topology

Using Theorem 7.6, we connect kfsm sets in $\Lambda_1(g)$ to their symbolic analogs in $\hat{\Lambda}_1(g)$ and obtain the following corollary.

COROLLARY 10.4. *A g-invariant set $Z \subset \Lambda_1(g)$ is kfsm if an only if $\iota_1(Z) \subset \hat{\Lambda}_1(g)$ is. Further, ι_1 induces homeomorphisms $\mathcal{C}_k(g) \to \hat{\mathcal{C}}_k(g)$ and $\mathcal{O}_k(g) \to \hat{\mathcal{O}}_k(g)$.*

Remark 10.5. All the comments in Remark 10.2 hold *mutatis mutandis* for symbolic kfsm sets.

10.3. *The HM construction and its symmetries.* We bring the HM construction back into play and take the projections from Ω_k to Ω_1.

Definition 10.6. Let $C_k(\omega, \vec{\nu}) = \hat{\pi}_k(B_k(\omega, \vec{\nu}))$ and $\mu_k(\omega, \vec{\nu}) = (\hat{\pi}_k)_*(\lambda_k(\omega, \vec{\nu}))$

We know from Theorem 9.5 that the HM construction provides a parameterization of $\mathcal{B}_k(g)$ and $\mathcal{N}_k(g)$. The goal now is to get a parameterization of the kfsm sets and their invariant measures in S^1, that is, of $\hat{\mathcal{C}}_k(g)$ and $\hat{\mathcal{O}}_k(g)$. For this we need to understand the symmetries inherent in the HM construction.

Recall that the left shift on the parameter ν is $\tau(\nu_1, \ldots, \nu_k) = (\nu_2, \ldots, \nu_k, \nu_1)$. There are two types of symmetries to be considered. The first is when different $\vec{\nu}$ give rise to the same $C_k(\omega, \vec{\nu})$. For minimal $C_k(\omega, \vec{\nu})$ this happens if and only if the $\vec{\nu}$s are shifts of each other as is stated in parts (a) and (d) in the lemma below. The second sort of symmetry happens when some $C_k(\omega, \vec{\nu})$ is also a $C_j(\omega, \vec{\nu}\,')$ for some $j < k$, which is to say the map $\hat{\pi}_k : B_k(\omega, \vec{\nu}) \to C_k(\omega, \vec{\nu})$ is not one-to-one. In the minimal case this happens if and only if $\tau^j(\vec{\nu}) = \vec{\nu}$ as is stated in parts (b) and (c) below.

LEMMA 10.7. *Fix $k > 0$ and assume $\vec{\nu}$ is allowable for ω.*

(a) *For all j, $B_k(\omega, \tau^j(\vec{\nu})) = \hat{T}_k^j(B_k(\omega, \nu))$ and so $C_k(\omega, \tau^j(\vec{\nu})) = C_k(\omega, \nu)$.*

(b) *If $\tau^j(\vec{\nu}) = \vec{\nu}$ for some $0 < j < k$ then*

$$B_k(\omega, \vec{\nu}) = \hat{T}_k^j(B_k(\omega, \vec{\nu})) \tag{10.1}$$

and $C_k(\omega, \vec{\nu}) = C_j(\omega, \vec{\nu}\,')$ where $\vec{\nu}\,' = (\nu_1, \ldots, \nu_j)$.

(c) *If $B_k(\omega, \vec{\nu})$ is minimal and (10.1) holds then $\vec{\nu} = \tau^j(\vec{\nu})$. If $\vec{\nu} \neq \tau^j(\vec{\nu})$ for all $0 < j < k$, then $\hat{\pi}_k : B_k(\omega, \vec{\nu}) \to C_k(\omega, \vec{\nu})$ is a homeomorphism.*

(d) *If $B_k(\omega, \vec{\nu})$ and $B_k(\omega, \vec{\nu}\,')$ are minimal and $C_k(\omega, \vec{\nu}) = C_k(\omega, \vec{\nu}\,')$, then for some j, $\vec{\nu}\,' = \tau^j(\vec{\nu})$.*

Proof. The fact that $B_k(\omega, \tau^j(\vec{\nu})) = \hat{T}_k^j(B_k(\omega, \nu))$ is an easy consequence of the HM construction, and since $\hat{\pi}_k \hat{T}^k = \hat{\pi}_k$ we have $C_k(\omega, \tau^j(\vec{\nu})) = C_k(\omega, \nu)$, proving (a) The first part of (b) follows directly from (a), using the given fact that $\tau^j(\vec{\nu}) = \vec{\nu}$.

For the second part of (b), first note that if $\{X_i\}$ is the address system for k and $(\omega, \vec{\nu})$ then since $\tau^j(\vec{\nu}) = \vec{\nu}$, we have $T_k^j(X_i) = X_{i+2j}$. This implies that under the quotient $S_k \to S_j$, $\{X_i\}$ descends to an allowable HM address system on S_j using $(\omega, \vec{\nu}\,')$. Thus using the dynamics R_ω on both address systems, the corresponding entries of $B_k(\omega, \vec{\nu})$ and $B_j(\omega, \vec{\nu}\,')$ are equal mod 2 and so $C_k(\omega, \vec{\nu}) = C_j(\omega, \vec{\nu}\,')$.

To prove the first part of (c), as remarked in Remark 7.11, if $B_k(\omega, \nu)$ is minimal it is uniquely ergodic. Thus if (10.1) holds, then $\lambda_k(\omega, \vec{\nu}) = \lambda_k(\omega, \tau^j(\vec{\nu}))$, and since λ_k is injective by Theorem 9.5, $\vec{\nu} = \tau^j(\vec{\nu})$. Now for the second part of (c), certainly $\hat{\pi}_k : B_k(\omega, \vec{\nu}) \to C_k(\omega, \vec{\nu})$ is continuous and onto, so assume it is not injective. Then there exist $\underline{s}, \underline{t} \in B_k(\omega, \vec{\nu})$ with $\underline{s} \neq \underline{t}$ and $\hat{\pi}_k(\underline{s}) = \hat{\pi}_k(\underline{t})$. Thus for some $0 < j' < k$, $\underline{t} = \hat{T}_k^{j'}(\underline{s})$, and so if $j = k - j'$, then $B_k(\omega, \vec{\nu}) \cap T_k^j B_k(\omega, \vec{\nu}) \neq \emptyset$. But by assumption

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$B_k(\omega, \vec{\nu})$ is minimal and so $B_k(\omega, \vec{\nu}) = T_k^j B_k(\omega, \vec{\nu})$ and so $\vec{\nu} = \tau^j(\vec{\nu})$, a contradiction. Thus $\hat{\pi}_k : B_k(\omega, \vec{\nu}) \to C_k(\omega, \vec{\nu})$ is injective, as required.

For part (d), $C_k(\omega, \vec{\nu}) = C_k(\omega, \vec{\nu}\,')$ implies that

$$\bigcup_{i=1}^{k} T^i(B_k(\omega, \vec{\nu})) = \hat{\pi}^{-1}(C_k(\omega, \vec{\nu})) = \hat{\pi}^{-1}(C_k(\omega, \vec{\nu}\,')) = \bigcup_{i=1}^{k} T^i(B_k(\omega, \vec{\nu}\,')).$$

Since each of $T^i(B_k(\omega, \vec{\nu}))$ and $T^i(B_k(\omega, \vec{\nu}\,'))$ is minimal, for some j, $T^j(B_k(\omega, \vec{\nu})) = B_k(\omega, \vec{\nu}\,')$, and so by part (c), $\tau^j(\vec{\nu}) = \vec{\nu}\,'$. $\square$

Remark 10.8. It is possible that if $B_k(p/q, \vec{\nu})$ is a cluster of periodic orbits, π_k could be injective on some of them and not on others.

10.4. *Continuity and injectivity.* Let $\overline{\mathrm{HM}_k(g)} = \mathrm{HM}_k(g)/\tau$ with equivalence classes denoted by $[\vec{\nu}]$. Note that $\tau^j(\vec{\nu}) \in \mathcal{D}_k$ for some j is resonant if and only if $\vec{\nu}$ is, so we may call $[\vec{\nu}]$ resonant or non-resonant.

Since the τ-action preserves slices, we define $\overline{\mathrm{HM}}_{k\omega} = \mathrm{HM}_{k\omega}(g)/\tau$. The ω-slices of $\mathcal{C}_k(g)$ and $\mathcal{O}_k(g)$ are defined in the obvious way. If $(p/q, \vec{\nu})$ is a pure parameter so is $\tau^j(\vec{\nu})$ for any j and so we define $\overline{\mathrm{Pure}}(k, p/q) = \mathrm{Pure}(k, p/q)/\tau$. Note that $\overline{\mathrm{Pure}}(k, p/q)$ is all $[\vec{\nu}]$ such that $B_k(p/q, \vec{\nu})$ is a single periodic orbit, it is *not* all $[\vec{\nu}]$ such that $C_k(p/q, \vec{\nu}) = \hat{\pi}_k B_k(p/q, \vec{\nu})$ is a single periodic orbit.

Definition 10.9. Lemma 10.7 implies that $(\iota_1)^{-1} \circ C_k$ induces a map $\theta_k : \overline{\mathrm{HM}}_k(g) \to \mathcal{C}_k(g)$ and that $(\iota_1)_*^{-1} \circ \mu_k$ induces a map $\beta_k : \overline{\mathrm{HM}}_k(g) \to \mathcal{O}_k(g)$. The induced maps on slices are $\theta_{k\omega} : \overline{\mathrm{HM}}_{k\omega}(g) \to \mathcal{C}_{k\omega}(g)$ and $\beta_{k\omega} : \overline{\mathrm{HM}}_{k\omega}(g) \to \mathcal{O}_{k\omega}(g)$.

THEOREM 10.10. *Assume $g \in \mathcal{G}$, for each $k > 0$.*

(a) *The map θ_k is onto, continuous at non-resonant values and discontinuous at resonant values. Restricted to an irrational slice, it is injective, continuous at non-resonant values, and discontinuous at resonant values. Restricted to a rational slices, it is injective on the pure lattice.*

(b) *The map β_k is a homeomorphism when restricted to irrational slices and pure rational lattices.*

Proof. By construction we have the following commuting diagram:

$$\begin{tikzcd} \mathrm{HM}_k(g) \arrow[r, "B_k"] \arrow[d, "\sim"'] & \hat{\mathcal{B}}_k(g) \arrow[r, "\iota_k^{-1}"] \arrow[d, "\hat{\pi}_k"] & \mathcal{B}_k(g) \arrow[d, "\pi_k"] \\ \overline{\mathrm{HM}}_k(g) \arrow[r, "C_k"] & \hat{\mathcal{C}}_k(g) \arrow[r, "\iota_1^{-1}"] & \mathcal{C}_k(g) \end{tikzcd} \tag{10.2}$$

The vertical maps are all onto and continuous, while by definition the composition of the bottom horizontal maps is θ_k. In a slight abuse of notation, the map C_k in the diagram denotes the map induced on equivalence classes in $\overline{\mathrm{HM}}_k(g)$ by C_k. Since ι_k and ι_1 are homeomorphisms we need only consider C_k and μ_k. The fact that these are continuous follows from Lemma 9.10 and the just stated properties of the diagram, as do the various continuity assertions in the theorem. We prove the discontinuity result for C_k on irrational slices. The other discontinuity assertions follow similarly.

Assume $(\alpha, \vec{\nu})$ is resonant with $\alpha \notin \mathbb{Q}$. From Lemma 9.3 and its proof we have a sequence $(\alpha, \vec{\nu}^{(i)}) \to (\alpha, \vec{\nu})$ so that $B_k(\alpha, \vec{\nu}^{(i)}) \to Z$, and an $\underline{s} \in Z \setminus B_k(\alpha, \vec{\nu})$ with $\underline{s}$ non-recurrent. In the quotients, $[\vec{\nu}^{(i)}] \to [\vec{\nu}]$ and $C_k(\alpha, \vec{\nu}^{(i)}) \to \hat{\pi}_k(Z)$ by continuity. We need to show that $\hat{\pi}_k(Z) \neq C_k(\alpha, \vec{\nu})$ Now if $\pi_k(\underline{s}) \in \hat{\pi}_k(Z) \setminus C_k(\alpha, \vec{\nu})$ we are done, so assume $\pi_k(\underline{s}) \in C_k(\alpha, \vec{\nu})$. Thus for some $\underline{t} \in B_k(\alpha, \vec{\nu})$, $\pi_k(\underline{s}) = \pi_k(\underline{t})$, and so by Lemma 5.3(e), for some j, $\hat{T}_k^j(\underline{s}) = \underline{t}$. This implies that the action of σ_k on $\mathrm{Cl}(o(\underline{s}, \sigma_k))$ is conjugated to that on $\mathrm{Cl}(o(\underline{t}, \sigma_k))$ by $\hat{T}_k^j$. But by Theorem 8.5, $\mathrm{Cl}(o(\underline{t}, \sigma_k))$ is a minimal set and thus so is $\mathrm{Cl}(o(\underline{s}, \sigma_k))$, and so $\underline{s}$ is recurrent, a contradiction, yielding the discontinuity.

To show C_k is injective on the sets indicated, assume $C_k(\omega, \vec{\nu}) = C_k(\omega, \vec{\nu}\,')$ with either $\omega = p/q$ and $\vec{\nu}, \vec{\nu}\,'$ in the pure lattice or $\omega \notin \mathbb{Q}$. In either case $B_k(\omega, \vec{\nu})$ and $B_k(\omega, \vec{\nu}\,')$ are minimal and since $\hat{\pi}_k$ is a semi-conjugacy, $C_k(\omega, \vec{\nu})$ and $C_k(\omega, \vec{\nu}\,')$ are also. Thus by Lemma 10.7(d), for some j, $\vec{\nu} = \tau^j \vec{\nu}\,'$ and so $[\vec{\nu}] = [\vec{\nu}\,']$.

Now for part (b), there is a diagram similar to (10.2) for β_k. Since Denjoy minimal sets and individual periodic orbits are uniquely ergodic, the injectivity asserted for μ_k follows from that of C_k just proved. Continuity and surjectivity follow from the diagram and Lemma 9.10. □

Remark 10.11. We remark on the relationship of pure parameters to C_k and B_k. As a shorthand we indicate symbolic periodic orbits by their repeating block. A simple computation shows that $B_2(2/5, (3/5, 3/5)) = 01223 \cup 00123$ and so $C_2(2/5, (3/5, 3/5)) = 01001$. Note that, as required, $\hat{T}_2(01223) = 00123$ and 01001 is the 2/5-Sturmian (as defined in the next section). Now $B_2(2/5, (4/5, 2/5)) = 00123$ and so $C_2(2/5, (4/5, 2/5)) = 01001 = C_2(2/5, (3/5, 3/5))$. A further computation shows that both $\mu_2(2/5, (4/5, 2/5))$ and $\mu_2(2/5, (3/5, 3/5))$ are the unique invariant measure on 01001 and thus μ_2 is not injective on rational slices of $\overline{\mathrm{HM}}_k$ despite the fact that it is injective on rational slices of HM_k. The underlying explanation is that being a pure parameter requires B_k to be a single periodic orbit, not that C_k be one.

Definition 10.12. Let $\mathcal{Q}_k = P_k/\tau$ where $P_k \subset \mathbb{R}^{k+1}$ is the standard k-dimensional simplex and τ is the shift. Equivalence classes in $\mathcal{Q}_k$ are denoted $[\cdot]$. For $\hat{\eta} \in \hat{\mathcal{O}}_k(g)$ with $\rho(\hat{\eta}) = \omega$ from Theorem 10.10 we may find an $\vec{\nu}$ with $\hat{\eta} = \mu_k(\omega, \vec{\nu})$. The skewness of $\hat{\eta}$ is defined as $\overline{\gamma}(\hat{\eta}) := [\gamma(\lambda_k(\omega, \vec{\nu}))]$. Note that by Lemma 10.7 this is independent of the choice of $\mu_k(\omega, \vec{\nu})$. And also for $\eta \in \mathcal{O}_k(g)$ via $\overline{\gamma}(\eta) = \overline{\gamma}((\iota_k)_*(\eta))$.

Remark 10.13. On an irrational quotient slice $\mathcal{O}_{k,\alpha}$, let $\overline{\gamma}_1 = k\overline{\gamma} - \alpha\mathbb{1}$. Then $\overline{\gamma}_1$ is the inverse of β_k and may be viewed as a parameterization of $\hat{\mathcal{O}}_{k,\omega}$ by skewness as in Remark 9.14. Also as in that remark, skewness also provides a parameterization of the quotient of the pure parameters.

10.5. *Sturmian minimal sets, the case $k = 1$.* We will need the special and much-studied case of symbolic kfsm sets for $k = 1$. When $k = 1$ there is only one allowable choice for ν, namely $\nu = 1 - \omega$, and so we write $C_1(\omega)$ for $C_1(\omega, 1 - \omega) = B_1(\omega, 1 - \omega)$. When ω is rational $C_1(\omega)$ is a single periodic orbit and when ω is irrational it is a semi-Denjoy minimal set. These minimal sets (and associated sequences) have much historical importance and an abundance of literature (see [2] for a survey). Their main importance here is as an indicator of when a given number is in the rotation set.

Definition 10.14. The minimal set $C_1(\omega) \subset \Sigma_2^+$ is called the Sturmian minimal set, with rotation number ω which may be rational or irrational.

Because there are many definitions in the literature, to avoid confusion we note that here 'Sturmian' refers to a minimal set and not a sequence, and it is subset of the *one-sided* shift Σ_2^+. The next result is standard and we remark on one proof in Remark 13.6.

LEMMA 10.15. *$\omega \in \rho(\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle)$ if and only if $C_1(\omega) \subset \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$. If $0 \le \omega_1 < \omega_2 \le 1$, then in Σ_2^+,*

$$\min C_1(\omega_1) < \min C_1(\omega_2) < \max C_1(\omega_1) < \max C_2(\omega_2).$$

Definition 10.16. For a fixed k, let $\vec{\nu}_s(\omega)$ be defined by $(\vec{\nu}_s(\omega))_i = 1 - \omega$ for $i = 1, \ldots, k$.

Remark 10.17. Since $\tau(\vec{\nu}_s) = \vec{\nu}_s$ it follows directly from Lemma 10.7 that for any k, $C_k(\omega, \vec{\nu}_s) = C_1(\omega)$, the Sturmian minimal set with rotation number ω.

11. *Structure of* $\mathrm{HM}_k(g)$

One obvious property of $\mathrm{HM}_k(g)$ is the symmetry $\tau(\mathrm{HM}_k(g)) = \mathrm{HM}_k(g)$ for all k. The full structure of $\mathrm{HM}_k(g)$ for a general $g \in \mathcal{G}$ is quite complicated and will be saved for future papers. Here we focus on the structure near the diagonal in $\mathcal{D}_k$.

11.1. *Irrationals on the diagonal.*

We parameterize the diagonal $\Delta_k \subset \mathcal{D}_k$ by ω using $\vec{\nu}_s(\omega) \in \Delta_k$ as defined in the previous section, and so

$$\Delta_k = \{\vec{\nu}_s(\omega) \colon 0 \le \omega \le 1\}.$$

For $g \in \mathcal{G}$ the next result asserts that for each irrational $\alpha \in \mathrm{Int}(\rho(g))$ there is some $\delta = \delta(\alpha)$ so that the neighborhood $N_\delta(\vec{\nu}_s(\alpha)) \subset \mathrm{HM}_k(g)$. It gives the proof of Theorem 1.2(a).

THEOREM 11.1. *Assume $g \in \mathcal{G}$ and $k > 0$.*

(a) $\mathrm{HM}_k(g) \cap \Delta_k = \{\vec{\nu}_s(\omega) \colon \omega \in \rho(g)\}$.

(b) *If $\alpha \notin \mathbb{Q}$ with $\alpha \in \mathrm{Int}(\rho(g))$, there exists a $\delta > 0$ so that $N_\delta(\vec{\nu}_s(\alpha)) \subset \mathrm{HM}_k(g)$.*

(c) *If $\alpha \in \mathrm{Int}(\rho(g)) \setminus \mathbb{Q}$, then $\mathcal{O}_k(g)$ contains a $(k-1)$-dimensional topological disc consisting of unique invariant measures each supported on a member of a family of kfsm semi-Denjoy minimal sets with rotation number α.*

Proof. Assume $\hat{\Lambda}_1(g) = \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$. For (a) $C_k(\omega, \vec{\nu}_s(\omega)) = C_1(\omega)$, the Sturmian minimal set with rotation number ω, and from Lemma 10.15, $C_1(\omega) \subset \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$ if and only if $\omega \in \rho(\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle) = \rho(g)$.

For (b) note that the pair $(\alpha, \vec{\nu}_s)$ is non-resonant. We will first show that if $\alpha \in \mathrm{Int}(\rho(g))$ then there exists an $\epsilon > 0$, so that $\mathrm{HD}(C_k(\alpha, \vec{\nu}_s(\alpha)), C_k(\omega, \vec{\nu})) < \epsilon$ implies $C_k(\omega, \vec{\nu}) \subset$

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$\langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$. Pick $\alpha_1, \alpha_2 \in \mathrm{Int}(\rho(g))$ with $\alpha_1 < \alpha < \alpha_2$. Thus by Lemma 10.15, in Σ_2^+,

$$\underline{\kappa}_0 < \min C_1(\alpha_1) < \min C_1(\alpha) < \max C_1(\alpha) < \max C_1(\alpha_2) < \underline{\kappa}_1,$$

and let

$$\epsilon = \min\{d(\min C_1(\alpha_1), \min C_1(\alpha)), d(\max C_1(\alpha), \max C_1(\alpha_2))\}.$$

Thus $\mathrm{HD}(C_k(\alpha, \vec{\nu}_s(\alpha)), C_k(\omega, \vec{\nu})) = \mathrm{HD}(C_1(\alpha), C_k(\omega, \vec{\nu})) < \epsilon$ implies that the compact, invariant set $C_k(\omega, \vec{\nu})$ satisfies $\min C_1(\alpha_1) < C_k(\omega, \vec{\nu}) < \max C_1(\alpha_2)$ and so $C_k(\omega, \vec{\nu}) \subset \langle \underline{\kappa}_0, \underline{\kappa}_1 \rangle$.

Using the continuity of C_k at non-resonant irrationals from Theorem 10.10(a), there is a $\delta > 0$ so that $\|(\omega, \vec{\nu}) - (\alpha, \vec{\nu}_s)\| < \delta$ implies $C_k(\omega, \vec{\nu}) \subset N_\epsilon(C_k(\alpha, \vec{\nu}_s))$, and so $(\omega, \vec{\nu}) \in \mathrm{HM}(g)$.

Since $\tau(N_\delta(\vec{\nu}_s(\alpha))) = (N_\delta(\vec{\nu}_s(\alpha)))$, the neighborhood descends to one in $\overline{\mathrm{HM}}_k(g)$ and β_k is a homeomorphism on irrational slices of $\overline{\mathrm{HM}}_k(g)$ (Theorem 10.10(b)), yielding (c). □

12. *Rational slices*

In this section we study rational slices in the HM parameter and in $\mathcal{B}_k(g)$ and $\mathcal{C}_k(g)$. As proved in Theorem 8.5, each $B_k(p/q, \vec{\nu})$ is a collection of periodic orbits. They each have period $qk/\gcd(p,k)$ and all have rotation number p/q. Collectively as a set they are kfsm. Note that this is stronger than each periodic orbit being individually kfsm. The invariant measure $\lambda_k(p/q, \vec{\nu})$ is a convex combination of the unique measures supported on each periodic orbit.

12.1. *Periods in Ω_1.*

The next lemma examines how the periods of B_k can change after projection to C_k via $\hat{\pi}_k$.

LEMMA 12.1. *Fix $k > 0$ and $p/q \in \mathbb{Q}$ and assume $\vec{\nu}$ is allowable for p/q. If $\tau^j(\vec{\nu}) = \vec{\nu}$ with $0 < j \le k$ and it is the least such j, then the period of $C_k(p/q, \vec{\nu})$ is $jq/\gcd(j,p)$.*

Proof. Recall from Theorem 8.5 that the period of $B_k(p/q, \vec{\nu})$ is $kq/\gcd(k,p)$. If $j = k$ by Lemma 10.7(c), $\hat{\pi}_k : B_k(p/q, \vec{\nu}) \to C_k(p/q, \vec{\nu})$ is injective, and since $\sigma_1 \hat{p}_k = \hat{\pi}_k \sigma_k$, $B_k(p/q, \vec{\nu})$ and $C_k(p/q, \vec{\nu})$ have the same period. Now if $j < k$ by Lemma 10.7(b), $C_k(p/q, \vec{\nu}) = C_j(p/q, \vec{\nu}\,')$ where $\vec{\nu}\,' = (\nu_1, \ldots, \nu_j)$. Since j is the least such, $\hat{\pi}_j : B_j(p/q, \vec{\nu}\,') \to C_j(p/q, \vec{\nu}\,')$ is injective and $C_j(p/q, \vec{\nu}\,')$ has period $jq/\gcd(j,p)$. □

12.2. *The rational structure theorem.*

The theorem in this section describes in more detail how the measures on p/q-kfsm sets vary with the parameter.

In the HM construction fix k, $0 < p/q < 1$ with $\gcd(p,q) = 1$, and an allowable $\vec{\nu}$. We often suppress dependence on these choices and so $R = R_{p/q}$, etc. Let $N = qk/\gcd(p,k)$ so N is the period of R acting on S_k. Recall that the address intervals are $X_j = [\ell_j, r_j]$ for $j = 0, \ldots, 2k-1$ and so $r_j = \ell_{j+1}$. The good set is G and the itinerary map is ζ. When we write $\zeta(x)$ it is implicitly assumed that $x \in G$.

The orbit of 0, $o(0, R)$, partitions S_k into N pieces, each of width $k/N = \gcd(p,k)/q$. Thus $J = [k - \gcd(p,k)/q, 1)$ is a fundamental domain for the action of R on S_k in the

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sense that $S_k = \bigcup_{i=0}^{N-1} R^i(J)$ as a disjoint union. Thus for each $0 \le p \le 2k-1$ there is a unique $0 \le m < N$ with $\ell_p \in R^m(J)$, and then let $d_p = R^{-m}(\ell_p)$. Note that since $|X_{2j+1}| = p/q$, $d_{2j} = d_{2j-1}$, and that all d_{2j+1} as well as both endpoints of J are *not* in G. Finally, for $j = 0, 1, \ldots, 2k-1$ and $x \in J \cap G$, let $M_j(x) = \{0 \le i < N \colon \zeta(x)_i = j\} = \{i \colon R^i(x) \in X_j\}$.

LEMMA 12.2. *Assume $x, x' \in J \cap G$.*

(a) *$\zeta(x) = \zeta(x')$ if and only if $M_{2j+1}(x) = M_{2j+1}(x')$ for all $j = 0, \ldots, k-1$.*
(b) *For each j, $M_{2j+1}(x) = M_{2j+1}(x')$ if and only if x and x' are in the same component of $J - \{d_{2j+1}\}$.*
(c) *For each k, $\#M_{2j}(x) = \#M_{2j}(x')$ if and only if x and x' are in the same component of $\Sigma \setminus \{d_{2j-1}, d_{2j+1}\}$ where Σ is the circle $\Sigma = J/\sim$ with $(k - 1/N) \sim k$.*

Proof. First note that both endpoints of J are not in G so they are out of consideration for x and x' in what follows.

For (a) one implication is obvious. For the other, it suffices to show that the collection of $M_{2j+1}(x)$ determines $\underline{s} = \zeta(x)$. By Remark 7.11 we know that $\underline{s} \in \Omega_k$ and so its one-step transitions are governed by (5.1). If $s_i = 2j+1$ then $s_{i+1} = 2j+2$ or $2j+3$ and we know which depending on whether $i+1 \in M_{2j+3}$ or not. Similarly, if $s_i = 2j$ then s_{i+1} is determined by whether $i+1 \in M_{2j+1}$ or not. Thus $\underline{s}$ is determined, completing the proof of (a).

For (b), first note that $|X_{2k-1}| = p/q$ and $(p/\gcd(p,k))(k/N) = p/q$. Thus X_{2k-1} is exactly filled with $p/\gcd(p,k)$ iterates of J with disjoint interiors. Thus $M_{2k-1}(x) = M_{2k-1}(x')$ for all $x \in J \cap G$. Thus we only consider $0 \le j < k-1$. If i is such that $R^i(J) \subset X_{2j+1}$, then $i \in M_{2j+1}(x)$ for all $x \in J$, and if $R^i(\mathrm{Int}(J)) \cap X_{2j+1} = \emptyset$ then $i \notin M_{2j+1}(x)$ for all $x \in J \cap G$. If $\ell_{2j+1} \in R^i(\mathrm{Int}(J))$, then for $x > d_{2j+1}$ in J, we have $i \in M_{2j+1}(x)$ and for $x < d_{2j+1}$, $i \notin M_{2j+1}(x)$. The last situation to consider is $r_{2j+1} \in R^i(\mathrm{Int}(J))$. Since $|X_{2k+1}| = p/q$, we have $R^i(d_{2j+1}) = r_{2j+1}$ and so then for $x > d_{2j+1}$ in J, we have $i \notin M_{2j+1}(x)$ and for $x < d_{2j+1}$, $i \in M_{2j+1}(x)$, completing the proof of (b).

If $d_{2j-1} = d_{2j+1}$ every $x \in \Sigma \setminus \{d_{2j-1}\}$ has the same number of indices in M_{2j}, so assume that $d_{2j-1} < d_{2j+1}$, with the other inequality being similar. If i is such that $R^i(J) \subset X_{2j}$ then $i \in M_{2j}(x)$ for all $x \in J$. If i is such that $R^i(J) \cap X_{2j} = \emptyset$ then $i \notin M_{2j}(x)$ for all $x \in J$. If i is such that $\ell_{2j} \in R^i(J)$ then $i \in M_{2j}(x)$ if and only if $x > d_{2j-1}$ in J. If i is such that $\ell_{2j+1} = r_{2j} \in R^i(J)$ then $i \in M_{2j}(x)$ if and only if $x < d_{2j+1}$ in J, finishing the proof. □

COROLLARY 12.3. *If for some m, the non-empty connected components of $\Sigma \setminus \bigcup_{j=0}^{k-1}\{d_{2j+1}\}$ are $K_1, \ldots, K_m$, then $B_k(p/q, \nu)$ consists of exactly m distinct periodic orbits $P_1, \ldots, P_m$ with $\zeta(x) \in P_j$ if and only if $x \in o(K_j, R)$. Further, $\lambda_k(p/q, \nu) = \sum N|K_j|\delta_j$ with Υ_j the unique invariant probability measure supported in P_j and $N = qk/\gcd(p,k)$.*

Proof. As noted above, J is a fundamental domain for the action of R on S_k and so it suffices to study $\zeta(x)$ for $x \in J$.

Combining Lemma 12.2(a) and (b), we have that for $x \in \Sigma$, $\zeta(x) = \zeta(x')$ if and only if x and x' are in the same component K_j. Further, using Lemma 12.2(c), $\zeta(x)$ and $\zeta(x')$ can be on the same σ-orbit if and only if they are in the same component K_j, proving the first sentence of the corollary. The second sentence follows from the definition of λ_k, the fact that $S_k = \bigcup_{i=1}^N R^i(J)$, and that R preserves Lebesgue measure. □

12.3. *The pure lattice and the structure of* $\mathrm{HM}_{kp/q}$. We now describe the pure affine lattice in more detail with an eye towards counting the number of p/q-periodic kfsm sets. For this a new method of specifying the address system in S_k will be useful. We fix a k and an $\omega = p/q$ and sometimes suppress dependence on them

Recall that a pair $(p/q, \vec{\nu})$ specifies an address system $\{X_j(p/q, \vec{\nu})\}$ with each $X_j(p/q, \vec{\nu}) = [\ell_j, r_j]$. For each $i = 1, \ldots, k-1$ let ξ_i be the signed displacement of the address system from its totally symmetric position given by $(\omega, \vec{\nu}_s(\omega))$. Thus

$$\xi_i(\vec{\nu}) = (\nu_1 + \cdots \nu_i) - i(1-\omega). \tag{12.1}$$

Since in the HM construction X_{2k} is fixed for all $\vec{\nu}$, the vector $\vec{\xi}(\omega)$ is $(k-1)$-dimensional and so $\vec{\xi} : \mathcal{D}_{k,p/q} \to \vec{\xi}(\mathcal{D}_{k,p/q})$ is an affine map from the simplex $\sum \nu_i = k(1 - p/q)$ to a subset of $\mathbb{R}^{k-1}$. Note that $\vec{\xi}(\vec{\nu}_s) = \vec{0}$.

LEMMA 12.4. *Given k and p/q, there exists a $\vec{\eta}$ with $\|\vec{\eta}\|_\infty \le \gcd(p,k)/(2q)$ so that $\vec{\nu} \in \mathcal{D}_{k,p/q}$ is a pure parameter for p/q if and only if $\vec{\xi}(p/q, \vec{\nu}) \in \vec{\eta} + (\gcd(p,k)/q)\mathbb{Z}^{k-1}$ in $\vec{\xi}(\mathcal{D}_{k,p/q})$.*

Proof. Theorem 12.3 implies that $B_k(p/q, \vec{\nu})$ is a single periodic orbit if and only if no d_{2j-1} is in the interior of J. This happens if and only if all ℓ_{2j-1} are contained in $o(0, R_{p/q})$. Now $o(0, R_{p/q})$ divides S_k evenly into subintervals of length $\gcd(p,k)/q$. For each $j = 1, \ldots, k-1$ let m_j be such that $R_{p/q}^{m_j}(0)$ is the point on $o(0, R_{p/q})$ that is closest to ℓ_{2j-1} and define $\eta_j = \ell_{2j-1} - R_{p/q}^{m_j}(0)$. Thus $\|\vec{\eta}\|_\infty \le \gcd(p,k)/(2q)$ and $\vec{\nu}$ is pure if and only if $\phi(\vec{\nu}) \in \vec{\eta} + (\gcd(p,k)/q)\mathbb{Z}^{k-1}$. □

Definition 12.5. The set $L = \vec{\eta} + (\gcd(p,k)/q)\mathbb{Z}^{k-1} \cap \vec{\xi}(\mathcal{D}_{k,p/q})$ is called the p/q-pure affine lattice as is its pre-image $\vec{\xi}^{-1}(L) \subset \mathbb{R}^{n-1}$.

12.4. *Sub-resonance and the size of clusters*

Definition 12.6. When $\omega = p/q$, the pair $(p/q, \nu)$ is called *sub-resonant* if for some $qk/\gcd(p,k) > n > 1$ and $j \neq j'$, $R_\omega^n(\ell_j) = \ell_{j'}$.

It follows from Theorem 12.3 that the number of sub-resonances in $(p/q, \vec{\nu}) \in \mathcal{D}_{k,p/q}$ controls the number of distinct periodic orbits in a cluster $B_k(p/q, \vec{\nu})$. No sub-resonance corresponds to k distinct periodic orbits and when all the ℓ_j are on a single $R_{p/q}$ orbit, $B_k(p/q, \vec{\nu})$ is a single periodic orbit. In the latter case, $(p/q, \vec{\nu})$ is a pure parameter.

The set of sub-resonance parameters is a finite collection of codimension-one affine subspaces in $\mathcal{D}_{k,p/q}$. Thus the set of parameters with no sub-resonance case is an open, dense and full measure subset of $\mathcal{D}_{k,p/q}$. It follows then that in $\mathrm{HM}_{k,p/q}$ the typical parameter corresponds to a cluster of k periodic orbits. It also follows that, restricted to $\mathrm{HM}_{k,p/q}$, the assignment $\vec{\nu} \mapsto B_k(p/q, \vec{\nu})$ is constant, and thus is continuous on connected components of the no sub-resonance parameters and is discontinuous at the sub-resonance parameters.

12.5. *Estimating the number of p/q-kfsm sets.* For a given $g \in \mathcal{G}$ the number of points from the pure p/q-lattice $\mathrm{Pure}_{k,p/q}$ contained in $\mathrm{HM}_{k,p/q}(g)$ tells us how many distinct periodic orbits there are in $\hat{\mathcal{B}}_k(g)$. So by Lemma 10.7 it tells us how many distinct periodic p/q-kfsm sets g has. We get an estimate for this number using the continuity properties of B_k from Theorem 9.5 and the relationship of kfsm sets in S_k to those in S^1. The next result proves Theorem 1.2(b).

THEOREM 12.7. *If $\alpha \in \mathrm{Int}(\rho(g))$, $\alpha \notin \mathbb{Q}$, $k > 0$, and p_n/q_n is a sequence of rationals in lowest terms with $p_n/q_n \to \alpha$, then there exists a $C > 0$ so that for sufficiently large n the number of distinct periodic p_n/q_n-kfsm sets in $\Lambda_1(g)$ is greater than or equal to Cq_n^{k-1}.*

Proof. By Theorem 9.5(b) there is an $\epsilon_1 > 0$ so that $N_{\epsilon_1}(\vec{\nu}_s(\alpha)) \subset \mathrm{HM}_k(g)$, where recall that $\vec{\nu}_s(\alpha)$ is the Sturmian $\vec{\nu}$ for α on the diagonal of $\mathcal{D}_k$. Since $\vec{\xi}$ is a homeomorphism there is an ϵ-ball H in the max norm with $\epsilon > 0$ about $\vec{0}$ in $\vec{\xi}(\mathcal{D}_k)$ with $\vec{\xi}^{-1}(H) \subset \mathrm{HM}_k(g)$. Thus if $|p_n/q_n - \alpha| < \epsilon$ there is a ϵ-ball in the max norm, that is, a $(k-1)$-dimensional hypercube H_1, about $(p_n/q_n, \vec{0})$ in $\vec{\xi}(\mathcal{D}_{k,p_n/q_n})$ with $\vec{\xi}^{-1}(H_1) \subset \mathrm{HM}_{k,p_n/q_n}(g)$.

We next estimate the number of pure resonance $\vec{\nu}$ in H_1. By Lemma 12.4, the pure $\vec{\nu}$ form an affine lattice with linear separation $\gcd(p_n, k)/q_n$. Thus for p_n/q_n close enough to α, the number of lattice points in H_1 is larger than

$$\left(\frac{\epsilon q_n}{\gcd(p_n, k)}\right)^{k-1} \geq \left(\frac{\epsilon q_n}{k}\right)^{k-1}$$

since $\gcd(p_n, k) \leq k$. Thus since $\vec{\xi}$ is a homeomorphism the same estimate holds for the number of pure lattice points in $\vec{\xi}^{-1}(H_1) \subset \mathrm{HM}_{k,p_n/q_n}(g)$. By Theorem 9.5 this tells us how many distinct periodic p_n/q_n are in $\hat{\mathcal{B}}_k(g)$ and thus in $\mathcal{B}_k(g)$ by Theorem 7.6.

To project this estimate to kfsm sets in S^1, recall from Theorem 10.10 that $\theta_k : \overline{\mathrm{HM}}_{k,p_n/q_n}(g) \to \mathcal{C}_{k,p_n/q_n}(g)$ is injective on the pure lattice. The projection $\mathrm{Pure}(k, p_n/q_n) \to \overline{\mathrm{Pure}}(k, p_n/q_n)$ is at most k to 1 and so the number of distinct p_n/q_n periodic orbits in $\mathcal{C}(g)$ is greater than or equal to

$$\frac{1}{k}\left(\frac{\epsilon}{k}\right)^{k-1} q_n^{k-1}.$$

□

Remark 12.8. Using Lemma 12.1 for a pure $(p/q, \vec{\nu})$, if there is a symmetry of the form $\tau^j(\vec{\nu}) = \vec{\nu}$ for some $0 < j < k$, then the period of the $C_k(p/q, \vec{\nu})$ counted in the theorem is $jq/\gcd(j, p)$. In the typical case of no such symmetry the period is $kq/\gcd(k, p)$. So, for example, when p and k are relatively prime, the counted periodic orbit has rotation type (pk, qk), and when k divides p, the rotation type is (p, q). By making judicious choices of the sequence $p_n/q_n \to \alpha$, one can control the rotation types of the counted periodic orbits.

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13. *Parameterization via the interpolated family of maps*

We return now to the heuristic description of kfsm sets in the Introduction using a family of interpolated semi-monotone maps, and prove results and connections to the HM parameterization. Since we are mainly developing a heuristic, some details are left to the reader. In many ways this perspective is better for studying kfsm sets, while the HM construction is better for measures. Initially the parameterization depends on the map $\tilde{g} \in \mathcal{G}$ but using the model map we will get a uniform parameterization.

13.1. *The family of k-fold interpolated maps for $g \in \mathcal{G}$.*

Fix $g \in \mathcal{G}$ with preferred lift $\tilde{g}$. For $y \in [g(x_{\min} + n), g(x_{\max} + n)]$ there is a unique $x \in [x_{\min} + n, x_{\max} + n]$ with $\tilde{g}(x) = y$. Denote this x by $b_n(y)$ (b for branch). Let $L_g = \tilde{g}(\min(\Lambda_\infty(g) \cap I_0))$ and $U_g = \tilde{g}(\max(\Lambda_\infty(g) \cap I_{-1}))$ with the I_i as defined in §5.1. Note that, from the definition of the class $\mathcal{G}$, $0 \le L_g < U_g \le 1$ and, by equivariance, $L_g + j = \tilde{g}(\min(\Lambda_\infty(g) \cap I_{2j}))$ and $U_g + j = \tilde{g}(\max(\Lambda_\infty(g) \cap I_{2j-1}))$.

Definition 13.1. For $\vec{c} \in \mathbb{R}^k$ define $\tilde{c} \in \mathbb{R}^k$ via $\tilde{c}_j = c_j + j - 1$ for $j = 1, \ldots, k$.

Fix $k > 0$. For $\vec{c} \in [L_g, U_g]^k$ and for $j = 1, 2, \ldots, k$ define $\tilde{H}_{k\vec{c}}(x)$ on $[b_{-1}(0), b_{-1}(0) + k]$ as

$$\tilde{H}_{k\vec{c}}(x) = \begin{cases} \tilde{c}_j & \text{when } x \in [b_{j-2}(c_j), b_{j-1}(c_j)], \\ \tilde{g}(x) & \text{otherwise,} \end{cases}$$

and extend to $\tilde{H}_{k\vec{c}} : \mathbb{R} \to \mathbb{R}$ so that $\tilde{H}_{k\vec{c}}(x + k) = \tilde{H}_{k\vec{c}}(x) + k$. See Figure 1. Next define $H_{k\vec{c}} : S_k \to S_k$ as the descent of $H_{k,\vec{c}}$ to S_k.

Example: The model map. For the model map f_m, $x_{\min} = 0$, $x_{\max} = 1/2$, $L = 0$, $U = 1/2$ and $b_j(y) = (y + 2j)/3$.

Given a compact $Z \subset \Lambda_k(g)$, for $j = 0, \ldots, k$ let

$$\ell'_j(Z) = \tilde{g}_k(\max\{Z \cap I_{2j-1}\}) - (j-1) \quad \text{and} \quad r'_j(Z) = \tilde{g}_k(\min\{Z \cap I_{2j}\}) - (j-1).$$

If for some j we have $\ell'_j < L_g$ let $\ell_j = L_g$, otherwise let $\ell_j = \ell'_j$. Similarly, if for some j we have $r'_j > R_g$ let $r_j = R_g$, otherwise let $r_j = r'_j$. Not that these rs and ℓs are unrelated to those in §9.1.

THEOREM 13.2. *Assume $Z \subset \Lambda_k(g)$ is compact and invariant. The following assertions are equivalent.*

(a) *Z is a kfsm set.*
(b) *For $j = 1, \ldots, k$, $\ell_j(Z) \le r_j(Z)$.*
(c) *$Z \subset P(H_{k\vec{c}})$ for*

$$\vec{c} \in \prod_{j=1}^{k} [\ell_j(Z), r_j(Z)], \tag{13.1}$$

thus $(\tilde{g}_k)_{|Z} = (H_{k\vec{c}})_{|Z}$.

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Proof. If for some j, $\ell_j(Z) > r_j(Z)$ then g restricted to Z does not preserve the cyclic order, and so (a) implies (b). Assertion (c) implies (a) since invariant sets in non-decreasing maps are always kfsm. Finally, (b) says that $Z \subset P(H_{k\vec{c}})$ for $\vec{c}$ in the given range. □

Definition 13.3. For $Z \in \mathcal{B}_k(g)$, let

$$\mathrm{Box}_g(Z) = \prod_{j=1}^{k} [\ell_j(Z), r_j(Z)],$$

and so $\mathrm{Box}_g(Z) \subset [L_g, U_g]^k$.

Remark 13.4

(a) Nothing in the theorem requires Z to be recurrent. If it is then $Z \in \mathcal{B}_k(g)$ and so by Theorem 8.5, $\iota_k(Z) = B_k(\omega, \vec{\nu})$ where $\omega = \rho_k(Z)$ and $\vec{\nu} \in \mathcal{D}_{k\omega}$.

(b) When Z is a periodic orbit or cluster, $\mathrm{Box}_g(Z)$ is k-dimensional. When Z is a periodic orbit cluster its box is equal to the intersections of the boxes of its constituent single periodic orbits.

(c) When Z is a semi-Denjoy minimal set contained in $P(H_{k\vec{c}})$, recall that a tight flat spot of Z is one for which both endpoints of a flat spot of $H_{k\vec{c}}$ are in Z. The dimension of $\mathrm{Box}_g(Z)$ is the same as the number of *loose* flat spots in Z. Since, by Lemma 3.5(b), Z cannot have k tight flat spots, the dimension of $\mathrm{Box}_g(Z)$ is between 0 and $k-1$.

(d) Note that in contrast, in the HM parameterization, each single periodic orbit or semi-Denjoy minimal set corresponds to just one point.

(e) When $\alpha \notin \mathbb{Q}$, if $Z = \iota_k^{-1}(B_k(\alpha, \vec{\nu}))$ from the HM construction then the number of loose flat spots of Z is the same as the number of resonances of $(\alpha, \vec{\nu})$, that is, $j \neq j'$ with $R_\alpha^N(\ell_j) = \ell_{j'}$ for some $n > 1$, which is then the same as the dimension of $\mathrm{Box}_g(Z)$.

13.2. *Non-recurrence and kfsm sets that hit the negative-slope region.* Throughout this paper we have assumed that the kfsm sets were recurrent and avoided the negative-slope region. In this section we use the interpolated maps to motivate and explain these assumptions.

Assume now that Z is a kfsm set for some $g \in \mathcal{G}$ and Z contains points in the negative-slope region of g. It still follows that Z is an invariant set of some $H_{k\vec{c}}$. Let Z' be the maximal recurrent set in $P(H_{k\vec{c}})$. A *gap* of Z' is a component of the complement of Z' that contains a flat spot of $H_{k\vec{c}}$. In formulas, a gap is an interval $(\max\{Z' \cap I_{2j-1}\}, \min\{Z' \cap I_{2j}\})$ for some j. Since g_k acting on Z is semi-monotone, Z can contain at most one point p_j in the negative-slope region within each gap.

There are two cases. In the first, which may happen for both rational and irrational rotation numbers, for all j' there is some n so that $f^i(p_{j'}) \notin \{p_j\}$ for all $i \geq n$. This implies that $H^i_{k\vec{c}}(p_j) \in P(H_{k\vec{c}})$ for all $i \geq n$, and so, by Lemma 3.5(c), there is an n' so that $H^i_{k\vec{c}}(p_j) \in Z'$ for all $i \geq n'$. Thus in this case negative-slope orbits add no additional recurrent dynamics. In Figure 4 the disks give part of a periodic kfsm set and the squares show additional homoclinic points to this kfsm set in the negative- and positive-slope region.

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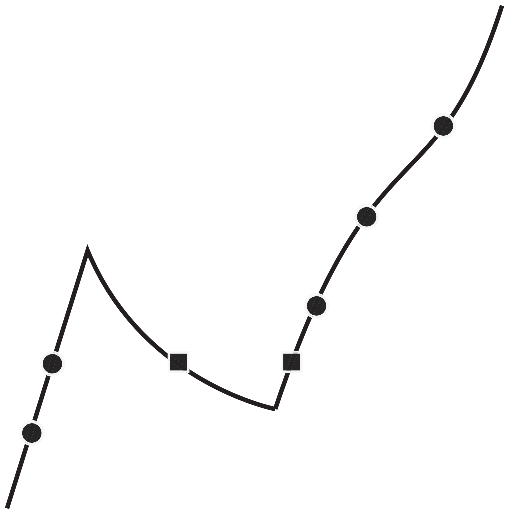

FIGURE 4. A semi-monotone set with homoclinic points.

The second case holds for just rational rotation numbers and occurs when some p_j is a periodic point; this does add new recurrent kfsm sets. Since the periodic points on the endpoint of a gap always return, there is, in fact, a periodic point in the negative-slope region of each gap of Z'. By adjusting $\vec{c}$ we can assume that all these gap periodic orbits are also superstable periodic orbits of $H_{k\vec{c}}$. Since the periodic points in Z' are all unstable the periodic points of Z' must alternate with these gap periodic points. In particular, the number of gap periodic orbits equals the number of periodic orbits in Z'. Thus the addition of the negative-slope periodic orbits just adds a factor of two to the basic estimates of §12.5.

We again use Figure 4, but this time to discuss the holes in the space of recurrent kfsm sets. Let H_{c_0} be an interpolated map whose flat spot contains the homoclinic points indicated by squares. Assume we are in the $k = 1$ case and so each interpolated map H_c contains exactly one recurrent semi-monotone set Z_c in its positive-slope region. As c_n increases to c_0, the sets Z_{c_n} converge in the Hausdorff topology not just to Z_{c_0}, but to that set union the boxed point shown in the positive slope region. A similar phenomenon happens as c_n decreases to c_0. This phenomenon also clearly happens for loose gaps of semi-Denjoy minimal sets and for all k. This is the geometric explanation of the holes in $\mathcal{B}_k(g)$ and the discontinuity of B_k discussed in §9.1.

13.3. *The rotation number diagram.* Fix $g \in \mathcal{G}$. Since $H_{k\vec{c}} : S_k \to S_k$ we define $R_k(\vec{c}) = k\rho(D_k \circ H_{k\vec{c}} \circ D_k^{-1})$. Thus if $Z \subset P(H_{k\vec{c}})$ is compact invariant then $\rho_k(Z) = R_k(\vec{c})$. We treat R_k as a function $R_k : [L_g, U_g]^k \to \mathbb{R}$.

Let $\mathbb{R}^k_+ = \{\vec{u} \in \mathbb{R}^k : \text{ all } u_i > 0\}$. The open projective positive cone in $\mathbb{R}^k$ is $Q_k = \{\vec{u} \in \mathbb{R}^k_+ : \|\vec{u}\|_2 = 1\}$. For a given k and ω, define $\varphi_{-,\omega}, \varphi_{+,\omega} : Q_k \to \mathbb{R}^+$ as

$$\varphi_{-,\omega}(\vec{u}) = \min\{t \in \mathbb{R}^+ : R_k(t\vec{u} + \vec{L}) = \omega\},$$
$$\varphi_{+,\omega}(\vec{u}) = \max\{t \in \mathbb{R}^+ : R_k(t\vec{u} + \vec{L}) = \omega\},$$

where $\vec{L} = (L_g, L_g, \ldots, L_g)$. So $\varphi_{-,\omega}$ and $\varphi_{+,\omega}$ give the top and bottom edges of the level set $R_k^{-1}(\omega)$ when viewed from the origin.

THEOREM 13.5. *Assume $g \in \mathcal{G}$ and construct R_k as above.*

(a) *R_k is continuous function and is non-decreasing in t along any line $\vec{c} = t\vec{u} + \vec{v}$ with all $v_i \geq 0$.*

(b) *For all ω the functions $\varphi_{-,\omega}$ and $\varphi_{+,\omega}$ are continuous.*

(c) *For rational ω, $\varphi_{-,p/q} < \varphi_{+,p/q}$, while for $\alpha \notin \mathbb{Q}$, $\varphi_{-,\alpha} = \varphi_{+,\alpha}$. Thus each level set $R_k^{-1}(p/q)$ is homeomorphic to a $(k-1)$-dimensional open disk product a non-trivial closed interval, while each $R_k^{-1}(\alpha)$ is homeomorphic to a $(k-1)$-dimensional open disk.*

(d) *$\rho(g) = [\rho(H_{L_g}), \rho(H_{U_g})] = \rho(\Lambda_1(g), g)$.*

Proof. Part (a) follows directly from Lemma 3.1(b) and (c). For (b) assume to the contrary that φ_- is not continuous. Then there is a sequence $\vec{u}_n \to \vec{u}_0$ with $\varphi_-(\vec{u}_n) \not\to \varphi_-(\vec{u}_0)$. Passing to a subsequence if necessary, there is some t_0 with $\varphi_-(\vec{u}_n)\vec{u}_n + \vec{L} \to t_0\vec{u}_0 + \vec{L}$. By the continuity of R_k, $R_k(t_0\vec{u}_0 + \vec{L}) = \omega$, and by the non-convergence assumption, there is some $t' < t_0$ with $R_k(t'\vec{u}_0 + \vec{L}) = \omega$. Thus again by the continuity of R_k for n large enough there is some $t_n'' < \varphi_-(\vec{u}_n)$ with $R_k(t_n''\vec{u}_n + \vec{L}) = \omega$, a contradiction. Therefore, φ_- is continuous; the continuity of φ_+ is similar.

For (c), pick any t_0 and $\vec{u}_0$ with $R_k(t_0\vec{u}_0 + \vec{L}) = p/q$ and let $\vec{c} = t_0\vec{u}_0 + \vec{L}$. Then by Lemma 3.5, $H_{\vec{c}}$ has a periodic orbit $Z \subset P(H_{\vec{c}})$. Since Z is a finite set there is a non-trivial interval I so that $t \in I$ implies $R_k(t\vec{u}_0 + \vec{L}) = p/q$ and so $\varphi_{-,p/q} < \varphi_{+,p/q}$.

To complete (c), assume to the contrary that for some $\vec{u}_0$, $\varphi_{-,\alpha}(\vec{u}_0) < \varphi_{+,\alpha}(\vec{u}_0)$. Thus by the continuity of R_k there is an open ball $N \subset R_k^{-1}(\alpha)$. Pick $\vec{c} \in N$ and let Z be the semi-Denjoy minimal set in $P(H_{\vec{c}})$ guaranteed by Lemma 3.5 which has at least one tight gap, say the gap associated with c_1, the first coordinate of $\vec{c}$. Let y be the x-coordinate of the right-hand endpoint of this gap and so $\tilde{g}_k(y) = c_1$. Since Z is minimal under $H_{\vec{c}}$ there are points $z \in Z$ with $z > y$ and arbitrarily close to y which have a $n > 0$ with $y < H_{\vec{c}}^n(z) < z$. Now let $c_1' = \tilde{g}_k(z)$ and $\vec{c}' = (c_1', c_2, \ldots, c_k)$ and we have that $H_{\vec{c}'}^n(z) < z$, which says that the nth iterate of the first coordinate flat spot of $H_{\vec{c}'}$ is in that flat spot. Thus $H_{\vec{c}'}$ has a periodic orbit and so $R_k(\vec{c}') \neq \alpha$ for some $\vec{c}'$ arbitrarily close to $\vec{c}$, a contradiction.

For (d), assume $k = 1$ and $\rho(g) = [\rho_1, \rho_2]$. Let H_T be the semi-monotone map constructed from g to have a single flat spot of height $\tilde{g}(x_{\max})$ and H_B similarly constructed to have a single flat spot of height $\tilde{g}(x_{\min})$. Since $H_T \geq \tilde{g}$, we have $\rho(H_T) \geq \rho_1$. Now by Lemma 3.5, there is a compact invariant $Z \subset P(H_T)$ and so $g_{|Z} = (H_T)_{|Z}$ and so $\rho(H_T) = \rho(Z, g) \in \rho(g)$ and so $\rho(H_T) = \rho_1$. Similarly, $\rho(H_B) = \rho_2$. Note that by definition of H_U, the compact invariant $Z \subset P(H_T)$ also satisfies $Z \subset P(H_U)$ and so $\rho(H_T) = \rho(H_U)$. Similarly, $\rho(H_B) = \rho(H_L)$. Thus $\rho(g) = [\rho(H_L), \rho(H_U)]$. Finally, consider the entire family H_c for $c \in [L, U]$. Since $\rho(H_c)$ is continuous in c, for each $\omega \in [\rho(H_L), \rho(H_U)]$ there is a c with $\rho(H_c) = \omega$. Further, for each c there is a compact invariant $Z_c \subset P(H_c)$ and $Z_c \subset \Lambda_1(g)$, and thus $\omega \in \rho(\Lambda_1(g)) \subset \rho(g)$ □

Remark 13.6

(a) Note that $H_T(x) \leq x+1$ and $H_B(x) \geq x$, and thus $g \in \mathcal{G}$ implies $\rho(g) \subset [0, 1]$. Further, it follows from (d) that the image of each R_k is $\rho(g)$.

(b) Part (b) deals only with the part of the level sets of R_k in the open set $(L_g, U_g)^k$. The extension to all of $[L_g, U_g]^k$ is technical and not very illuminating, so we leave it to the interested reader.

(c) Let τ act on $\vec{c}$ as the left cyclic shift. It easily follows that $R_k(\tau(\vec{c})) = R_k(\vec{c})$.

(d) When $k = 1$ there is a one-dimensional family H_c for $c \in [L_g, U_g]$. The rotation number $R_1(c)$ is non-decreasing in c and assumes each irrational value at a point and each rational value on an interval by (a) and (c). For each c there is a unique recurrent $Z_c \subset P(H_c)$ and $\iota_1(Z_c)$ is the Sturmian minimal set with the given rotation number. This, along with the geometry of the family H_c, gives the proof of Lemma 10.15.

13.4. *Comparing $g \in \mathcal{G}$ to the model map.* In this section we use the interpolation parameter $\vec{c}$ to parameterize all the $Z \in \mathcal{B}_k(g)$ for a general $g \in \mathcal{G}$. Notice that for the model map, $\hat{\Lambda}_k(f_m)$ is all of Ω_k. Thus $\hat{\mathcal{B}}_k(g) \subset \hat{\mathcal{B}}_k(f_m)$ and we can pass back to $\mathcal{B}(g)$ using the inverse of the itinerary map. Thus we can use a subset of the interpolation parameters of the model map to parameterize $\mathcal{B}(g)$ using the symbolic representation of a kfsm set as the link. This subset turns out to be a square of the form $[L', U']^k$. In this section we often add an additional f or g subscript to indicate which map f_m or g is involved.

Since $\iota_{k,g}(\Lambda_k(g)) = \hat{\Lambda}_k(g) \subset \Omega_k = \hat{\Lambda}_k(f)$ we may define $\psi' : \Lambda_k(g) \to \Lambda_k(f)$ by $\psi' = \iota_{kf} \circ \iota_{kg}^{-1}$. By Theorem 6.1, ψ' is an orientation-preserving homeomorphism onto its image as well as a conjugacy. It thus induces a map $\overline{\psi} : \mathcal{B}_k(g) \to \mathcal{B}_k(f)$.

Recall that the parameters for the model map are $[L_f, U_f]^k = [0, 1/2]^k$. For a map $\phi : [a, b] \to [a, b]$ extend it to the Cartesian product as $\phi^{(k)} = (\phi, \phi, \ldots, \phi)$.

THEOREM 13.7. *Given $g \in \mathcal{G}$ and $k > 0$, construct the interpolation parameters $[L_g, U_g]$. There exist an interval $[L', U'] \subset [0, 1/2]$ and an orientation-preserving homeomorphism $\phi : [U_g, L_g] \to [L', U']$ so that for all $Z \in \mathcal{B}_k(g)$, $\phi^{(k)}(\mathrm{Box}_g(Z))) = \mathrm{Box}_f(\overline{\psi}(Z))$, and for all $\omega \in \rho(g)$, $\phi^{(k)}\rho_{k,g}^{-1}(\omega) = \rho_{k,f}^{-1}(\omega)$.*

Proof. Construct ψ' as above. Its properties imply that

$$\psi'(\ell_j(Z)) = \ell_j(\overline{\psi}(Z)) \quad \text{and} \quad \psi'(r_j(Z)) = r_j(\overline{\psi}(Z)) \tag{13.2}$$

for all $Z \subset \mathcal{B}_k(g)$ and $j = 1, \ldots, k$. Let $L' = \psi'(L_g)$ and $U' = \psi'(U_g)$. Then ψ' restricts to $\psi : \Lambda_k(g) \cap [L_g, U_g]^k \to \Lambda_k(f) \cap [L', U']^k$. Since $\psi T_k = T_k \psi$ and $\Lambda_k(g) \cap [L_g, U_g]^k$ is compact we can extend ψ equivariantly to a homeomorphism $\Psi : [L_g, U_g]^k \to [L', U']^k$ which, using (13.2), satisfies $\Psi \circ \mathrm{Box}_g = \mathrm{Box}_f \circ \overline{\psi}$. Finally, since $\Psi \circ \tau = \tau \circ \Psi$ (recall τ is the left cyclic shift) there is a $\phi : [U_g, L_g] \to [L', U']$ with $\Psi = \phi^{(k)}$. □

This result implies that the ρ_k-diagram for g looks like a k-dimensional cube cut from inside the ρ_k-diagram of the model map and perhaps rescaled.

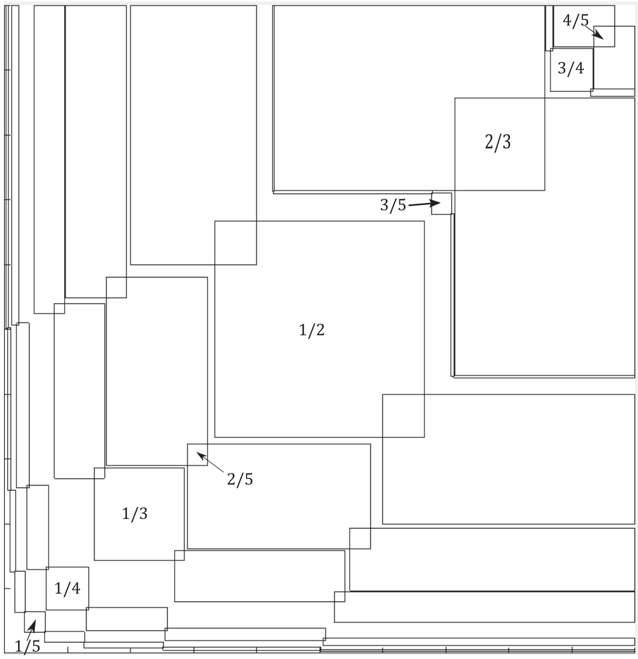

FIGURE 5. The rotation number diagram for the model map with $k = 2$, reparameterized for clarity.

13.5. *The case $k = 2$: numerics.* Figure 5 shows the $k = 2$ rotation number diagram for the model map f_m. Each connected union of rectangles is the level set of some rational. Rationals with denominator less than 6 are shown. Only the center rectangle is labeled for each rational. Each rectangle in the figure corresponds to a different 2-fold semi-monotone periodic orbit. The intersections of these rectangles correspond to $H_{\vec{c}}$ which have a cluster of two periodic orbits.

The computation of this diagram used a discrete version of the HM construction. The construction depends on integers p, q, μ with $0 < p/q < 1$, p and q relatively prime, and $0 \leq \mu \leq 2(q - p)$. The discrete circle is the finite cyclic group $\mathbb{Z}/2q\mathbb{Z} = \mathbb{Z}_{2q}$ and it is acted on by $R_p : n \mapsto n + p$. The address intervals are $X'_0 = [1, \mu]$, $X'_1 = [\mu + 1, \mu + p]$, $X'_2 = [\mu + p + 1, 2q - p]$, and $X'_3 = [2q - p + 1, 2(q - p)]$. Let $B'(p, q, \mu)$ be the itinerary of the point 1 under R_p.

Using Theorem 8.5(b), when p is odd, R_p has a single period $2q$ orbit in $\mathbb{Z}_{2q}$. Expanding the points in $\mathbb{Z}_{2q}$ to intervals in the circle as in the proof of Theorem 8.5(c), we see that by varying μ the construction generates all the symbolic p/q-periodic 2-fold semi-monotone sets in Ω_2.

Now when p is even, R_p has a pair of period q orbits. When μ is odd, these generate different periodic orbits $B'(p, q, \mu)$. However, μ even corresponds to a pure parameter and so varying μ through the even μ generates all the symbolic p/q-periodic 2-fold semi-monotone sets in Ω_2.

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The next step is to use $B'(p, q, \mu)$ to compute its symbolic box as in Corollary 14.1 below. Finally, we take the inverse of the itinerary map for the model map to get a box in the $\vec{c}$ parameter. Because the map f_m has uniform slope of three in its positive-slope region the formula for this inverse is $\underline{s} \in \Sigma_2^+$,

$$\iota_1^{-1}(\underline{s}) = \sum_{j=0}^{\infty} \frac{s_j}{3^{j+1}}. \tag{13.3}$$

14. *Symbolic kfsm sets and the map $z \mapsto z^n$*

Using the model map, the characterization of 'physical' kfsm sets in Theorem 13.2 can be transformed into a characterization of symbolic kfsm sets. For compact $\hat{Z} \subset \Omega_k$ for $j = 1, \ldots, k$, define

$$\hat{\ell}_j(Z) = \sigma_k(\max\{\hat{Z} \cap [2j-1]\}) \quad \text{and} \quad \hat{r}_j(Z) = \sigma_k(\min\{\hat{Z} \cap [2j]\}).$$

Since ι_k is order-preserving and onto for the model map we have the following corollary.

COROLLARY 14.1. *Assume $\hat{Z} \subset \Omega_k$ is compact and shift invariant. The following assertions are equivalent.*

(1) *$\hat{Z}$ is kfsm.*

(2) *For $j = 1, \ldots, k$, $\hat{\ell}_j(\hat{Z}) \leq \hat{r}_j(\hat{Z})$ with indices reduced mod $2k$.*

If Z is recurrent we know that each $Z \in \mathcal{B}_k(f)$ has $\iota_k(\hat{Z}) = B_k(\omega, \vec{\nu})$ for some allowable $(\omega, \vec{\nu})$ which yields an indirect connection between the interpolated semi-monotone maps and HM parameterization.

There is a well-known connection between the dynamics of $d_n : z \mapsto z^n$ and the full shift on n symbols. This yields a connection of the symbolic kfsm sets as described by this corollary to invariant sets of the circle on which the action of d_n is semi-monotone, sometimes called *circular orbits*.

In Figure 6 we show the conditions forced by Corollary 14.1 as flat spots in the graph of d_n for $k = 3$ and $n = 6$. There are two classes of flat spots. Those in class A are forced by the condition that $\hat{Z} \subset \Omega_k$ and thus satisfies (5.1). These are the intervals of width $1/6$, $[1/18, 2/9]$, $[7/18, 5/9]$ and $[13/18, 8/9]$. These conditions are satisfied by all symbolic kfsm sets in the corollary. The other three flat spots in class B are determined by the conditions in part (b) of the corollary and vary with the symbolic kfsm set. Note that adding all the flat spots yields a degree-one semi-monotone circle map as expected. See figures in [**21**, **34**].

This figure also illustrates a clear difference between the kfsm sets for bimodal circle maps and circular orbits for d_n. Specifically, the kfsm sets correspond to a specific subclass of circular orbits for d_{2k}. On the other hand, there is clearly a tight relationship between the theories which needs to be investigated. Perhaps the degree reduction process described in [**8**, **34**] would be a good place to start.

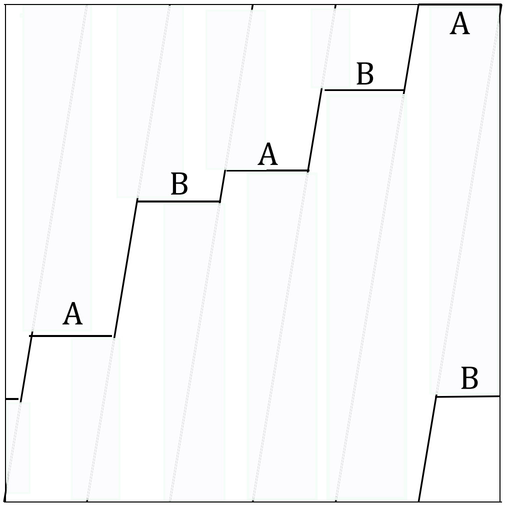

FIGURE 6. The semi-monotone map corresponding to a symbolic 3-fold semi-monotone set interpolated into $z \mapsto z^6$.

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